Solvable spin models on one-dimensional chain have found a variety of applications in the description of stationary and non-equilibrium processes. It is pointed out that some of these models being solved by the quantum inverse scattering method can be transformed one to another by an appropriate twist of the underlying dynamical symmetry quantum algebra. Connections among spectra and eigenfunctions of the twist related hamiltonians are discussed.
1 Introduction

Integrable models\(^1\) solved by the quantum inverse scattering method (QISM)\(^2,3,4\) have a wide range of applications in quantum field theory and statistical mechanics. Recently some solvable spin models have found interesting applications in the description of non-equilibrium processes\(^5-9\) and renormalization group method.\(^10\) One of the characteristic features of these spin models is that corresponding hamiltonians are often non-hermitian.\(^7,11\) We point out in this paper, that in the theory of quantum groups\(^12,13\) there exists an interesting transformation (twist) which relates at first sight different spin models in the framework of the QISM.\(^11,14\) The explicit knowledge of this transformation element (F-matrix) leads to explicit relations of different physically relevant properties of spin models (spectrum, eigenvectors).

Particular twist transformations were applied for the Heisenberg chain of spin 1/2 to get so-called 7-vertex model.\(^11,15\) We write down general formulas of hamiltonian transformations under the twist procedure, and apply them to the spin models used in Ref. 7 to describe certain reaction-diffusion processes. Although the changes of hamiltonians due to the twisting are relatively simple, the structure of their eigenstates (Bethe states) and construction of these Bethe states can be more complicated. An isotropic sl(2) Gaudin model deformed by a jordanian classical r-matrix is considered as an example.

2 Reaction-diffusion processes and spin systems

Starting, probably, from the paper on the Ising model with random interaction,\(^5\) the spin systems on one-dimensional chain were used to describe also different reaction-diffusion processes. The basis vectors \(|\alpha>, \alpha = 0, 1, \ldots, N - 1\) of an N dimensional state space \(V\) at a given site of the chain are identifying with different species of particles or molecules \(A_\alpha\), while spin hamiltonian coefficients are given by the transition rates of reactions corresponding to the operator structure: diffusion (to the right or to the left), coagulation, annihilation, etc (see details in Ref. 7). The state of the system is given by a probability distribution \(P(\{\alpha_j\})\), which is in one to one correspondence with normalized vectors of the spin state space \(\mathcal{H}\)

\[
\mathcal{H} = \otimes_k V_k, \quad |P\rangle = \sum P(\{\alpha_j\}) \otimes_k |\alpha_k\rangle
\]

with non-negative coefficients in the expansion over the basis vectors. The main problems related to this non-equilibrium system are its time evolution, given by a master equation, and the evaluation of different expectation values or correlation functions.
There is a very special requirement on the spin hamiltonian $H$ entering the evolution equation (master equation):

$$\frac{d}{dt} |P\rangle = -H |P\rangle ,$$  \hspace{1cm} (1)

where $|P\rangle$ is the state and $H$ is the spin hamiltonian. This requirement (conservation of probability) reads:

$$\langle \text{all}|H = 0 , \quad \langle \text{all} = \sum_{\{\sigma_j\}} \langle \{\sigma_j\} |,$$

(2)

where $\{\sigma_j\}$ are all basis vectors of tensorial structure $\otimes_j \langle \sigma_j\rangle$. It is interesting to note that considering quantities like the mean concentration of a type $A$ particles, are given by expressions different from usual quantum mechanical expectation values, e.g.

$$c_A(t) = \langle \text{all}|\Pi_A \exp(-Ht) |P\rangle ,$$

where $\Pi_A = \sum_k |\alpha_A\rangle_k \langle \alpha_A|_k$ is the projectors on the states corresponding to the particle $A$.

## 3 Twisting of quantum groups and integrable models

One of the main algebraic ingredients of the quantum inverse scattering method (QISM) is the theory of quantum groups.\textsuperscript{13} A quantum group (in this context better to call it a quantum algebra $\mathcal{A}$) is a particular case of the Hopf algebra i.e. it is an associative algebra with certain additional maps: the counit $\epsilon$, the antipode $S$ and the coproduct $\Delta$. The latter one is an algebra homomorphism from the quantum algebra $\mathcal{A}$ to the tensor square of this algebra $\mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ so that

$$\Delta(a) = \sum b_i \otimes c_i , \quad a, b_i, c_i \in \mathcal{A} .$$

The coproduct map $\Delta$ satisfies several axioms and as a consequence it defines a representation of $\mathcal{A}$ in the tensor product of its representations $\mathcal{V}_q \otimes \mathcal{V}_q$. One can consider composition of any number of coproducts $\Delta^{(n)} : \mathcal{A} \rightarrow \mathcal{A}^\otimes n$,

$$\Delta^{(n)} = (id^\otimes n \cdot 2 \otimes \Delta) \circ \cdots (id \otimes \Delta) \circ \Delta,$$

(3)

and the image does not depend on the order of $(n-2)$ maps $\Delta$’s (coassociativity property), e.g. $(id \otimes \Delta) \circ \Delta(a) = (\Delta \otimes id) \circ \Delta(a)$.

A very special and characteristic property of quantum algebras used in the QISM, is the existence of a universal $R$-matrix $R \in \mathcal{A} \otimes \mathcal{A}$ intertwining the
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Coproduct $\Delta$ with the opposite coproduct $\Delta' = \tau \circ \Delta$, $\Delta'(a) = \tau(\sum b_i \otimes c_i) = \sum c_i \otimes b_i$: $R\Delta = \Delta'R$. By fixing the set of representations $\rho_n, n = 1, 2, \ldots, M$, of the quantum algebra, an $L$-operator defining integrable model on chain with $M$ sites, can be obtained from the universal R-matrix:

$$L_{0n} = (\rho_0 \otimes \rho_n)R, \quad R = \sum R^{(1)}_i \otimes R^{(2)}_i, \quad R^{(j)}_i \in \mathcal{A},$$

$$T = L_{0N}L_{0N-1}\cdots L_{01} = (\rho_0 \otimes \prod_{i=1}^{N} \rho_n)(id \otimes \Delta^{(N)})R,$$

where for simplicity the spectral parameter dependence of the $L$-operator and the monodromy matrix $T(\lambda)$ is not displayed. The hamiltonian of the spin model is extracted from the matrix $T(\lambda)$ by quantum trace formulas. In particular, the generating function of mutually commuting integrals of the motion (the transfer matrix) is given just be the trace of $T(\lambda)$: $t(\lambda) = trT(\lambda)$.

There is quite a useful transformation of coproduct in the theory of quantum groups which is called twisting. It is defined by an invertible element $F \in \mathcal{A} \otimes \mathcal{A}$, $F = \sum f_i^{(1)} \otimes f_i^{(2)}$, $f_i^{(j)} \in \mathcal{A}$ and the similarity transformation of the original coproduct (the antipode $S$ is also transformed)

$$\Delta_i(a) = F\Delta(a)F^{-1}.$$  

However, to stay in the realm of quantum groups the twisted coproduct has to satisfy the Hopf algebra axioms. It is sufficient that the twisting element satisfies the equations

$$(\epsilon \otimes id)F = (id \otimes \epsilon)F = 1.$$  

$$F_{12}(\Delta \otimes id)F = F_{23}(id \otimes \Delta)F,$$

where the indices define a non-trivial embedding of $F$ into $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$, e.g. $F_{23} = 1 \otimes F$. The twist equation (8) is difficult to solve in full generality. Still many twisting elements are known explicitly. Few of them will be used below to explain connections between different integrable models. A nontrivial character of these connections is due to the QISM formalism and the following transformation property of the universal $R$-matrix of the quantum group

$$R^{(t)} = F_{21}R,$$  

where $F$ is the twisting element.
Often the choice of representation (e.g. fundamental) results in a simple structure of the $R$-matrix (as in the case of hamiltonians found in Ref. 7)

$$R(u) = (\rho \otimes \rho) R = uR - \frac{1}{u} R_{21}^{-1},$$

where $\mathcal{R}$ is the universal $R$-matrix of the quantum affine algebra $A$, while $R$ is constant $R$-matrix of finite dimensional quantum algebra $U_q(g) \subset A$. The corresponding hamiltonian is expressed in terms of $R$, provided $R - R_{21}^{-1} = \omega \mathcal{P}$ is permutation operator in $V_\rho \otimes V_\rho$.

$$H = \sum_n h_{n-1n} = \sum_n (\mathcal{P}_{n-1n} R_{n-1n}) = \sum_n \tilde{R}_{n-1n}.$$  

(11)

The twisting element $\mathcal{F}$ in representation space $V_\rho \otimes V_\rho$, is also a matrix $F = (\rho \otimes \rho) \mathcal{F}$, and according to (9) the hamiltonian density changes by a similarity transformation

$$h^{(t)}_{n-1n} = \mathcal{P}_{n-1n}(F_{n-1n} R_{n-1n} F^{-1}_{n-1n}) = F_{n-1n} \tilde{R}_{n-1n} F^{-1}_{n-1n}.$$  

(12)

Although this similarity transformation depends on the site number $n$ there exists a global operator $K$ acting on $\bigotimes_{n=1}^M V_n$ giving similarity transformation between original and twisted hamiltonians with free end boundary conditions

$$H^{(t)} = \sum_{n=2}^M h^{(t)}_{n-1n} = K(\sum_{n=2} h_{n-1n}) K^{-1},$$

(13)

$$K = (\rho \otimes \rho \otimes \cdots \otimes \rho) \left( \mathcal{F}_{12}(\Delta \otimes \text{id}) \mathcal{F} \cdots (\Delta^{(M-2)} \otimes \text{id}) \mathcal{F} \right).$$  

(14)

Once we know the operator $K$ or even just its existence, the conclusion about the coincidence of spectra of $H^{(t)}$ and $H$ can be drawn.

Let us point out those models and/or $R$-matrices for which twist elements relating these models with well-known ones, can be found.

1. Non-hermitian free fermion model ($XX\xi$) which was analyzed in detail in Ref. 7 (B.5), (B.8), is related by a scaling to the standard XY model twisting by the element $\mathcal{F} = \exp(\xi \chi^- \otimes \chi^-)$, where $\chi^-$ is a nilpotent generator of $gl_q(1|1)$ having a primitive coproduct after an additional twist of Reshetikhin type and transformation of $Z_2$-graded $R$-matrix to the usual one. Due to the embedding of $gl_q(1|1) \subset gl_q(m|n)$ one can apply this twist to $R$-matrices corresponding to the higher rank quantum super-algebras (cf Ref. 7 (C.16)). Hence, some integrable models have a hidden super-algebra symmetries.
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2. Spin model with three-dimensional space of states at each site. Its hamiltonian is given by the Cremmer–Gervais $R$-matrix\textsuperscript{17}

$$H^{(t)} = \sum_{n=1}^{M-1} h_{nn+1}^{(t)},$$

where the hamiltonian density is given by the matrix acting in $C_3 \otimes C_3$ ($\omega = q - 1/q$):

$$qI - h_{nn+1}^{(t)} = \begin{pmatrix}
q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & q\nu & 0 & q/p^2 & 0 & 0 & 0 & 0 & 0 \\
0 & p & 0 & \omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & q & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/p & 0 & 0 & 0 & 0 \\
0 & 0 & p^2/q & 0 & -p^2\nu/q & 0 & 0 & 0 & 0 & \omega & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & p & 0 & \omega & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 & 0 & 0 \\
\end{pmatrix}$$

Using the basis matrices $(E_{ab})_{ij} = \delta_{ai}\delta_{bj}$, the density can be written as:

$$qI - h_{nn+1}^{(t)} = \sum_{i=1}^{3} qE_{ii} \otimes E_{ii} + (pE_{21} \otimes E_{12} + \frac{1}{p}E_{12} \otimes E_{21}) + q\nu(pE_{12} \otimes E_{32} - (p/q)^2E_{32} \otimes E_{12}) + (p^2/q E_{31} \otimes E_{13} + q/p^2 E_{13} \otimes E_{31}) + \omega \sum_{a<b} E_{bb} \otimes E_{aa}.$$ 

The universal twist element $\mathcal{F}$ has been proposed in Ref.\textsuperscript{18}. It is interesting that the probability conservation condition (2) requires $p = q$, while the parameter $\nu$ is arbitrary.

The extra non-zero elements in twisted $R$-matrices (e.g. with $\nu$-dependence in the Cremmer–Gervais $R$-matrix) vs to the standard quantum group $R$-matrices\textsuperscript{2,3,4} result in a more complicated structure of the commutation relations of the monodromy matrix $T$ entries, and create difficulties in constructing the eigenvectors of the hamiltonian by an algebraic Bethe Ansatz (cf Ref.\textsuperscript{19–21}). These difficulties will be discussed in the next Section using the deformation of the isotropic $sl(2)$ Gaudin model by a jordanian classical $r$-matrix. The latter model can be obtained from the deformed $XXX$ spin chain\textsuperscript{11} by a quasi-classical limit.
4 On Bethe Ansatz for a deformed Gaudin model

Gaudin models can be constructed using any classical \( r \)-matrix. Let us start with the classical \( r \)-matrix of the trigonometric \( XXZ \)-Gaudin model
\[
\hat{r}_{XXZ}(\lambda) = (\sinh \lambda)^{-1} \left( e^{\lambda r_{DJ}} + e^{-\lambda (r_{DJ})_{21}} \right),
\]
where \( r_{DJ} = h \otimes h + X^+ \otimes X^- \) is the classical \( r \)-matrix of the quantum algebra \( sl_q(2) \). The similarity transformation with \( A(y) = \exp y X^+ \) gives
\[
\hat{r}_{XXZ}(\lambda) = \text{Ad} (\exp y X^+) \hat{r}_{XXZ}(\lambda) = r_{XXZ}(\lambda) - 2 y r_j,
\]
\[ r_j = h \otimes X^+ - X^+ \otimes h. \]
The additional term, the jordanian \( r \)-matrix \( r_j \), leads to changes in the commutation relations for trigonometric loop algebra
\[
[L_1(\lambda), L_2(\nu)] = -[\hat{r}_{XXZ}(\lambda - \nu), L_1(\lambda) + L_2(\nu)],
\]
where as usual the \( L \)-operator \( L(\nu) \) is a matrix in the fundamental representation
\[
\left( \begin{array}{cc} h(\nu) & X^-(\nu) \\ X^+(\nu) & -h(\nu) \end{array} \right).
\]
However the resulting transfer matrix \( t(\lambda) = \frac{1}{2} \text{tr} L(\lambda)^2 \) is just the similarity transformation of the initial \( t(\lambda) \). Hence, their spectra coincide, and their eigenfunctions are related by the same similarity transformation in the quantum space \( \text{Ad} (\otimes_k A_k(y)) \).

To get an isotropic \( XXX \)-Gaudin model\(^{22} \) from the trigonometric model one has to do a scaling limit:
\[
\lambda \to \epsilon \lambda, \quad y \to \xi / \epsilon, \quad \epsilon \to 0.
\]
The \( L \)-operator has the usual expression in terms of the loop algebra generators \( h(\lambda), X^+(\lambda), X^-(\lambda) \). However due to the jordanian term in the \( r \)-matrix their commutation relations are more complicated
\[
[h(\lambda), h(\nu)] = -\xi (X^+(\lambda) - X^+(\nu)),
\]
\[
[h(\lambda), X^-(\nu)] = \frac{X^+(\lambda) - X^-(\nu)}{\lambda - \nu} - 2 \xi h(\nu),
\]
\[
[h(\lambda), X^+(\nu)] = \frac{X^+(\lambda) - X^+(\nu)}{\lambda - \nu},
\]
\[
[X^+(\lambda), X^-(\nu)] = -2 \frac{h(\lambda) - h(\nu)}{\lambda - \nu} - 2 \xi X^+(\lambda),
\]
\[
[X^-(\lambda), X^-(\nu)] = 2 \xi (X^-(\lambda) - X^-(\nu)).
\]

\(^{22}\) M. Chaichian and P. Kulish
This algebra is a quasi-classical limit of the deformed Yangian $\mathcal{Y}_\xi(sl(2))$.\footnote{spin Hamiltonians}{317}

The transfer matrix $t(\lambda)$ has finite scaling limit, as well as its spectrum despite of the singular behavior of $y \rightarrow \xi/\epsilon$. Hence the spectrum of the deformed XXX-Gaudin model coincides with the original one. However, the similarity transformation of the original creation operators $X^-(\nu)$ in the quantum space do not have finite scaling limit, and the same is true for the non-trivial eigenvectors. By this reason their construction requires a recurrence algebraic Bethe Ansatz.\footnote{e}{19,21}

Let us pointed out that realization of the commutation relations (17) can be given as usual in terms the local generators $h_k$, $X^\pm_k$ and the inhomogeneity parameters $z_k$ related to the sites $k = 1, 2, \ldots, N$ of the chain. The realization is

$$
h(\lambda) = \sum_{k=1}^{N} \left( \frac{h_k}{\lambda - z_k} \right) - \xi X^+_g \lambda, \quad X^-(\lambda) = \sum_{k=1}^{N} \left( \frac{X^-_k}{\lambda - z_k} \right) + 2\xi X^+_g \lambda, \quad (18)
$$

with unchanged $X^+(\lambda)$ and $X^+_g = \sum_{k=1}^{N} X^+_k$. Taking the highest spin vector $\Omega$ as the bare vacuum (it is stable under the similarity transformation)

$$
h(\lambda)\Omega = \rho(\lambda)\Omega, \quad X^+(\lambda)\Omega = 0, \quad \otimes k A_k(y)\Omega = \Omega, \quad (19)
$$

one can obtained the eigenvectors of $t(\lambda)$ acting on $\Omega$ by the lowering generators $X^-(\nu_j)$. However, due to their non-commutativity among themselves (17) the “multi-particle” creation operators have elaborated structure (cf Ref. 19 and 21)

$$
t(\lambda)\Psi(\nu_1, \nu_2, \ldots, \nu_M) = \Lambda(\lambda; \{\nu_j\})\Psi(\nu_1, \nu_2, \ldots, \nu_M), \quad (20)
$$

$$
\Psi(\nu_1, \nu_2, \ldots, \nu_M) = B_M(\nu_1, \nu_2, \ldots, \nu_M)\Omega, \quad (21)
$$

$$
B_1(\nu) = X^-(\nu), \quad B_2(\nu_1, \nu_2) = X^-(\nu_1)X^-(\nu_2) - 2\xi X^-(\nu_1), \quad (22)
$$

$$
B_3(\nu_1, \nu_2, \nu_3) = X^-(\nu_1)X^-(\nu_2)X^-(\nu_3) - 2\xi (2X^-(\nu_1)X^-(\nu_2) + X^-(\nu_2)X^-(\nu_3)) + 8\xi^2 X^-(\nu_1).
$$

These operators are obtained by commuting $t(\lambda)$ with the ordered product of $X^-(\nu_j)$ and correcting the result by a lower order products to get eigenvector provided the Bethe equations on the set of quasimomenta $\{\nu_j\}_{j=1}^{M}$ are satisfied

$$
\rho(\nu_j) - \sum_{k \neq j}^{M} \frac{1}{\nu_j - \nu_k} = 0. \quad (23)
$$

The eigenvalue of $t(\lambda)$ is the same as for the $sl(2)$-invariant Gaudin model\footnote{c}{22}

$$
\Lambda(\lambda; \{\nu_j\}) = (\rho(\lambda) - \sum_{k=1}^{M} \frac{1}{\lambda - \nu_k})^2 - \frac{\partial}{\partial \lambda} (\rho(\lambda) - \sum_{k=1}^{M} \frac{1}{\lambda - \nu_k}),(24)
$$
It is instructive to note that these creation operators are symmetric functions of quasimomenta \( \nu_j \). The commutation relation of \( t(\lambda) \) with the product of \( X^- (\nu_j) \) can be obtained by using the quantum \( M \)-operator

\[
[t(\lambda), L(\nu)] = [M(\lambda - \nu), L(\nu)],
\]

(25)

where

\[
M(\lambda - \nu) = - \text{tr} \left( r_{12}(\lambda - \nu) \frac{L_1(\lambda)}{1} \right) - \frac{1}{2} \text{tr} \left( r_{12}^2(\lambda - \nu) \right)
\]

(26)

with the second term as quantum corrections due to non-commutativity of the quantum \( L \)-operator entries (16). In particular, we have

\[
t(\lambda) X^- (\nu) = X^- (\nu) t(\lambda) + 2 \frac{X^- (\lambda h(\nu) - X^- (\nu) h(\lambda)}{\lambda - \nu}
\]

\[
-2\xi \left( X^- (\nu) - \xi \right) X^+ (\lambda) - 2\xi \left( h(\lambda) h(\nu) + h(\nu) h(\lambda) \right).
\]

It is interesting to note that the transfer matrix \( t(\lambda) \) commutes with the global generator \( X^g \), and the constructed eigenstates \( \Psi(\nu_1, ..., \nu_M) \) are the highest spin vectors

\[
X^g \Psi(\nu_1, ..., \nu_M) = 0.
\]

The complete set of the creation operators and correlation functions, as well as connection with the Knizhnik-Zamolodchikov equation and deformed conformal field theory will be studied in the near future.

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