Non-compact quantum groups and quantum Harish-Chandra modules

D. Shklyarov\textsuperscript{a} *, S. Sinel’shchikov\textsuperscript{a} *, A. Stolin\textsuperscript{b}, L. Vaksman\textsuperscript{a} *

\textsuperscript{a}Mathematics Division, Institute for Low Temperature Physics & Engineering
47 Lenin Ave, 61164 Kharkov, Ukraine

\textsuperscript{b}Chalmers University of Technology, Göteborg University, Department of Mathematics
412 96, Göteborg, Sweden

An important problem of the quantum group theory is to construct and classify the Harish-Chandra modules; it is discussed in this work. The way of producing the principal non-degenerate series representations of the quantum group $SU_{a,n}$ is sketched. A q-analogue for the Penrose transform is described.

A general theory of non-compact quantum groups which could include, for instance, the quantum group $SU_{2,2}$, does not exist. However, during the recent years, a number of problems on non-commutative geometry and harmonic analysis on homogeneous spaces of such 'groups' was solved. In these researches, the absent notion of non-compact quantum group was replaced by Harish-Chandra modules over quantum universal enveloping algebra $U_q\mathfrak{g}$. This work approaches an important and still open problem in the theory of quantum groups, the problem of constructing and classifying quantum Harish-Chandra modules. A construction of the principal non-degenerate series of quantum Harish-Chandra modules is described in the special case of the quantum group $SU_{2,2}$. The notion of quantum Penrose transform is investigated.

The last named author is grateful to V. Akulov for numerous discussions of geometric aspects of the quantum group theory.

Everywhere in the sequel $\mathfrak{g}$ stands for a simple complex Lie algebra and $\{\alpha_1, \alpha_2, \ldots, \alpha_l\}$ for its system of simple roots with the standard ordering. The field $\mathbb{C}(q)$ of rational functions of the deformation parameter $q$ normally works as a ground field (when solving the problems of harmonic analysis, it is more convenient to assume $q \in (0, 1)$ and to set $\mathbb{C}$ as a ground field).

A background in quantum universal enveloping algebras was made up by V. Drinfeld and M. Jimbo in mid-80-ies. The principal results of this theory at its early years are expounded in the review [1] and in the lectures [2]. We inherit the notation of these texts; in particular, we use the standard generators $\{E_j, F_j, K_j^{\pm 1}\}_{j=1,2,\ldots,l}$ of the Hopf algebra $U_q\mathfrak{g}$ and relatively prime integers $d_j$, $j = 1, 2, \ldots, l$, which symmetrize the Cartan matrix of $\mathfrak{g}$ (note that $d_j = 1$, $j = 1, 2, \ldots, l$, in the case $\mathfrak{g} = sl_{l+1}$). We restrict ourselves to considering $\mathbb{Z}^l$-admissible $U_q\mathfrak{g}$-modules $V$ i.e. those admitting a decomposition into a sum of weight subspaces

$$V = \bigoplus_{\mu} V_\mu, \quad \mu = (\mu_1, \mu_2, \ldots, \mu_l) \in \mathbb{Z}^l,$$

$$V_\mu = \{v \in V| K_j^{\pm 1}v = q^{d_j \mu_j}v, \ j = 1, 2, \ldots, l\}.$$

Recall that some of the simple roots $\alpha_1, \alpha_2, \ldots, \alpha_l$ determine Hermitian symmetric spaces of non-compact type [3]. (The coefficients of such simple roots in an expansion of the highest root of $\mathfrak{g}$ is 1.) For example, in the case $\mathfrak{g} = sl_{l+1}$ all simple roots possess this property. Choose one such root $\alpha_{j_0}$ and introduce the notation $U_q\mathfrak{k}$ for the Hopf subalgebra generated by $K_j^{\pm 1}$, $E_j$, $F_j$, $K_j^{\pm 1}$, $j \neq j_0$. Of course, every $U_q\mathfrak{g}$-module $V$ is also a $U_q\mathfrak{k}$-module.

A finitely generated $\mathbb{Z}^l$-admissible $U_q\mathfrak{g}$-module $V$ is called quantum Harish-Chandra module if

1. $U_q\mathfrak{k}$-module $V$ is a sum of finite dimensional
simple $U_q\mathfrak{k}$-modules,

2. every finite dimensional simple $U_q\mathfrak{k}$-module $W$ occurs with finite multiplicity
   \[(\dim \text{Hom}_{U_q\mathfrak{k}}(W, V)) < \infty\).

There are several methods of construction and classification of classical Harish-Chandra modules [4]. A similar problem for quantum Harish-Chandra modules is still open. In our opinion, it is among the most important problems of the quantum group theory.

To describe obstacles that appear this way, consider the Hermitian symmetric space $SU_{2,2}/S(U_2 \times U_2)$. It is determined by the Lie algebra $\mathfrak{g} = \mathfrak{sl}_4$ and the specified simple root $\alpha_{10} = \alpha_2$ of this Lie algebra.

Our primary desire is to construct a q-analogue for the principal non-degenerate series of Harish-Chandra modules. This interest is partially inspired by Casselman’s theorem [14] which claims that every classical simple Harish-Chandra module admits an embedding into a module from the the principal non-degenerate series. A well known method of producing this series is just the inducing procedure from a subalgebra. Regretfully, this subalgebra has no q-analogue (this obstacle does not appear if one substitutes the subalgebra $U\mathfrak{t}$, thus substituting a subject of research, cf. [5]). Fortunately, a quantization is available for another less known method of producing the principal non-degenerate series. We describe this method in a simple special case of quantum $SU_{2,2}$.

Let $G = SL_4(\mathbb{C})$, $K = S(GL_2 \times GL_2)$, $B \subset G$ be the regular Borel subgroup of upper triangular matrices, and $X = G/B$ the variety of complete flags. It is known that there exists an open $K$-orbit in $X$, which is also an affine algebraic variety. The regular functions on this orbit constitute a Harish-Chandra module of the principal non-degenerate series. The regular differential forms of the highest degree form another module of this series. The general case is essentially approached by considering a generic homogeneous bundle on $X$ and subsequent restricting it to the open $K$-orbit. Note that the space of highest degree differential forms admits a $G$-invariant integral, which allows one to produce the principal non-degenerate series of unitarizable Harish-Chandra modules.

The above construction procedure can be transferred to the quantum case and leads to the principal non-degenerate series of unitarizable quantum Harish-Chandra modules.

In the classical representation theory, the above interplay between $K$-orbits on the variety and the theory of Harish-Chandra modules constitutes a generic phenomenon. In the theory of Beilinson-Bernstein, simple Harish-Chandra modules are derived from the so called standard Harish-Chandra modules. Furthermore, every series of standard Harish-Chandra modules is associated to a $K$-orbit in $X = G/B$. However, given an orbit $Q$ of a codimension $s > 0$, one should consider the local cohomology $H^s_Q(X, F)$ instead of functions on $Q$, with $F$ being a sheaf of sections of a homogeneous bundle (here our description of a standard module is somewhat naive but hopefully more plausible (the precise construction is expounded in [4])).

Our conjecture is that the standard quantum Harish-Chandra modules can be produced via some q-analogue of local cohomology $H^s_Q(X, F)$. An immediate obstacle that appears this way is in a lack of critical background in non-commutative algebraic geometry [6,7].

Probably the case of a closed $K$-orbit $Q \subset X$ is the simplest and most important one. Note that closed $K$-orbits are related to discrete series of Harish-Chandra modules, which are of an essential independent interest [8].

To conclude, consider a very simple example of a closed $K$-orbit in the space of incomplete flags which leads to the well known ‘ladder representation’ of the quantum $SU_{2,2}$ and a q-analogue of the Penrose transform [9].

We start with a purely algebraic description of the corresponding simple quantum Harish-Chandra module, to be succeeded with its two geometric realizations. Quantum Penrose intertwines these geometric realizations and is given by an explicit integral formula.

Consider the generalized Verma module $M$ over $U_q\mathfrak{sl}_4$ given by its single generator $v$ and the relations
The structure of a $U\mathfrak{sl}_4$-module in the space $\mathbb{C}[\text{Mat}_{2,2}]$ of polynomials on the space of matrices is introduced via embedding into the space of rational functions on the Grassmann variety of two-dimensional subspaces in $\mathbb{C}^4$:

\[
\psi\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right) \mapsto \\
\psi\left(\frac{1}{t_{(3,4)}^i t_{(1,4)}^j} t_{(3,4)}^1 t_{(2,3)}^j - \frac{1}{t_{(3,4)}^i t_{(1,4)}^j} t_{(3,4)}^1 t_{(2,4)}^j\right)
\]

(1)

Here $t_{(i,j)} = t_{1i} t_{2j} - t_{1j} t_{2i}$, $i < j$, and $t_{ij}$ are the generators of the algebra $\mathbb{C}[\text{Mat}_{2,4}]$ of polynomials on $2 \times 4$ matrices. This realization is can be transferred onto the quantum case, with the ordinary wave equation being replaced by its $q$-analogue

\[
\frac{\partial^2 \psi}{\partial \alpha \partial \delta} - q \frac{\partial^2 \psi}{\partial \beta \partial \gamma} = 0,
\]

and ordinary matrices by quantum matrices (cf. [10]). It is worthwhile to note that we use $U\mathfrak{sl}_4$-invariant differential calculus on the quantum space of $2 \times 2$ matrices.

The isomorphism of the two geometric realizations of the ladder representation is unique up to a constant multiple. In the case $q = 1$ it is given by the Penrose transform. It can be defined explicitly by the following integral formula:

\[
\int f(u_1, u_2, u_3, u_4) \mapsto \\
\int f\left(\sum_{i=1}^2 \zeta_i t_{1i}, \sum_{i=1}^2 \zeta_i t_{2i}, \sum_{i=1}^2 \zeta_i t_{13}, \sum_{i=1}^2 \zeta_i t_{14}\right) \, d\nu(\zeta),
\]

where it is implicit that

\[
(\zeta_1 t_{13} + \zeta_2 t_{23})^{-1} = \frac{1}{\zeta_2 t_{23}} \cdot \sum_{k=0}^{\infty} (-1)^k \left(\frac{\zeta_1 t_{13}}{\zeta_2 t_{23}}\right)^k,
\]

\[
(\zeta_1 t_{14} + \zeta_2 t_{24})^{-1} = \sum_{k=0}^{\infty} (-1)^k \left(\frac{\zeta_2 t_{24}}{\zeta_1 t_{14}}\right)^k \cdot \frac{1}{\zeta_1 t_{14}}
\]

and the $U\mathfrak{sl}_4$-invariant integral is given by

\[
\int \left(\sum c_{ij} \zeta_1^{i} \zeta_2^{j}\right) \, d\nu \overset{\text{def}}{=} c_{-1,-1}
\]

(a passage from the Plücker coordinates $t_{(i,j)}$ to polynomials on $\text{Mat}_{2,2}$ is described by (1)). For
example, in the case of a lowest weight vector $\frac{1}{u_3 u_4}$ we have

$$\frac{1}{u_3 u_4} \mapsto \frac{1}{t_{13} t_{24} - t_{14} t_{23}} \mapsto 1.$$  

To pass from the classical case to the quantum one it suffices to replace the ordinary space $\mathbb{C}^2$ with its quantum analogue: $\zeta_1 \zeta_2 = q \zeta_2 \zeta_1$, the ordinary product in $\sum \zeta_i t_{ij}$ with the tensor product and to order multiples in the above formulae in a proper way.

We thus get the quantum Penrose transform, which is an isomorphism of the two geometric realizations of the quantum Harish-Chandra module $L$.

It is well known [11] that the quantum Harish-Chandra module $L$ is unitarizable. The second geometric realization of this module allows one to find the corresponding scalar product as in [12] (using an analytic continuation of the scalar product involved into the definition of the holomorphic discrete series [13]).

The precise formulations and complete proofs of the results announced in this work will be placed to the Eprint Archives (http://www.arXiv.org/find/math).

REFERENCES