1.) Write a routine in double precision using the Lax-Wendroff scheme to solve the linear advection equation
\[
\frac{\partial w}{\partial t} = -c \frac{\partial w}{\partial x},
\]
where \(w\) is the quantity being advected, \(t\) is time, \(x\) is the (one) spatial dimension and \(c\) is the velocity of advection. The Lax-Wendroff scheme was outlined in class and also in section 5.1.5 of Vesely. It can be summarized as follows:
\[
w_{k+1}^{n+1} = w_k^n - \frac{\Delta t c}{\Delta x} (w_{k+1/2}^{n+1/2} - w_{k-1/2}^{n+1/2}),
\]
where, for example,
\[
w_{k+1/2}^{n+1/2} = \frac{1}{2} (w_{k+1}^n + w_k^n) - \frac{\Delta t c}{2\Delta x} (w_{k+1}^n - w_k^n).
\]
The stability, CFL condition, is \(a = (\Delta t |c|/\Delta x) < 1\). If your spatial grid spans \((x_1, x_K)\), note that you need to include an extra ‘ghost’ or boundary zone on each end, so that your arrays should include indices 0 through \(K+1\).

a) Test your routine with CFL parameter, \(a = 0.99\), by advecting the function with initial conditions \(w(x,0) = \sin(2\pi x(K-1)/K)\), where \(x_1 = 0\) and \(x_K = 1.0\). Apply the propagation velocity \(c = \pm 1.0\) (both values) and let \(K = 256\). Apply periodic boundary conditions, so that \(x_0 = x_K\) and \(x_{K+1} = x_1\) at each time. Since the advection speed is unity, the function should lie precisely on top of the initial conditions after \(t = 1, 2, 3, \ldots\) Confirm graphically that this is so for \(t = 1, 2\). If your routine is working properly the initial and advected functions should be virtually indistinguishable at these times. Also plot \(w\) at intermediate times, \(t = 0.3, 0.5, 0.67\) to confirm that advection is taking place. Note that your end time will usually be only approximately the desired value, since the actual time will be constrained by the CFL parameter.

b) Now run your code on the same set up until \(t = 0.5\), using CFL parameters \(a = 0.99\) and \(a = 1.1\). Plot the solutions, describe and explain the comparative behaviors.

c) Now advect the initial function \(w(x,0) = 1\) for \(0<x<1/2\) and \(w(x,0) = 0\) elsewhere; that is a ‘top hat’ distribution. Again use \(K = 256\), periodic boundaries and \(a = 0.99\), with \(c = +1\). Plot the solutions at \(t = 0, 1, 2, 3, 4\). Make a similar plot for a run with \(a = 0.50\). How do they compare? Which solution is more accurate (that is, which ‘a’ value ‘works better’)?