1.) Write a routine in double precision using the Lax-Wendroff scheme to solve the linear advection equation

\[ \frac{\partial w}{\partial t} = -c \frac{\partial w}{\partial x}, \]

where \( w \) is the quantity being advected, \( t \) is time, \( x \) is the (one) spatial dimension and \( c \) is the (assumed constant) velocity of advection. The Lax-Wendroff scheme was outlined in class and also in chapter 5 of Vesely. It can be summarized as follows:

\[
w^{n+1}_k = w^n_k - \frac{\Delta t c}{\Delta x} (w^{n+1/2}_{k+1/2} - w^{n+1/2}_{k-1/2}), \]

where, for example,

\[
w^{n+1/2}_{k+1/2} = \frac{1}{2} (w^n_{k+1} + w^n_k) - \frac{\Delta t c}{2\Delta x} (w^n_{k+1} - w^n_k). \]

The stability, CFL condition, is \( a = (\Delta t|c|/\Delta x) < 1 \). If your spatial grid spans \((x_1,x_K)\), note that you need to include a ‘ghost’ or boundary zone on each end (total 2), so that your arrays should include indices 0 through \( K+1 \).

a) Test your routine with CFL parameter, \( a = 0.99 \), by advecting the function with initial conditions \( w(x,0) = \sin[2\pi x(K-1)/K] \), where \( x_1 = 0 \) and \( x_K = 1.0 \). Apply the propagation velocity \( c = \pm 1.0 \) (both values) and let \( K = 256 \). Apply periodic boundary conditions, so that \( x_0 = x_K \) and \( x_{K+1} = x_1 \) at each time. Since the advection speed is unity, the function should lie precisely on top of the initial conditions after \( t = 1,2,3, \ldots \) Confirm graphically that this is so for \( t = 1,2 \). If your routine is working properly the initial and advected functions should be virtually indistinguishable at these times. Also plot \( w \) at intermediate times, \( t = 0.3, 0.5, \) 0.67 to confirm that advection is taking place. Note that your end time will usually be only approximately the desired value, since the actual time will be constrained by the CFL parameter. Unless the specified time interval divided by \( a \), is an integer, your end time will not exactly match.

b) Now run your code on the same set up until \( t = 0.5 \), using CFL parameters \( a = 0.99 \) and \( a = 1.1 \). Plot the solutions, describe and explain the comparative behaviors.

c) Now advect the initial function \( w(x,0) = 1 \) for \( 0 < x < 1/2 \) and \( w(x,0) = 0 \) elsewhere; that is a ‘top hat’ distribution. Again use \( K = 256 \), periodic boundaries and \( a = 0.99 \), with \( c = +1. \) Plot the solutions at \( t = 0,1,2,3,4 \). Make a similar plot for a run with \( a = 0.50 \). How do they compare? Which solution is more accurate (that is, which ‘\( a \)’ value ‘works better’)?

2.) This problem addresses the same linear advection equation using the finite volume methods discussed in class.

a) Write a routine in double precision applying a slope-limited, upwinded finite volume scheme to solve the linear advection equation

\[ \frac{\partial w}{\partial t} = -v \frac{\partial w}{\partial x}, \]

where \( w \) is the quantity being advected, \( t \) is time, \( x \) is the spatial dimension and \( v \) is the advection velocity (so is signed). As discussed in class such a scheme takes the form

\[
\langle w^{n+1}_k \rangle = \langle w^n_k \rangle - \Delta t \left( \frac{v}{\Delta x} (\langle w^{n+1/2}_{k+1/2} \rangle - \langle w^{n+1/2}_{k-1/2} \rangle) \right).
\]
where the angular brackets indicate a spatial average over a cell of width $\Delta x$ between $x_{k-1/2}$ and $x_{k+1/2}$, while the overbar represents a time average over the interval $\Delta t$ between $t_n$ and $t_{n+1}$. The time averaged values of $w$ are constructed as

$$\bar{w}_{k+1/2} = \begin{cases} 
\langle w_k \rangle + \frac{1}{2} \sigma_k \Delta x \left(1 - \frac{\Delta tv}{\Delta x}\right), & v > 0 \\
\langle w_{k+1} \rangle - \frac{1}{2} \sigma_{k+1} \Delta x \left(1 + \frac{\Delta tv}{\Delta x}\right), & v < 0 
\end{cases},$$

where $\sigma_k$ is the (suitably selected) slope of $w$ for cell $k$.

To eliminate numerical oscillations it is desirable to select the slope so that it is so-called “TVD”. A simple way to do that is the so-called “minmod” limited slope. In that case, if $(\langle w \rangle_{k+1} - \langle w_k \rangle)/\Delta x$ and $(\langle w \rangle_k - \langle w_{k-1} \rangle)/\Delta x$ have the same sign, one sets $\sigma_k$ to the smaller of the two. Otherwise $\sigma_k=0$ (since the cell is near an extremum).

b) Apply this routine to advect the top-hat function $w(x,t=0) = 1$ for $0.25 \leq x \leq 0.75$, else = 0, with $v = \pm 1$ (both values). Let your $x$ variable span the interval $[0,1]$ using $N=256$ cells. You will need two ghost or boundary cells on each end of your space in order to define the slopes for the cells 1 and $N$. Thus the total number of $w_k$ and $x_k$ will be $256+4 = 260$. Assume periodic boundary conditions, so that $w_0 = w_N$, $w_{N+1} = w_1$, $w_{-1} = w_{N-1}$, $w_{N+2} = w_2$, assuming the “real” space is spanned by $1 \leq k \leq N$. Advance the function over a time interval $t = 0$ to approximately $t = 2$ using a CFL constraint $a = |\Delta t v/\Delta x| = 0.99$. Plot results at $t = 0, 0.5, 1, 1.5, 2.0$. Note that because of the CFL constraint all your times will be only approximate. If your routine is working properly the solutions at $t = 1, 2$ should be virtually the same as the initial conditions, except the ‘corners’ of the hat will have ‘rounded’. There should be no oscillations. Estimate the width in cells of the corners of the evolved hat (say, values more than 5% from initial corner values). Does the width increase over time? Now double the spatial resolution. How many cells span the corners compared to the lower resolution case? That is, how does finer spatial resolution influence the ‘sharpness’ of a discontinuity?

c) Apply the above routine to the following problem. Let $w(x,t=0) = 0$ everywhere inside the interval $[0,1]$. In the two left ghost cells, $k = 0, -1$, let $w(x,t) = \sin[2\pi(t-x_k)/v]$. Let $v = 1$. Assume a continuous boundary on the right, so that $w_{N+1} = w_{N+2} = w_N$. Advance your solution to approximately $t = 2$. Illustrate it with plots at $t = 0, 0.5, 1.0, 1.5$ and $2.0$. If your code is working properly you should see a wave of unit amplitude propagate in from the left boundary and fill the simulation space after time $t = 1$. Compare this expectation with your result.