Adiabatic elimination and reduced probability distribution functions in spatially extended systems with a fluctuating control parameter

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We obtain the stationary probability distribution functions of the order parameter near onset for the one-dimensional real Ginzburg-Landau and Swift-Hohenberg equations with a fluctuating control parameter. A perturbative expansion in the intensity of the fluctuations leads to a hierarchy of Fokker-Planck equations for conditional probability distribution functions that relate components of the order parameter that evolve in different time scales. Successive integration leads to a Fokker-Planck equation for the slowest mode, which we solve analytically for the models studied. In all cases, the probability distribution function above onset is of the form $P(A_0)\propto A_0^{\delta}e^{-\gamma A_0^2}$, where A_0 is the slow component of the order parameter and the values of δ and γ depend explicitly on the intensity of the fluctuations. Knowledge of $P(A_0)$ allows the calculation of an effective bifurcation threshold and of the moments of A_0 above threshold.

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I. INTRODUCTION

We obtain the reduced probability distribution function near threshold in two models widely used to study pattern formation in extended systems: the real Ginzburg-Landau and Swift-Hohenberg equations when the control parameter has a fluctuating component. We allow the control parameter to have a small component that is periodic in space, but random in time. For the real Ginzburg-Landau equation, we also consider variations of the control parameter that are random in both space and time. In all three cases, we focus exclusively on the one-dimensional case, and recover known results concerning the dependence of the location of the instability threshold as a function of the intensity of the fluctuations in the control parameter. However, we are also able to obtain analytically the probability distribution of the order parameter near threshold by systematic elimination of degrees of freedom.

Progress in applying classical bifurcation theory to n-dimensional (or infinite dimensional) dynamical systems often involves the introduction of low-dimensional invariant manifolds (the center manifolds) on which the system displays the essential elements of the bifurcations under study [1–3]. The dynamic evolution off this manifold is treated as secondary, as in many cases the system of interest decays exponentially fast to the manifold for any initial condition that is not on (but close to) it. Accordingly, the original dynamical variables can be classified as fast or slow close to the bifurcation, and the former adiabatically eliminated to obtain a reduced description valid near the bifurcation. As an illustration, consider the following system of equations,

$$\frac{d}{dt}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & -\lambda \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} + \begin{bmatrix} -cA^3+\cdots \\ dA^3+\cdots \end{bmatrix} \tag{1}$$

near the bifurcation point $\alpha=0$, where the trivial solution $A=B=0$ loses stability. Provided that $\lambda\sim\mathcal{O}(1)$ there are two different time scales in Eq. (1). This becomes apparent if one introduces a slow time scale $T=\alpha t$, and the scaling $A\sim\mathcal{O}(\alpha^{1/2})$ and $B\sim\mathcal{O}(\alpha^{3/2})$, with $\alpha\ll 1$. Then $dB/dT\sim\mathcal{O}(\alpha^{5/2})\ll-\lambda B+dA^3$, and therefore the evolution of the so called fast variable B is confined to the center manifold $B_{cm}=dA^3/\lambda$ over the slow time scale T. As a result, B can be adiabatically eliminated from the dynamics, and Eq. (1) then reduces to the normal form equation for a pitchfork bifurcation.

For large (or infinite)-dimensional systems it is often more useful to model the effect of the fast variables as random sources (the ''thermal bath'') [4], and to interpret the bifurcation as a phase transition (in the thermodynamic limit). However, in the case of pattern forming systems, the characteristic scale of these fluctuations of thermal origin is much too small at the (macroscopic) energy scale of the slow variables, and they are usually negligible [5]. Nevertheless, it is often argued that other stochastic effects (not of microscopic origin) may enter the description of the system that are related to other degrees of freedom that cannot be completely controlled or specified [6,7]. For example, one can imagine that the control parameter for a particular bifurcation has a small random component. A case in point is Rayleigh-Bénard convection when the temperature control of the bounding solid surfaces is not perfect, and small spatial or temporal inhomogeneities may be present during the experiment. Although an idealization, it is useful to phenomenologically model the resulting temperature difference across the boundaries as a random function of space and time.

We focus in this paper on the case of fluctuations of external origin that enter the governing equations as random contributions to the system parameters. In a number of cases of interest, the existence of random parametric dependence still preserves the essential separation of time scales that allows a center manifold reduction in the stochastic case, and an analysis along the lines of classical bifurcation theory. We elaborate here on earlier work by Knobloch and Wiesenfeld [8], van den Broeck *et al.* [9], Xu and Roberts [10], and our own [11] that concerned systems without any spatial dependence. Consider again the system in Eq. (1) but now allow fluctuations in the control parameter $\alpha+\xi(t)$, where $\xi(t)$ is a Gaussian, white process of zero mean and intensity κ. Equation (1) now defines a stochastic process for the joint probability density $\mathcal{P}(A,B;t)$ at time t. The process of reduction of Eq. (1) to its normal form in the deterministic case motivates in the stochastic case the decomposition

$$\mathcal{P}(A,B;t)=p(B|A;t)P(A;t), \tag{2}$$

where $p(B|A;t)$ is the conditional probability density of B given a value of A. For small κ, it is anticipated that $p(B|A;t)$ and $P(A;t)$ will evolve over different time scales, thus allowing their separate determination within a perturbative expansion in κ. This is reminiscent of the separation of time scales in the underlying deterministic problem in that the fluctuations of B occur in a fast scale compared to the fluctuations in A. Once the equation for $P(A;t)$ has been obtained after integrating out the fluctuations in B, we find the stationary probability density $P(A)$, which can then be used to determine the location of the effective threshold in the stochastic case, as well as the moments of A above threshold. In short, $P(A)$ is a δ function at $A=0$ below threshold, whereas above threshold there exists another normalizable solution that has non vanishing moments.

We present in this paper the calculation of the probability distribution function on the center manifold of two widely studied equations that model pattern formation in one-dimensional systems. Our results are an extension of the calculations of Refs. [12–16] that addressed the location of the bifurcation threshold in each of the cases when the control parameter is random. Although the methodology can be more widely applied, we focus solely on the Ginzburg-Landau and Swift-Hohenberg [17] equations in one spatial dimension. We first reduce the evolution close to threshold to a set of coupled stochastic differential equations, which we solve recursively following the approach of Ref. [11]. The approximate probability distribution function on the center manifold is obtained, and is used to determine the location of the bifurcation threshold as a function of the intensity of the fluctuations. In general, we find a shift in the location of the onset, and a nonuniversal dependence of the order parameter on the distance away from threshold. Both results are seen to be a consequence of resonant interaction between randomness and the fast variables that produces slowly varying contributions, and hence corrections to the evolution on the underlying center manifold. In particular, our results for the location of the bifurcation threshold agree with earlier results obtained by direct linearization of the equations for the statistical moments that were given by Becker and Kramer [13], and Röder *et al.* [15]. Sections II and III describe our results for the Ginzburg-Landau and Swift-Hohenberg equations when the control parameter is periodic in space and randomly modulated in time. We finally consider the Ginzburg-Landau equation with an order parameter that is a random function of both space and time in Sec. IV.

II. REAL GINZBURG-LANDAU EQUATION WITH A TIME DEPENDENT, SPATIALLY PERIODIC CONTROL PARAMETER

We first consider the one-dimensional real Ginzburg-Landau equation in a spatially extended system. Its associated amplitude equation is the normal form for a pitchfork bifurcation. Motivated by earlier work by Röder *et al.* [15], we consider the case in which the control parameter of the bifurcation is modulated in space with wave number Q, with an amplitude that is a random function of time. In terms of a scalar field $\psi(x,t)$ the equation that we study reads,

$$\frac{\partial}{\partial t}\psi(x,t)=[\alpha+\xi(t)\cos(Qx)]\psi(x,t)-c\psi^3(x,t)+\frac{\partial^2}{\partial x^2}\psi(x,t), \tag{3}$$

where α and $c>0$ are real, and $\xi(t)$ is assumed to be a Gaussian white process of zero mean and variance $\langle\xi(t)\xi(t')\rangle=2\kappa\delta(t-t')$. The choice $Q\sim\mathcal{O}(1)$ allows a separation of time scales between the fundamental response (uniform in space), and harmonic response to the control parameter modulation (of wave number Q, and its higher harmonics).

In the deterministic limit of $\kappa=0$, Eq. (3) admits uniform stationary solutions $\psi=0$ (stable for $\alpha\leqslant 0$) and $\psi=\pm\sqrt{\alpha/c}$ (stable for $\alpha>0$), the bifurcation point being defined by $\alpha=0$. For the special case in which $\xi(t)$ is a constant, a time dependent solution may be found by considering a power series expansion

$$\psi(x,t)=\sum_{n=0}^{\infty}A_n(t)\cos(nQx), \tag{4}$$

where the amplitudes $A_n(t)$ are proportional to increasing powers of α, and hence a mode reduction is possible near onset ($\alpha\to 0$).

Our analysis of the stochastic case begins with the same expansion of $\psi(x,t)$ in power series, with the amplitudes $A_n(t)$ being stochastic processes in time. We first analyze the case in which only the first two terms of Eq. (4) are retained. A coupled system of ordinary stochastic differential equations for the two amplitudes follows,

$$\frac{d}{dt}\begin{bmatrix}A_0\\A_1\end{bmatrix}=\begin{bmatrix}\alpha A_0\\(\alpha-Q^2)A_1\end{bmatrix}-c\begin{bmatrix}A_0^3+3A_0A_1^2/2\\3A_0^2A_1+3A_1^3/4\end{bmatrix}+\xi\begin{bmatrix}0&1/2\\1&0\end{bmatrix}\times\begin{bmatrix}A_0\\A_1\end{bmatrix}. \tag{5}$$

Near onset (the location of which is yet unknown) $\alpha/|\alpha-Q^2|\ll 1$, suggesting that A_0 is a slow variable. We now introduce an expansion in the small amplitude of the noise

$\sqrt{\kappa}$, and anticipating the magnitude of the shift in the bifurcation threshold with κ, we further assume that α and κ will be of the same order at threshold: $\overline{\kappa}=\kappa/\epsilon^2$, $\overline{\alpha}=\alpha/\epsilon^2$, $\overline{A}_0=A_0/\epsilon$, and $\overline{A}_1=A_1/\epsilon^2$, with $\epsilon\ll 1$. These definitions imply that $A_0\gg A_1$ and that $\alpha\sim A_0^2$, expressions that are analogous to those found in the deterministic limit. Expressed in terms of the scaled variables, the Fokker-Planck equation that corresponds to Eq. (5) reads,

$$\partial_t\mathcal{P}=-\epsilon^2\frac{\partial}{\partial\overline{A}_0}\left\{\left[\left(\overline{\alpha}+\frac{\overline{\kappa}}{2}\right)\overline{A}_0-c\left(\overline{A}_0^3+\epsilon^2\frac{3\overline{A}_0\overline{A}_1^2}{2}\right)\right]\mathcal{P}\right\}+\epsilon^4\frac{\partial^2}{\partial\overline{A}_0^2}\left[\frac{\overline{\kappa A_1^2}}{4}\mathcal{P}\right]-\frac{\partial}{\partial\overline{A}_1}\left\{\left[\left(\epsilon^2\overline{\alpha}-Q^2+\epsilon^2\frac{\overline{\kappa}}{2}\right)\overline{A}_1-3c\epsilon^2\left(\overline{A}_0^2\overline{A}_1+\epsilon^2\frac{\overline{A}_1^3}{4}\right)\right]\mathcal{P}\right\}+\frac{\partial^2}{\partial\overline{A}_1^2}(\overline{\kappa A_0^2}\mathcal{P})+2\epsilon^2\frac{\partial^2}{\partial\overline{A}_0\partial\overline{A}_1}\left(\frac{\overline{\kappa A_1 A_0}}{2}\mathcal{P}\right), \tag{6}$$

where $\mathcal{P}(\overline{A}_0,\overline{A}_1;t)$ is the joint probability density at time t. To lowest order $[\mathcal{O}(1)]$, Eq. (6) reduces to

$$\partial_t p_1(\overline{A}_1|\overline{A}_0;t)=\frac{\partial}{\partial\overline{A}_1}\left[Q^2\overline{A}_1 p_1(\overline{A}_1|\overline{A}_0;t)+\overline{\kappa A_0^2}\frac{\partial p_1(\overline{A}_1|\overline{A}_0;t)}{\partial\overline{A}_1}\right], \tag{7}$$

where we have introduced the decomposition $p_1(\overline{A}_1|\overline{A}_0;t)=\mathcal{P}(\overline{A}_0,\overline{A}_1;t)/P(\overline{A}_0;t)$, with $p_1(\overline{A}_1|\overline{A}_0;t)$ the conditional probability density of $\overline{A}_1$ given $\overline{A}_0$. Therefore the conditional probability evolves over a time scale of $\mathcal{O}(1)$, and at this scale it relaxes to a stationary density given by

$$p_1(\overline{A}_1|\overline{A}_0)=\sqrt{\frac{Q^2}{2\pi\overline{\kappa A_0^2}}}\exp\left[-\frac{Q^2}{2\overline{\kappa A_0^2}}\overline{A}_1^2\right]. \tag{8}$$

The amplitude $\overline{A}_1$ follows a Gaussian distribution with zero mean and variance $\sigma^2=\overline{\kappa A_0^2}/Q^2$. At this order, the nonlinear terms in the Langevin equation for A_1 are negligible.

We next obtain an equation for $P(\overline{A}_0;t)$ by integrating Eq. (6) over $\overline{A}_1$. Terms involving partial derivatives with respect to $\overline{A}_1$ vanish, leaving the following equation for $P(\overline{A}_0;t)$

$$\partial_t P(\overline{A}_0;t)=-\epsilon^2\frac{\partial}{\partial\overline{A}_0}\left\{\left[\left(\overline{\alpha}+\frac{\overline{\kappa}}{2}\right)\overline{A}_0-c\overline{A}_0^3-\epsilon^2\frac{3}{2}c\overline{A}_0\langle\overline{A}_1^2|\overline{A}_0\rangle\right]P(\overline{A}_0;t)\right\}+\epsilon^4\frac{\partial^2}{\partial\overline{A}_0^2}\left[\frac{\overline{\kappa}\langle\overline{A}_1^2|\overline{A}_0\rangle}{4}P(\overline{A}_0;t)\right], \tag{9}$$

where

$$\langle\overline{A}_1^2|\overline{A}_0\rangle=\int_{-\infty}^{+\infty}d\overline{A}_1\overline{A}_1^2 p_1(\overline{A}_1|\overline{A}_0)=\frac{\overline{\kappa A_0^2}}{Q^2}. \tag{10}$$

Keeping the lowest order terms only [up to $O(\epsilon^2)$], Eq. (9) reduces to

$$\epsilon^2\partial_T P(\overline{A}_0;T)=-\epsilon^2\frac{\partial}{\partial\overline{A}_0}\left\{\left[\left(\overline{\alpha}+\frac{\overline{\kappa}}{2}\right)\overline{A}_0-c\overline{A}_0^3\right]P(\overline{A}_0;T)\right\}, \tag{11}$$

where we have introduced a slow time scale $T=\epsilon^2 t$. The solution of Eq. (11) is $P(\overline{A}_0;T)=\delta(\overline{A}_0-\sqrt{f(T)})$, where

$$f(T)=\frac{\overline{\alpha}+\overline{\kappa}/2}{c\{1+[(\overline{\alpha}+\overline{\kappa}/2)/c\overline{A}_0^{0^2}-1]\exp[-2(\overline{\alpha}+\overline{\kappa}/2)T]\}},$$

and $\overline{A}_0^0$ is the initial value of $\overline{A}_0$. At this order, the slow variable A_0 effectively satisfies the equation

$$\frac{d\overline{A}_0}{dT}=(\overline{\alpha}+\overline{\kappa}/2)\overline{A}_0-c\overline{A}_0^3. \tag{12}$$

Its evolution is purely deterministic, although the stochastic modulation has renormalized the linear part of the equation. Equation (12) also follows by averaging the Langevin equation for A_0 over the fast time scale. Explicitly,

$$\frac{dA_0}{dt}=\alpha A_0-c\left(A_0^3+\frac{3A_0\langle A_1^2\rangle}{2}\right)+\langle A_1\xi\rangle/2, \tag{13}$$

where we have approximated the temporal average over the fast time scale by an ensemble average. By making use of the Furutsu-Novikov theorem [18,19], we find

$$\langle A_1\xi\rangle\approx\langle A_1\rangle\langle\xi\rangle+\kappa\left\langle\frac{\delta A_1}{\delta\xi}\right\rangle=\kappa A_0, \tag{14}$$

implying that the correlation of $A_1\xi$ itself evolves over the slow time scale. We finally arrive at Eq. (12) by combining the last two equations and noting that $\langle A_1^2\rangle A_0/A_0^3\ll 1$.

In addition, we can obtain the location of the bifurcation point by considering the long time, stationary solution of Eq. (11) or, equivalently, that of Eq. (12). We find

$$\alpha_c=-\kappa/2, \tag{15}$$

and conclude that the bifurcation point is shifted relative to the deterministic threshold $\alpha=0$ by an amount that equals $\langle A_1\xi\rangle$, the correlation between the fast variable A_1 and the noise ξ.

The functional form $P(\overline{A_0};T)=\delta(\overline{A_0}-\sqrt{f(T)})$ also implies that the term $\overline{\alpha A_0}+\overline{\kappa A_0}/2-c\overline{A_0^3}$ on the right-hand side of Eq. (9) vanishes in the limit $T\to\infty$. This is consistent with the observation that, except for an initial transient, both drift and diffusion terms in that equation are of higher order in ϵ ($\mathcal{O}(\epsilon^4)$). As a result, the deterministic evolution of A_0 at $\mathcal{O}(\epsilon^2)$ becomes stochastic at $\mathcal{O}(\epsilon^4)$. In order to obtain $P(\overline{A_0};T)$ to that order in ϵ, it is now necessary to keep an additional term in the expansion of $\psi(x,t)$, that which has an amplitude $A_2\sim\mathcal{O}(\epsilon^3)$. We note at this point that a systematic expansion in ϵ can be developed, as successive amplitudes $A_n(t)$ in Eq. (4) will be proportional to higher powers of ϵ. At the order we consider now, terms only up to A_2 will contribute to the probability density of A_0. Equation (3) now leads to,

$$\frac{d}{dt}\begin{bmatrix}A_0\\A_1\\A_2\end{bmatrix}=\begin{bmatrix}\alpha A_0\\(\alpha-Q^2)A_1\\(\alpha-4Q^2)A_2\end{bmatrix}-c\begin{bmatrix}A_0^3+3A_0A_1^2/2+3A_1^2A_2/4+3A_0A_2^2/2\\3A_0^2A_1+3A_1^3/4+3A_0A_1A_2+3A_1A_2^2/2\\3A_0^2A_2+3A_0A_1^2/2+3A_1^2A_2/2+3A_2^3/4\end{bmatrix}+\xi\begin{bmatrix}0&1/2&0\\1&0&1/2\\0&1/2&0\end{bmatrix}\begin{bmatrix}A_0\\A_1\\A_2\end{bmatrix}. \tag{16}$$

The coefficients $(\alpha-Q^2)$ and $(\alpha-4Q^2)$ multiplying A_1 and A_2 on the right-hand side of Eq. (16) are both of $O(1)$. As a result, we expect both variables to evolve over the same (fast) time scale. We first obtain from Eq. (16) the Fokker-Planck equation satisfied by the joint probability density $\mathcal{P}(A_0,A_1,A_2;t)=P(A_0;t)p_1(A_1|A_0;t)p_2(A_2|A_0,A_1;t)$. Integration of that equation over both A_1 and A_2 then yields the reduced equation

$$\partial_t P(\overline{A_0};t)=-\epsilon^2\frac{\partial}{\partial\overline{A_0}}\left\{\left[\left(\overline{\alpha}+\frac{\overline{\kappa}}{2}\right)\overline{A_0}-c\overline{A_0^3}+\epsilon^2\frac{\overline{\kappa}\langle\overline{A_2}|\overline{A_0}\rangle}{4}-\epsilon^2\frac{3}{2}c\overline{A_0}\langle\overline{A_1^2}|\overline{A_0}\rangle\right]P(\overline{A_0};t)\right\}+\epsilon^4\frac{\partial^2}{\partial\overline{A_0^2}}\left[\frac{\overline{\kappa}\langle\overline{A_1^2}|\overline{A_0}\rangle}{4}P(\overline{A_0};t)\right], \tag{17}$$

where

$$\overline{A_2}=A_2/\epsilon^3$$

and

$$\langle\overline{A_2}|\overline{A_0}\rangle=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}d\overline{A_2}d\overline{A_1}\overline{A_2}p_1\times(\overline{A_1}|\overline{A_0})p_2(\overline{A_2}|\overline{A_0},\overline{A_1}).$$

Equation (17), which is valid to $O(\epsilon^4)$, contains one more term than our earlier result Eq. (9). We do not obtain an analytical expression for the conditional density $p_2(\overline{A_2}|\overline{A_0},\overline{A_1})$, but instead calculate its first moment directly from the Langevin equation for $\overline{A_2}$. Explicitly,

$$\epsilon^3\frac{d\langle\overline{A_2}\rangle}{dt}=\epsilon^3(-4Q^2+\epsilon^2\overline{\alpha})\langle\overline{A_2}\rangle-\epsilon^5 3c\left(\frac{\overline{A_0}\langle\overline{A_1^2}\rangle}{2}+\overline{A_0^2}\langle\overline{A_2}\rangle+\epsilon^2\frac{\langle\overline{A_1^2}\overline{A_2}\rangle}{2}+\epsilon^4\frac{\langle\overline{A_2^3}\rangle}{4}\right)+\frac{\epsilon^2}{2}\langle\xi\overline{A_1}\rangle. \tag{18}$$

Using the result (14), Eq. (18) leads to the stationary value of $\langle\overline{A_2}|\overline{A_0}\rangle$,

$$\langle\overline{A_2}|\overline{A_0}\rangle=\frac{\overline{\kappa A_0}}{8Q^2}+\mathcal{O}(\epsilon^2). \tag{19}$$

Combining this result with Eqs. (10) and (17), we finally obtain the stationary probability density

$$P(A_0)=\mathcal{N}|A_0|^{4Q^2(\alpha+\kappa/2-15\kappa^2/32Q^2)/\kappa^2}\times\exp\left[-\frac{2Q^2c}{\kappa^2}\left(1+\frac{3\kappa}{2Q^2}\right)A_0^2\right]. \tag{20}$$

This is the main result of this section.

If

$$\frac{4Q^2}{\kappa^2}\left(\alpha+\frac{\kappa}{2}-\frac{15\kappa^2}{32Q^2}\right)<-1,$$

then $P(A_0)$ is not normalizable, and hence it is not an admissible solution. In this range $P(A_0)=\delta(A_0)$ is the only solution. On the other hand, $P(A_0)$ has nonzero moments (and is normalizable), with

$$\mathcal{N}=\frac{[2Q^2c(1+3\kappa/2Q^2)/\kappa^2]^\nu}{\Gamma(\nu)},$$

$$\nu=\left[\frac{2Q^2}{\kappa^2}\left(\alpha+\frac{\kappa}{2}-\frac{15\kappa^2}{32Q^2}\right)+\frac{1}{2}\right]$$

for

$$\frac{4Q^2}{\kappa^2}\left(\alpha+\frac{\kappa}{2}-\frac{15\kappa^2}{32Q^2}\right)>-1.$$

This implies that, to second order in the noise intensity κ, the bifurcation occurs at

$$\alpha_c=-\frac{\kappa}{2}+\frac{7\kappa^2}{32Q^2}. \tag{21}$$

Note that above threshold $P(A_0)$ is not Gaussian, but has a dominant power law contribution at small A_0, and even an integrable divergence in the range $\alpha_c \leqslant \alpha < -\kappa/2 + 15\kappa^2/32Q^2$. Moments of the distribution near threshold grow as a power law of ϵ, but with nonuniversal exponents that depend on the intensity of the fluctuations. Keeping higher orders in ϵ in the expansion of ψ [Eq. (4)], and in the resulting Fokker-Planck equation (6), is not expected to lead to qualitative changes in the result just presented, Eq. (20). We only expect higher order corrections to the threshold location Eq. (21).

We finally mention that the result for the bifurcation threshold Eq. (21), agrees with that obtained by Röder *et al.* [15] by a different method.

III. SWIFT-HOHENBERG EQUATION WITH A TIME DEPENDENT CONTROL PARAMETER

The method described can be extended to a number of other model equations that are often used to describe pattern forming systems. We consider in this section a modified, one-dimensional stochastic Swift-Hohenberg equation [17,20–22]

$$\frac{\partial}{\partial t}\psi(x,t) = \left[\alpha - \left(k^2 + \frac{\partial^2}{\partial x^2}\right)^2\right]\psi(x,t) - c\psi^3(x,t) + \xi(t)\psi(x,t)\cos(Qx), \tag{22}$$

where $\langle \xi(t)\xi(t')\rangle = 2\kappa\delta(t-t')$. In the deterministic limit $\kappa=0$, the stationary solution $\psi=0$ is stable for negative values of α, while for $\alpha>0$ a periodic solution of wave number q is obtained such that the mode $q=k$ is the fastest growing mode in the linear regime above threshold [$\psi(x,t) \approx A(t)\cos(kx)$ when $0<\alpha\ll 1$]. In Eq. (22), the base solution couples to the imposed modulation of the control parameter $\xi(t)\cos(Qx)$, and additional modes of wave number $k \pm nQ$ (n integer) are excited. This suggests the following expansion of the time dependent solution

$$\psi(x,t) = \sum_{n=-\infty}^{\infty} A_n(t)\cos[(k+nQ)x], \tag{23}$$

where the $\{A_n\}$ are random processes in time, and we anticipate that $A_n/A_{n-1} \ll 1$ for small κ. We first truncate the series at $|n|=2$. Inserting the resulting expansion in Eq. (22) and grouping terms according to their periodicity yields the set of equations

$$\frac{d}{dt}\begin{bmatrix} A_0 \\ A_{+1} \\ A_{-1} \\ A_{+2} \\ A_{-2}\end{bmatrix} = \begin{bmatrix} \left\{\alpha - \frac{3}{2}c\left(\frac{A_0^2}{2} + A_{+1}^2 + A_{-1}^2 + A_{+1}A_{-1}\right)\right\}A_0 \\ \{\alpha - Q^2(2k+Q)^2\}A_{+1} \\ \{\alpha - Q^2(2k-Q)^2\}A_{-1} \\ \{\alpha - 16Q^2(k+Q)^2\}A_{+2} \\ \{\alpha - 16Q^2(k-Q)^2\}A_{-2}\end{bmatrix} + \frac{\xi}{2}\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} A_0 \\ A_{+1} \\ A_{-1} \\ A_{+2} \\ A_{-2}\end{bmatrix} + \mathrm{NL}, \tag{24}$$

where NL stands for additional nonlinear terms that do not affect the results presented below. Provided that $\alpha/Q^2|k \pm Q| \ll 1$ and $\alpha/Q^2|2k \pm Q| \ll 1$, A_0 evolves over a longer time scale than the other amplitudes, and the reduction procedure presented above can be used to obtain an approximate solution $\mathcal{P}(A_0, A_{\pm 1}, A_{\pm 2}) = P(A_0)p_1(A_{+1}, A_{-1}|A_0)p_2(A_{+2}, A_{-2}|A_{+1}, A_{-1}, A_0)$ to the time-independent Fokker-Planck equation that results from Eq. (24). Since the derivation follows closely that of Sec. II, details are omitted below. As was the case in the Ginzburg-Landau equation, we assume the scalings $\kappa \sim \alpha \sim \mathcal{O}(\epsilon^2), A_0 \sim \mathcal{O}(\epsilon), A_1 \sim \mathcal{O}(\epsilon^2), A_2 \sim \mathcal{O}(\epsilon^3)$ and first integrate the Fokker-Planck equation over A_{+2} and A_{-2}. To $O(1)$, this yields an equation for the reduced density $p_1(A_{+1}, A_{-1}|A_0)$. Its stationary solution on the fast scale reads,

$$p_1(A_{+1}, A_{-1}|A_0) = \sqrt{\frac{Q^2(16k^4 - Q^4)^2}{4\pi^2\kappa^2 A_0^4 k^2}}\exp\left\{-\frac{(4k^2+Q^2)}{8\kappa A_0^2 k^2} \times [(2k+Q)^2(4k^2+Q^2)A_{+1}^2 + (2k-Q)^2(4k^2+Q^2)A_{-1}^2 - 2(4k^2-Q^2)^2 A_{+1}A_{-1}]\right\}. \tag{25}$$

The conditional probability density of A_{+1} (resp. A_{-1}) given A_0, but independent of A_{-1} (resp. A_{+1}) is given by

$$p_\pm(A_{\pm 1}|A_0) = \int_{-\infty}^{+\infty} dA_{\mp 1} p_1(A_{+1}, A_{-1}|A_0) = \sqrt{\frac{2Q^2(2k \pm Q)^2}{\pi\kappa A_0^2}} \times \exp\left[-\frac{2Q^2(2k\pm Q)^2}{\kappa A_0^2}A_{\pm 1}^2\right]. \tag{26}$$

This expression implies that, to a first approximation, A_{+1} and A_{-1} satisfy the Langevin equations $dA_{\pm1}/dt = -Q^2(2k\pm Q)^2 A_{\pm1} + \xi A_0/2$ that define Ornstein-Uhlenbeck processes. Since the same noise $\xi(t)$ appears in the equations for A_{+1} and A_{-1}, both variables remain correlated and $p_1(A_{+1},A_{-1}|A_0) \neq p_+(A_{+1}|A_0)p_-(A_{-1}|A_0)$. The equation satisfied by $P(A_0)$ is now obtained by integrating the original Fokker-Planck equation over $A_{\pm2}$ and $A_{\pm1}$. Keeping terms up to $O(\epsilon^4)$, we find

$$0 = -\epsilon^2 \frac{\partial}{\partial \bar{A}_0}\left\{\left[\left(\overline{\alpha A_0} + \frac{\bar{\kappa}}{2}\right)\bar{A}_0 + \epsilon^2 \frac{\bar{\kappa}}{4}(\langle \bar{A}_{+2}|\bar{A}_0\rangle + \langle \bar{A}_{-2}|\bar{A}_0\rangle) - \frac{3}{4}c\bar{A}_0^3 - \epsilon^2 \frac{3}{2}c(\langle \bar{A}_{+1}^2|\bar{A}_0\rangle + \langle \bar{A}_{-1}^2|\bar{A}_0\rangle + \langle \bar{A}_{+1}\bar{A}_{-1}|\bar{A}_0\rangle)\right]P(\bar{A}_0)\right\} + \epsilon^4 \frac{\partial^2}{\partial \bar{A}_0^2}\left[\frac{\kappa}{4}\langle(\bar{A}_{+1}+\bar{A}_{-1})^2|\bar{A}_0\rangle P(\bar{A}_0)\right], \tag{27}$$

where the scaled variables are defined as in Sec. II. Expressed in the original set of variables, the solution to that equation reads [to $O(\epsilon^2)$],

$$P(A_0) = \delta(A_0 - \theta(\alpha)\sqrt{4[\alpha+\kappa/2]/3c}), \tag{28}$$

where $\theta(\alpha)=1$ if $\alpha > -\kappa/2$ and $\theta(\alpha)=0$ otherwise.

At this order, the evolution of A_0 is deterministic, with coefficients that depend on the intensity of the modulation. As was the case in Sec. II, the evolution of A_0 is stochastic at higher orders in ϵ. At $O(\epsilon^4)$ we find

$$P(A_0) = \mathcal{N}|A_0|^{\delta}\exp[-\gamma A_0^2], \tag{29}$$

where

$$\delta = \frac{4Q^2(4k^2-Q^2)^2(4k^2+Q^2)}{\kappa^2(16k^4+Q^4)}\left(\alpha + \frac{\kappa}{2} + \frac{\kappa^2(k^2+Q^2)}{128Q^2(k^2-Q^2)^2} - \frac{\kappa^2(16k^4+Q^4)}{2Q^2(4k^2-Q^2)^2(4k^2+Q^2)}\right),$$

and

$$\gamma = \frac{3cQ^2(4k^2-Q^2)^2(4k^2+Q^2)}{2\kappa^2(16k^4+Q^4)} \times \left\{1 + \frac{\kappa}{2Q^2}\left[\frac{2(4k^2+Q^2)^2+(4k^2-Q^2)^2}{(4k^2-Q^2)^2(4k^2+Q^2)}\right]\right\}.$$

In order to obtain Eq. (29), we have used the result $\langle A_{\pm2}\rangle = \kappa A_0/64Q^2(k\pm Q)^2$ that can be obtained by averaging the Langevin equations for $A_{\pm2}$. Our result manifestly breaks down when $|k|\to Q$ or $|2k|\to Q$, i.e., when the coefficients $\alpha - Q^2(2k\pm Q)^2$ or $\alpha - 16Q^2(k\pm Q)^2$ on the right-hand side of Eq. (24) are no longer of $\mathcal{O}(1)$. In that case, A_0 and $A_{\pm1}$ (or $A_{\pm2}$) evolve over similar time scales and the elimination procedure will fail.

The threshold for instability can be again found from the requirement that $P(A_0)$ be normalizable. In Eq. (29), this amounts to requiring that $\delta > -1$, a condition that gives as bifurcation threshold

$$\alpha_c = -\frac{\kappa}{2} - \frac{\kappa^2(k^2+Q^2)}{128Q^2(k^2-Q^2)^2} + \frac{\kappa^2(16k^4+Q^4)}{4Q^2(4k^2-Q^2)^2(4k^2+Q^2)}. \tag{30}$$

The bifurcation threshold is shifted relative to the deterministic case by an amount that depends on the intensity of the stochastic modulation κ and the wave number Q.

IV. GINZBURG-LANDAU EQUATION WITH SPATIAL AND TEMPORAL RANDOMNESS IN THE CONTROL PARAMETER

We now consider the case in which the control parameter α has a random component in both space and time. We restrict our study to the Ginzburg-Landau equation but expect a similar analysis to hold for the Swift-Hohenberg equation. Specifically, we consider the equation

$$\frac{\partial}{\partial t}\psi(x,t) = [\alpha + \xi(x,t)]\psi(x,t) - c\psi^3(x,t) + \frac{\partial^2}{\partial x^2}\psi(x,t), \tag{31}$$

where $c>0$ and $\langle \xi(x,t)\xi(x',t')\rangle = 2\kappa\delta(x-x')\delta(t-t')$. Since the stochastic modulation excites a response over a range of length scales, $\psi(x,t)$ cannot be adequately described by a small number of modes as was the case in previous examples. However, a connection with these earlier cases can be established by introducing a Fourier transform on a lattice. After discretizing Eq. (31) on a uniform lattice with N sites and spacing Δx, and introducing the discrete Fourier transform, we find a set of N equations for the Fourier coefficients

$$\frac{d\tilde{\psi}_q}{dt} = (\alpha - q^2)\tilde{\psi}_q - \frac{c}{N^2}\sum_{k_1,k_2}\tilde{\psi}_{k_1}\tilde{\psi}_{k_2}\tilde{\psi}_{q-k_1-k_2} + \frac{1}{N}\sum_{j=0}^{N-1}\sum_k \xi_j e^{ij\Delta x(q-k)}\tilde{\psi}_k, \tag{32}$$

where $\tilde{\psi}_q = \Sigma_{j=0}^{N-1} e^{ijq\Delta x}$, and where the wave number q can adopt the values $q = 0, \pm 2\pi/N\Delta x, \pm 2\pi/N\Delta x, \ldots, \pm\pi/\Delta x$, and summation over k, k_1 and k_2 includes all possible wave vectors within the Brillouin zone. The resulting variance of the noise on the discretized lattice is $\langle \xi_m(t)\xi_n(t')\rangle = (2\kappa/\Delta x)\delta_{m,n}\delta(t-t')$. Since ψ_j is real, $\tilde{\psi}_{-k} = \tilde{\psi}_k^*$. The equation for $\tilde{\psi}_0$ is

$$\frac{d\tilde{\psi}_0}{dt}=\alpha\tilde{\psi}_0-\frac{c}{N^2}\sum_{k_1,k_2}\tilde{\psi}_{k_1}\tilde{\psi}_{k_2}\tilde{\psi}_{-k_1-k_2}+\sum_{j=0}^{N-1}\frac{\xi_j}{N}\left[\tilde{\psi}_0+2\sum_{k>0}\left[\cos(jk\Delta x)\tilde{\psi}_k^R+\sin(jk\Delta x)\tilde{\psi}_k^I\right]\right], \tag{33}$$

whereas the equations for the real and imaginary parts $\tilde{\psi}_q^R$ and $\tilde{\psi}_q^I(q>0)$ are,

$$\frac{d\tilde{\psi}_q^R}{dt}=(\alpha-q^2)\tilde{\psi}_q^R-\frac{c}{N^2}\sum_{k_1,k_2}[\tilde{\psi}_{k_1}\tilde{\psi}_{k_2}\tilde{\psi}_{q-k_1-k_2}]^R+\sum_{j=0}^{N-1}\frac{\xi_j}{N}\cos(jq\Delta x)\times\left\{\tilde{\psi}_0+2\sum_{k>0}[\cos(jk\Delta x)\tilde{\psi}_k^R+\sin(jk\Delta x)\tilde{\psi}_k^I]\right\}, \tag{34}$$

and

$$\frac{d\tilde{\psi}_q^I}{dt}=(\alpha-q^2)\tilde{\psi}_q^I-\frac{c}{N^2}\sum_{k_1,k_2}[\tilde{\psi}_{k_1}\tilde{\psi}_{k_2}\tilde{\psi}_{q-k_1-k_2}]^I+\sum_{j=0}^{N-1}\frac{\xi_j}{N}\sin(jq\Delta x)\times\left\{\tilde{\psi}_0+2\sum_{k>0}[\cos(jk\Delta x)\tilde{\psi}_k^R+\sin(jk\Delta x)\tilde{\psi}_k^I]\right\}. \tag{35}$$

Near the bifurcation $\alpha/|\alpha-q^2|\ll 1$ for all nonzero wave vectors provided that $N\Delta x\sim\mathcal{O}(1)$. Therefore discretization introduces a privileged wave vector, $q=0$, that evolves on a lower time scale than all others. This is a manifestation of the appearance of order (or a uniform solution) at sufficiently long times, even in the stochastic case. Therefore the situation is analogous to that described in Secs. II and III, and we anticipate a shift in the location of the onset that can be directly attributed to statistical correlation between the noise and the Fourier amplitudes of the fast variables $(q\neq 0)$. This can be easily shown by averaging out the fast variables from the equation for $\tilde{\psi}_0$. From Eqs. (34) and (35), and the Furutsu-Novikov theorem, we find $\langle\tilde{\psi}_k^R\xi_j\rangle\approx(\kappa/N\Delta x)\cos(jk\Delta x)\tilde{\psi}_0$ and $\langle\tilde{\psi}_k^I\xi_j\rangle\approx(\kappa/N\Delta x)\sin(jk\Delta x)\tilde{\psi}_0$, leading to

$$\frac{d\tilde{\psi}_0}{dt}=\left(\alpha+\frac{\kappa}{\Delta x}\right)\tilde{\psi}_0-\frac{c}{N^2}\tilde{\psi}_0^3+\xi'(t)\tilde{\psi}_0, \tag{36}$$

where $\xi'=\Sigma_j\xi_j/N$, and we have retained only the dominant nonlinear term in Eq. (33). Equation (36) is the normal form equation for a pitchfork bifurcation with multiplicative noise, and it is known that the bifurcation threshold occurs when the coefficient of the linear term changes sign [23,24],

$$\alpha_c=-\frac{\kappa}{\Delta x}. \tag{37}$$

This result is in agreement with Ref. [13].

We finally mention that it is also possible to use the adiabatic reduction procedure presented above to derive expressions for the probability density $\mathcal{P}(\{\tilde{\psi}_q\})=P(\tilde{\psi}_0)\mathrm{p}(\{\tilde{\psi}_{q>0}^{R,I}\}|\tilde{\psi}_0)$. The calculation, which is similar to that of Sec. II, yields the expressions

$$p_q(\tilde{\psi}_q^{R,I}|\tilde{\psi}_0)=\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\prod_{k\neq q}d\tilde{\psi}_k p(\{\tilde{\psi}_{k>0}^{R,I}\}|\tilde{\psi}_0)=\sqrt{\frac{N\Delta xq^2}{\kappa\tilde{\psi}_0^2\pi}}\exp\left[-\frac{N\Delta xq^2}{\kappa\tilde{\psi}_0^2}(\tilde{\psi}_q^{R,I})^2\right] \tag{38}$$

for the fast variables, and

$$P(\tilde{\psi}_0)=\mathcal{N}|\tilde{\psi}_0|^{(N\Delta x/\kappa)(\alpha+\kappa/\Delta x)-1}\exp\left[-\frac{\Delta xc}{2\kappa N}\tilde{\psi}_0^2\right], \tag{39}$$

where

$$\mathcal{N}=\left[\frac{\Delta xc}{2\kappa N}\right]^{(N\Delta x/2\kappa)(\alpha+\kappa/\Delta x)}\Bigg/\Gamma\left[\frac{N\Delta x}{2\kappa}\left(\alpha+\frac{\kappa}{\Delta x}\right)\right].$$

The predictions of the reduction scheme have been tested numerically by integrating the Ginzburg-Landau equation [Eq. (31)] on a one-dimensional lattice with 128 sites, spacing $\Delta x=0.1$ and periodic boundary conditions. The integration was performed using a first order, explicit algorithm, with time step $\Delta t=0.0005$. Initial values for ψ at every lattice site were chosen randomly from a uniform distribution in the interval $[0,0.01]$. The numerical results represented by solid circles in Fig. 1 were averaged over 50 independent runs and correspond to the parameter values $\alpha=0,c=1$, and $\kappa=0.001$. The figure shows $p(|\tilde{\psi}_q^R|)$, the probability density of $\tilde{\psi}_q^R$ for any $\tilde{\psi}_0$, i.e., $p(|\tilde{\psi}_q^R|)=2p(\tilde{\psi}_q^R)=4\int_0^\infty d\tilde{\psi}_0 P(\tilde{\psi}_0)p_q(\tilde{\psi}_q^R|\tilde{\psi}_0)$. The solid lines correspond to predictions from Eqs. (40) and (41) for three different wave numbers. The discrepancy between the two sets of results increases slightly with q, in spite of the fact that our assumption of adiabaticity is in principle more accurate as q increases. The reason is that discretization errors in the numerical solution become more severe with increasing q. As shown by the dotted line in Fig. 1, the discrepancy is eliminated by simply replacing the factor q^2 in the expression for $p_q(\tilde{\psi}_q^R|\tilde{\psi}_0)$ by $-(2/\Delta x^2)[\cos(q\Delta x)-1]$, the latter expression being the discrete Fourier transform of the Laplacian operator in the nearest neighbor approximation (the same discretization used in the numerical integration).

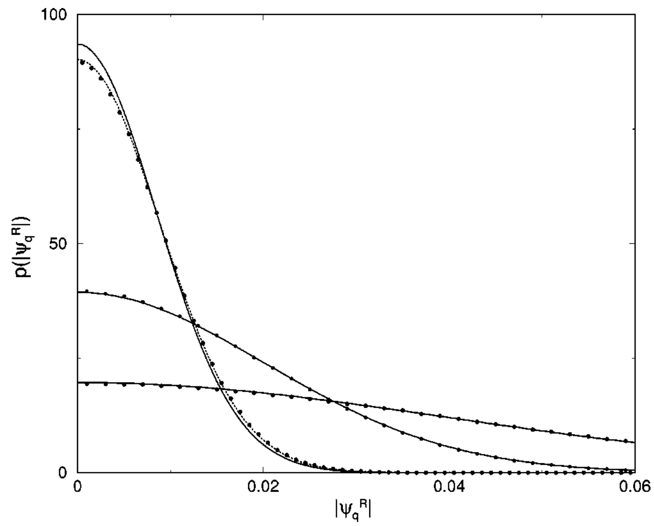

FIG. 1. Order parameter probability distribution function $p(|\tilde{\psi}_q^R|)$ obtained either from Eqs. (38) and (39) (solid lines), or by numerical integration of Eq. (31) (symbols). Three different values of q are shown, from top to bottom, $q=38\pi/N\Delta x, q=16\pi/N\Delta x$ and $q=8\pi/N\Delta x$. The dotted line is obtained by replacing q^2 in the expression for $p_q(\tilde{\psi}_q^R|\tilde{\psi}_0)$ by $-(2/\Delta x^2)[\cos(q\ \Delta x)-1]$.

V. DISCUSSION

A reduction procedure similar in spirit to a center manifold reduction can be developed for systems in which the control parameter is stochastically modulated provided that the intensity κ of the fluctuating component is small. A perturbation expansion in κ leads to a hierarchy of Fokker-Planck equations for conditional probability distributions that relate components of the order parameter field (e.g., Fourier modes) that evolve over different characteristic time scales. In the three cases studied, recursive integration of the equations governing the evolution of the conditional probability distribution functions leads to an effective equation for the evolution of the slowest mode A_0.

We show that, depending on the order of the expansion, the evolution of A_0 can be either deterministic with renormalized coefficients (the lowest order), or stochastic. The stationary solution for the probability $P(A_0)$ reveals that the bifurcation remains sharp (i.e., randomness in the control parameter does not lead to an imperfect bifurcation), but at a value of the deterministic control parameter that is shifted by an amount proportional to κ at lowest order. In addition, statistical moments of A grow as a power law of the distance away from threshold, but with an exponent that depends explicitly on κ.

We finally note that the shift in the location of the bifurcation threshold originates from statistical correlations between the fast variables that are eliminated and the random component of the control parameter. For example, in the case of the Swift-Hohenberg equation, the result $\alpha_c \simeq -\kappa/2$ follows immediately from the equation for A_0 [Eq. (24)] neglecting terms proportional to $A_{\pm 2}$, and replacing $A_{+1}\xi$ and $A_{-1}\xi$ by their statistical average. A similar conclusion is reached for the Ginzburg-Landau equation.

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