BOUNDARY STATES AND BROKEN BULK SYMMETRIES IN WA_r MINIMAL MODELS

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We review the free-field formalism for boundary states. The multi-component free-field formalism is then used to study the boundary states of (p', p) rational conformal field theories having a W symmetry of the type A_r. We show how the classification of primary fields for these models is obtained by demanding modular covariance of cylinder amplitudes and that the resulting modular S matrix satisfies all the necessary conditions. Basis states satisfying the boundary conditions are found in the form of coherent states and as expected we find that W violating states can be found for all these models. We construct consistent physical boundary states for all the rank 2 $(p+1, p)$ models (of which the already known case of the 3-state Potts model is the simplest example) and find that the W violating sector possesses a direct analogue of the Verlinde formula.

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1. Introduction

The Coulomb gas formalism [1, 2] provides a powerful method for calculating correlation functions and conformal blocks in minimal rational conformal field theories (CFTs) and boundary CFTs have been of great interest since Cardy's famous paper [3]. Recently it has been shown that free-field representations may be extended from bulk CFTs to systems with boundary(ies) [4, 5] in the case of the Virasoro diagonal minimal models and for the simplest non-diagonal case, the three state Potts model, where a multi-component Coulomb Gas formalism is required [6]. The boundary states appear as coherent states in the free-field formalism.

The three state Potts model is of particular interest because the conformal field theory describing its critical point is the simplest in which there is a higher dimensional chiral operator $W^{(3)}$ of dimension 3 [7]. There are six boundary states originally found by Cardy in which the $W^{(3)}$ current is conserved at the boundary [8] but in addition there are known to be two more states in which the W current is not conserved. Affleck et al [9] used fusion methods to establish these states while recently in [6] it was shown that precisely these states, and no others, appear also in the free-field formulation. This is all consistent with the general arguments given in [10] that there should be precisely eight conformally invariant boundary states in this model.

The free field formulation for the Potts model is just the simplest case of a whole family of W minimal models whose r component free-field representations are built on the Lie algebra A_r and are further characterized by two relatively prime integers p' and p. These models, denoted $W_{r+1}(p', p)$, have higher dimensional chiral operators $W^{(K)}$, $K = 3, \ldots r+1$ and thus an extended symmetry algebra of which the Virasoro algebra is a sub-algebra. The Potts model corresponds to $W_3(5, 4)$ and is the only member of the family which is also a Virasoro minimal model. It is to be expected that all the models will have boundary states which violate the higher symmetry as well as those that conserve it. Our aim here is to extend [6] and to study these boundary states. For general reviews of CFTs with W algebras the reader should consult [11–13].

This paper is organized as follows. We start in Section 2 with a collection of definitions and some results from the standard Coulomb Gas formalism that we need. In Section 3 we describe the classification of fields and the Felder complex for W minimal models and in Section 4 explain how to construct coherent state representations satisfying the Virasoro boundary conditions. Section 5 deals with the calculation of cylinder amplitudes and establishing which of the coherent states found previously are coupled to the bulk physics. The classification of primary fields in the $W_{r+1}(p', p)$ models is quite subtle [14] and we devote Section 6 to explaining how to use modular covariance of the cylinder amplitudes to do this. In Section 7 we consider the physical boundary states and annulus partition functions. The conservation or otherwise of W currents by the different possible boundary states is considered in Section 8 and, finally, in Section 9 we discuss some open issues.

2. Preliminaries

The usual Coulomb gas formalism [1, 2] can be extended to CFTs with a larger symmetry than the Virasoro algebra by introducing a multiple component scalar field [7] $\Phi^j(z, \bar{z})$, $j = 1 \dots r$, which is a vector in the root space of a finite dimensional Lie Algebra $\mathcal{A}$ of rank r. In this paper we will be mainly concerned with the algebra A_r and so will specialize to it straight away. Let us first fix some notation. The simple roots will be denoted $e_j, j = 1 \dots r$, and the corresponding dual weights $\omega_j, j = 1 \dots r$. We will use "$\cdot$" to denote multiplication of vectors and matrices in the root space. So the scalar product of two vectors u and v in the root space will be written $u \cdot v$, the product of two matrices $m_1 \cdot m_2$ and so on. The simple roots and dual weights then satisfy

$$e_j \cdot e_j = 2, \quad e_j \cdot e_{j+1} = -1, \quad e_j \cdot \omega_i = \delta_{i,j}. \tag{2.1}$$

The positive roots are given by

$$e_{jk} = e_j + \dots + e_k, \quad 1 \le j < k \le r. \tag{2.2}$$

The Weyl vector ρ is defined as

$$\rho = \sum_{j=1}^{r} \omega_j, \tag{2.3}$$

its square is

$$\rho^2 = \frac{1}{12} r(r+1)(r+2), \tag{2.4}$$

and the fundamental weights h_K, $K = 1 \dots r + 1$, satisfy

$$\begin{aligned} h_1 &= \omega_1, \\ h_K - h_{K+1} &= e_K. \end{aligned} \tag{2.5}$$

We denote the Weyl group of $\mathcal{A}$ by $\mathcal{W}$, an element of it by w, and let $\varepsilon_w = \det w$. The longest element of the Weyl group, w_0 is the unique element of $\mathcal{W}$ that maps the positive roots onto the negative roots. On the simple roots, dual weights and fundamental weights w_0 has the action

$$\begin{aligned} w_0\, e_i &= -e_{r-i+1}, \\ w_0\, \omega_i &= -\omega_{r-i+1}, \\ w_0\, h_K &= h_{N+1-K}, \end{aligned} \tag{2.6}$$

and we define its matrix representation in the root basis, S, by

$$S_{ij} = -\delta_{i+j,r+1}. \tag{2.7}$$

Finally I denotes the identity matrix.

The action for Φ takes the usual form

$$\mathcal{S}[\Phi] = \frac{1}{8\pi}\int d^2z\sqrt{g}\,\left(\partial_\mu\Phi\cdot\partial^\mu\Phi + 4i\alpha_0\rho\cdot\Phi R\right), \tag{2.8}$$

where R is the scalar curvature, g the metric, and α_0 a constant. We now split Φ into a holomorphic component $\phi(z)$ and an anti-holomorphic component $\bar\phi(\bar z)$. The field ϕ has mode expansion

$$\phi^j(z) = \phi_0^j - ia_0^j \ln z + i\sum_{n\neq 0}\frac{a_n^j}{n}z^{-n}, \tag{2.9}$$

and similarly for $\bar\phi$. Canonical quantization gives the usual commutation relations

$$\begin{aligned} [a_m^j, a_n^l] &= m\delta^{jl}\delta_{m+n,0}, \\ [\phi_0^j, a_0^l] &= i\delta^{jl}. \end{aligned} \tag{2.10}$$

Variation of the action with respect to the metric yields the energy-momentum tensor

$$T(z) = -2\pi T_{zz} = -\frac{1}{2} :\partial\phi\cdot\partial\phi: +2\,i\,\alpha_0\,\rho\cdot\partial^2\phi, \tag{2.11}$$

which has the usual expansion

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \tag{2.12}$$

where the operators

$$L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} :a_m\cdot a_{n-m}: -2\alpha_0(n+1)\rho\cdot a_n \tag{2.13}$$

obey the Virasoro algebra with central charge

$$c = r - 48\ \alpha_0^2 \rho^2. \tag{2.14}$$

Fock spaces $\mathcal{F}_\alpha$ are labeled by a vacuum $|\alpha\rangle$, which is an eigenvector of the a_0^j operator, and annihilated by the positive modes

$$\begin{aligned} a_0^j|\alpha\rangle &= \alpha^j|\alpha\rangle, \\ a_n^j|\alpha\rangle &= 0\,, \quad n > 0. \end{aligned} \tag{2.15}$$

The Fock space is formed by applying the creation operators to the vacuum,

$$a_{-n_1}^{j_1} a_{-n_2}^{j_2} \ldots a_{-n_p}^{j_p}|\alpha\rangle, \tag{2.16}$$

and different Fock spaces are related by

$$e^{i\beta\cdot\phi_0}|\alpha\rangle = |\beta + \alpha\rangle. \tag{2.17}$$

The chiral vertex operators $V_\alpha(z)$ are defined by

$$V_\alpha(z) = :e^{i\alpha\cdot\phi(z)}:\,, \tag{2.18}$$

and have conformal dimension given by

$$h(\alpha) = \frac{1}{2}\alpha\cdot(\alpha - 4\alpha_0\rho). \tag{2.19}$$

3. *W* minimal models

The $W_N(p', p)$ minimal models are defined for relatively prime integers p' and p such that $p' > p > N$ by

$$2\alpha_0 = \frac{p'-p}{\sqrt{pp'}}, \quad \alpha_+ = \frac{p'}{\sqrt{pp'}}, \quad \alpha_- = -\frac{p}{\sqrt{pp'}} \tag{3.1}$$

and have central charge

$$c = r\left(1 - \frac{(p'-p)^2}{pp'}(r+1)(r+2)\right). \tag{3.2}$$

Considerations requiring a consistent fusion algebra lead to the allowed values of α [11]

$$\begin{aligned} \alpha &= 2\alpha_0\rho - \frac{1}{\sqrt{pp'}}\,\lambda(m,n), \\ \lambda(m,n) &= -pm^i\omega_i + p'n^i\omega_i, \end{aligned} \tag{3.3}$$

where summation over repeated i is implied and the m^i and n^i are positive integers satisfying

$$\sum_i m^i < p', \quad \sum_i n^i < p\,. \tag{3.4}$$

So $m^i\omega_i$ and $n^i\omega_i$ are dominant weights and $\lambda(m,n)$ is a non-zero weight, although not necessarily dominant. The reader should note that there is not a primary field corresponding to each α given by this prescription. As we discuss directly below there is considerable degeneracy in this set-up and the appearance of copies related by Weyl transformation is to be expected. For example in $W_3(5,4)$, which is the critical three state Potts model and should have 6 primary fields, there are 18 λs satisfying these constraints; but $|\mathcal{W}| = 6$ and so we get three copies for each primary [a] (although there are six elements in the Weyl group). That something is amiss is even clearer if we look at $W_4(6,5)$ for which there are 40 λs satisfying (3.4) yet $|\mathcal{W}| = 24$. Actually these puzzles are resolved by considering the modular properties of the theory [14] which we will do in Section 6. For the moment it is sufficient that the λs certainly can be written in the form (3.3).

As mentioned above there is some degeneracy in the αs. Defining

$$\begin{aligned} \alpha^* &= 4\alpha_0\rho - \alpha\,, \\ \alpha_w &= 2\alpha_0\rho - \frac{1}{\sqrt{pp'}} w\,\lambda(m,n), \quad w \in \mathcal{W} \end{aligned} \tag{3.5}$$

it follows that

$$h(\alpha) = h(\alpha^*) = h(\alpha_w). \tag{3.6}$$

There are then two types of representation

(1) $\alpha^* \in \{\alpha_w\}$. This implies that

$$\lambda(m,n) = -w\,\lambda(m,n) \tag{3.7}$$

for some $w \in \mathcal{W}$. Since $\lambda(m,n)$ is a weight we can use the property that only for self-conjugate representations of SU(N)is the weight λ in the Weyl orbit of $-\lambda$. In this case there is just one self-conjugate

[a] This ambiguity resulted in the authors of [6] having to take an apparently arbitrary choice of λs on which to build the boundary states in this model. The choice that was made there is properly justified by the discussion in this paper.

primary field of conformal weight $h(\alpha)$. The highest weight in a self-conjugate representation is given by

$$\lambda(m,n) = \lambda^i \omega_i\,, \quad \lambda^i = \lambda^{N-i}. \tag{3.8}$$

It is clear that for $\lambda(m,n)$ to be self-conjugate either both or neither of $m^i\omega_i$ and $n^i\omega_i$ must be so. In fact the later case is excluded; (3.8) leads to the condition

$$p'(n^i - n^{N-i}) = p(m^i - m^{N-i}) \tag{3.9}$$

but there are no solutions to this for m, n in the range (3.4) if p', p are relatively prime.

(2) $\alpha^* \notin \{\alpha_w\}$. This implies that $\lambda(m,n)$ cannot be the weight of a self conjugate representation. There are thus *two* primary fields which are conjugates of each other, one built on $\lambda(m,n)$ and one on $-\lambda(m,n)$.

The vertex operators V_α operating on the SL(2,C) invariant vacuum generate states in a Fock space $\mathcal{F}(\lambda(m,n))$ (where α and λ are related as in in (3.3)), rather than the Verma module of the Virasoro primary field. This physical Hilbert space has to be constructed by a BRST procedure that was first described by Felder [15] and extended to the W_3 case in [16]. First define the set of operators

$$Q_k^{(j)} = B_k^j \left(\oint dz V_{\alpha_+ e_j}(z) \right)^k, \quad j = 1 \ldots r\,, \quad k < p\,, \tag{3.10}$$

where the B_k^j are non-zero constants (note that $h(\alpha_\pm e_i) = 1$). The $Q_k^{(j)}$ commute with the Virasoro algebra by construction and map

$$Q_k^{(j)} \mathcal{F}(\lambda) \to \mathcal{F}(\lambda - kp'e_j). \tag{3.11}$$

It is simple to check that if

$$k = n^j \bmod p \tag{3.12}$$

then the conformal dimensions of these two Fock spaces differ by an integer. The action on λ then amounts to

$$Q^{(j)} : \lambda(m,n) \to -pm^i\omega_i + p'w_{e_j}\, n^i\omega_i - Npp'e_j \tag{3.13}$$

where $N \in \mathbb{Z}$ is introduced to enforce the mod p condition in (3.12). Now $Q^{(j)}$ and $Q^{(j+1)}$ do not commute so we have to introduce further operators $Q^{(j,j+1)}$ such that

$$Q_{n^j}^{(j)} Q_{n^{j+1}}^{(j+1)} = Q_{n^j+n^{j+1}}^{(j,j+1)} Q_{n^j}^{(j)} \tag{3.14}$$

and with action on λ,

$$Q^{(j,j+1)} : \lambda(m,n) \to -pm^i\omega_i + p'w_{e_{jj+1}}\, n^i\omega_i - Npp'e_{jj+1}\,. \tag{3.15}$$

This operator in turn does not commute with (eg) $Q^{(j+2)}$ and so we iterate this process ending up with a set of operators $Q^{(j,k)}$ with action on λ given by

$$Q^{(j,k)} : \lambda(m,n) \to -pm^i\omega_i + p'w_{e_{jk}}\, n^i\omega_i - Npp'e_{jk}\,. \tag{3.16}$$

Starting from a given Fock space chosen according to the rules (3.3) one can now convince oneself that the action of the $Q^{j,k}$ generates an infinite complex of Fock spaces [b]

$$\mathcal{C}(\lambda) = \bigoplus_{\substack{w\in\mathcal{W}\\ N\in\mathbb{Z}^r}} \mathcal{F}(-pm^i\omega_i + p'w\, n^i\omega_i - pp'N^ie_i)\,. \tag{3.17}$$

It is possible to assemble from the $Q^{j,k}$ a nil-potent operator Q_B on $\mathcal{C}(\lambda)$ whose cohomology is the physical Hilbert space [c]

$$\mathcal{H} = \frac{\mathrm{Kernel}\, Q_B}{\mathrm{Image}\, Q_B}\,. \tag{3.18}$$

Expectation values are then calculated from alternating sums over the complex so for example the character of the Verma module is given by

$$\begin{aligned}\chi_{\lambda(m,n)}(q) &= \mathrm{Tr}\, q^{L_0 - c/24}\\ &= \frac{1}{\eta(\tau)^r}\sum_{\substack{w\in\mathcal{W}\\ N\in\mathbb{Z}^r}} \varepsilon_w q^{|p'wn^i\omega_i - pm^i\omega_i + pp'N^ie_i|^2/2pp'}\,.\end{aligned} \tag{3.19}$$

4. Coherent boundary states

Coherent boundary states may be defined in a straightforward generalization of the procedure for the one component Coulomb gas [4, 6]. First we introduce the states $|\alpha,\bar\alpha\rangle$ which are constructed by applying the vertex operator $V_\alpha(z)$ and its antiholomorphic counterpart, $\overline{V}_{\bar\alpha}(\overline{z})$ to the $SL(2,C)$-invariant vacuum $|0,0\rangle$,

$$|\alpha,\bar\alpha\rangle = \lim_{z,\overline{z}\to 0} \overline{V}_{\bar\alpha}(\overline{z})V_\alpha(z)|0,0\rangle = e^{i\overline{\alpha}\cdot\overline{\phi}_0}e^{i\alpha\cdot\phi_0}|0,0\rangle\,. \tag{4.1}$$

[b] This is hard to draw unless $r=2$ for which case it is described in detail in [6, 16].

[c] This was proved by Felder for the $r=1$ case; there seems to be no direct proof given in the literature of the general case, but there is also no evidence to the contrary and see [17] for a review.

These states satisfy

$$a_0^i|\alpha,\bar{\alpha}\rangle = \alpha^i|\alpha,\bar{\alpha}\rangle,$$
$$\bar{a}_0^i|\alpha,\bar{\alpha}\rangle = \bar{\alpha}^i|\alpha,\bar{\alpha}\rangle. \tag{4.2}$$

The corresponding bra states are given by

$$\langle\alpha,\bar{\alpha}| = \langle 0,0|e^{-i\overline{\alpha}\cdot\overline{\phi}_0}e^{-i\alpha\cdot\phi_0}. \tag{4.3}$$

The coherent state ansatz is given by

$$|B(\alpha,\bar{\alpha};\Lambda)\rangle = C_\Lambda\,|\alpha,\bar{\alpha}\rangle, \tag{4.4}$$
$$C_\Lambda = \prod_{k>0}\exp\left(\frac{1}{k}\,a_{-k}\cdot\Lambda\cdot\bar{a}_{-k}\right), \tag{4.5}$$

where Λ is a matrix to be determined by imposing the boundary condition

$$(L_n - \bar{L}_{-n})|B(\alpha,\bar{\alpha};\Lambda)\rangle = 0\,. \tag{4.6}$$

For positive n this gives the constraint

$$\Bigg(\;\frac{1}{2}\sum_{l=1}^{n-1}\overline{a}_{n-l}\cdot(\Lambda^T\cdot\Lambda - I)\cdot\overline{a}_{-l} + (a_0 - 2\alpha_0(n+1)\rho)\cdot\Lambda\cdot\overline{a}_{-n}$$
$$+(-\overline{a}_0 - 2\alpha_0(n-1)\rho)\cdot\overline{a}_{-n}\Bigg)\,|\alpha,\bar{\alpha}\rangle = 0\,, \tag{4.7}$$

and similarly for negative n. The constraint is satisfied provided

$$\Lambda^T\cdot\Lambda = I, \tag{4.8}$$
$$\Lambda\cdot\rho + \rho = 0, \tag{4.9}$$
$$\Lambda^T\cdot\alpha + 4\alpha_0\rho - \bar{\alpha} = 0. \tag{4.10}$$

The last of these conditions allows us to simplify our notation by defining

$$|B(\alpha;\Lambda)\rangle \equiv |B(\alpha,\bar{\alpha} = \Lambda^T\cdot\alpha + 4\alpha_0\rho;\Lambda)\rangle. \tag{4.11}$$

We next identify solutions for Λ. Using the form (3.3), the last constraint in (4.10) becomes

$$\Lambda^T(pn^i - p'm^i)\omega_i = -(p\bar{n}^i - p'\bar{m}^i)\omega_i\,. \tag{4.12}$$

Using the $\{\omega_i\}$ basis, we see that the simplest form for Λ, which we will denote Λ^ω, is one in which all the elements are integers; this guarantees that $\overline{\alpha}$ exists. In addition the vector $(1,\dots 1)$ must be an eigenvector of Λ^ω with eigenvalue -1 in order that (4.9) is satisfied so

$$\Lambda^\omega_{k1} + \Lambda^\omega_{k2} + \ldots + \Lambda^\omega_{kr} = -1, \quad k = 1,\dots r\,. \tag{4.13}$$

A little bit of care is necessary in implementing (4.8); recall that this is in an *orthogonal cartesian* basis because of the definition of the Heisenberg algebra (2.10). In the $\{\omega_i\}$ basis it becomes

$$\Lambda^{\omega\,T} A^{-1}\,\Lambda^{\omega} = A^{-1}, \quad \text{or, equivalently,} \quad \Lambda^{\omega}\,A\,\Lambda^{\omega\,T} = A\,, \tag{4.14}$$

where A is the Cartan matrix; picking out the diagonal elements gives

$$(\Lambda^{\omega}_{k1})^2 + (\Lambda^{\omega}_{k1} - \Lambda^{\omega}_{k2})^2 + \ldots + (\Lambda^{\omega}_{k\,r-1} - \Lambda^{\omega}_{kr})^2 + (\Lambda^{\omega}_{kr})^2 = 2\,. \tag{4.15}$$

Now (4.15) is a sum of squares so exactly two terms in the sum must be equal to unity. In conjunction with (4.13) this shows that each row of Λ^{ω} contains one element which is -1, all other elements being zero. Each row must be different, otherwise $\det \Lambda^{\omega} = 0$ which contradicts (4.14) because $\det A \neq 0$. Thus the action of Λ^{ω} on the l.h.s. of (4.14) is to permute the rows and columns of A. By inspection there are only two permutations that leave A invariant, the identity and reversal of the order of rows and columns, so there are only two solutions

$$\begin{aligned} &\Lambda^{\omega} = -I \\ \text{or} \quad &\Lambda^{\omega} = S\,. \end{aligned} \tag{4.16}$$

Equivalently, the action of Λ is simply the group of outer automorphisms on the Dynkin diagram for A_r.

5. States and decoupled states

The cylinder amplitudes between boundary states of the form (4.11) can be calculated by standard techniques and are given by

$$\begin{aligned} &\langle B(\beta;\Lambda_2)|\,q^{\frac{1}{2}(L_0+\overline{L}_0-\frac{c}{12})}\,|B(\alpha;\Lambda_1)\rangle = \\ &\qquad q^{h(\alpha)-\frac{c}{24}} \exp\left(\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{q^{kl}}{l}\mathrm{Tr}(\Lambda_1\Lambda_2^T)^l\right)\delta_{\alpha,\beta}\,\delta_{\alpha,\Lambda_1\Lambda_2^T\beta}\,. \end{aligned} \tag{5.1}$$

There are then three cases where the amplitude is non-zero;

$$\begin{aligned}\langle B(\alpha;\Lambda_1)|\, q^{\frac{1}{2}(L_0+\overline{L}_0-\frac{c}{12})}\,|B(\alpha;\Lambda_2)\rangle &= \frac{q^{h(\alpha)-\frac{c}{24}}}{\prod_{k>0}(1-q^k)^r}, \quad \Lambda_1=\Lambda_2,\\ &= \frac{q^{h(\alpha)-\frac{c}{24}}\,\delta_{\alpha,-S\alpha}}{\prod_{k>0}(1-q^{2k})^{\frac{r}{2}}}, \quad \Lambda_1\neq\Lambda_2, \quad r \text{ even}\\ &= \frac{q^{h(\alpha)-\frac{c}{24}}\,\delta_{\alpha,-S\alpha}}{\prod_{k>0}(1-q^k)(1-q^{2k})^{\frac{r-1}{2}}},\\ &\qquad \Lambda_1\neq\Lambda_2, \quad r \text{ odd.}\end{aligned} \tag{5.2}$$

Note that the states $|B(\alpha;S)\rangle$ can be written down for any α but they are completely decoupled from the theory unless

$$\alpha = -S\alpha \tag{5.3}$$

and, from (4.10), all non-decoupled states have the property

$$\alpha + \overline{\alpha} = 4\alpha_0\rho. \tag{5.4}$$

The constraint (5.3) implies that

$$\lambda(m,n) = -S\lambda(m,n) \tag{5.5}$$

which is uniquely satisfied by the highest (or lowest) weights of self-conjugate representations. Thus the only primary fields which have the second boundary state $|B(\alpha;S)\rangle$ associated with them are the self-conjugate ones.

The states $|B(\alpha;\Lambda)\rangle$ lie in the Fock space and the corresponding states that lie in the physical Hilbert space, $|\,\alpha;\Lambda\,\rangle\!\rangle$, are obtained by summing over the Felder complex

$$|\,\alpha;\Lambda\,\rangle\!\rangle = \sum_{\substack{w\in\mathcal{W}\\ N\in\mathbb{Z}^r}} \kappa_{wN}\,|B(2\alpha_0\rho - \frac{1}{\sqrt{pp'}}\,(p'wn^i\omega_i - pm^i\omega_i + pp'N^ie_i);\Lambda)\rangle\,, \tag{5.6}$$

where the κ_{wN} are constants of magnitude 1. There is a similar expression for the bra states but with κ_{wN} replaced by κ'_{wN} satisfying

$$\kappa_{wN}\kappa'_{wN} = \varepsilon_w\,. \tag{5.7}$$

From these states the physical cylinder amplitudes can be calculated; between identical in and out states these are simply the characters

$$\langle\!\langle\,\alpha;\Lambda\,|q^{\frac{1}{2}(L_0+\overline{L}_0-\frac{c}{12})}|\,\alpha';\Lambda\,\rangle\!\rangle = \chi_{\lambda(m,n)}(q)\delta_{\alpha,\alpha'}. \tag{5.8}$$

However for the self-conjugate fields there is a second non-zero amplitude

$$\widetilde{\chi}_{\lambda(m,n)}(q) = \langle\!\langle\, \alpha; -I \,| q^{\frac{1}{2}(L_0 + \overline{L}_0 - \frac{c}{12})} |\, \alpha; S \,\rangle\!\rangle. \tag{5.9}$$

Recall that, for such fields, $n^i\omega_i$ and $m^i\omega_i$ must be self-conjugate highest weights; non-zero contributions to this amplitude further require that $wn^i\omega_i$ and N^ie_i are self-conjugate. To solve these constraints introduce the basis for self-conjugate combinations of roots

$$d_k = e_k + e_{k+1} + \ldots + e_{N-k}, \quad k = 1, \ldots \lceil \tfrac{r}{2} \rceil \tag{5.10}$$

which, conveniently, is also orthogonal and denote by $\widetilde{\mathcal{W}}$ the abelian subgroup of $\mathcal{W}$

$$\bigotimes_{k=1}^{\lceil \frac{r}{2} \rceil} \{I, w_{d_k}\}. \tag{5.11}$$

Then the amplitude becomes

$$\widetilde{\chi}_{\lambda(m,n)}(q) = \frac{1}{\eta(\tau)^{r-2\lfloor \frac{r}{2} \rfloor} \eta(2\tau)^{\lfloor \frac{r}{2} \rfloor}} \sum_{\substack{w \in \widetilde{\mathcal{W}} \\ N \in \mathbb{Z}^{\lceil \frac{r}{2} \rceil}}} \varepsilon_w q^{|p' w n^i \omega_i - p m^i \omega_i + p p' N^i d_i|^2 / 2pp'}. \tag{5.12}$$

This expression can be simplified further but it is better to study the modular properties of the diagonal cylinder amplitudes first. At this point we still do not know the minimum self-consistent set of $\lambda(m,n)$ or, equivalently, the primary fields.

6. Modular properties and the primary fields

In this section we will show how modular covariance fixes the set of primary fields in these models. This was first done in [14]; the exact method we use here yields the results in a very convenient form for our subsequent study of the boundary states. The modular properties of the cylinder amplitudes can be examined using standard methods [18]; starting with the diagonal amplitudes, or equivalently the characters (3.19), the Poisson resummation formula gives

$$\chi_{\lambda(m,n)}(q) = \frac{1}{(pp')^{\frac{1}{2}r} \eta(\tau')^r \sqrt{\det A}} \times \sum_{\substack{w \in \mathcal{W} \\ \widetilde{N} \in \mathbb{Z}^r}} \varepsilon_w q'^{|\widetilde{N}^i \omega_i|^2 / 2pp} e^{i2\pi (\widetilde{N}^i \omega_i) \cdot (p' w n^i \omega_i - p m^i \omega_i)/pp'}, \tag{6.1}$$

where $\tau' = -1/\tau$ and $q' = e^{i2\pi\tau'}$. Now split the sum over the dual lattice

$$\mathcal{D} = \mathbb{Z}\omega_1 + \ldots + \mathbb{Z}\omega_r \tag{6.2}$$

into a sum over the scaled root lattice

$$\mathcal{R} = pp'(\mathbb{Z}e_1 + \ldots + \mathbb{Z}e_r) \tag{6.3}$$

and the quotient $\mathcal{Q} = \mathcal{D}/\mathcal{R}$.[d] Then it is the case that

$$\widetilde{N}^i\omega_i = pp'N^ie_i + b\,, \tag{6.4}$$

where $N \in \mathbb{Z}^r$ and $b = b^i\omega_i \in \mathcal{Q}$ and (6.1) becomes

$$\begin{aligned}\chi_{\lambda(m,n)}(q) = {} & \frac{1}{(pp')^{\frac{1}{2}r}\eta(\tau')^r\sqrt{\det A}} \\ & \times \sum_{\substack{w\in\mathcal{W}\\ N\in\mathbb{Z}^r\\ b\in\mathcal{Q}}} \varepsilon_w q'^{|pp'N^ie_i+b|^2/2pp'}\, e^{i2\pi b\cdot(p'wn^i\omega_i - pm^i\omega_i)/pp'}\,. \end{aligned} \tag{6.5}$$

All points in $\mathcal{Q}$ can be written $b = p'k^i\omega_i - p\ell^i\omega_i$. It is therefore convenient to introduce the unique integers r_0, s_0 such that

$$\begin{aligned} 1 &\le r_0 \le p-1\,, \\ 1 &\le s_0 \le p'-1\,, \\ 1 &= p'r_0 - ps_0\,, \end{aligned} \tag{6.6}$$

and define the operators

$$\begin{aligned} P_w &= p'r_0w - ps_0\,, \\ \overline{P}_w &= wP_{w^{-1}} = p'r_0 - ps_0w\,, \end{aligned} \tag{6.7}$$

which have the properties

$$\begin{aligned} P_w(p'k^i\omega_i - p\ell^i\omega_i) &= p'w\,k^i\omega_i - p\ell^i\omega_i \mod \mathcal{R}\,, \\ \overline{P}_w(p'k^i\omega_i - p\ell^i\omega_i) &= p'k^i\omega_i - pw\,\ell^i\omega_i \mod \mathcal{R}\,, \\ P_wP_{w'} &= P_{ww'} \mod \mathcal{R}\,, \\ \overline{P}_w\overline{P}_{w'} &= \overline{P}_{ww'} \mod \mathcal{R}\,. \end{aligned} \tag{6.8}$$

[d] There are many equivalent choices for $\mathcal{Q}$; in the following we take it to be the interior of a polyhedron centred on the origin.

Note that P_w is defined so that repeated application of it simply generates the Felder complex $\mathcal{C}(\lambda)$.

Now, suppose that

$$b = P_{\tilde{w}} b \bmod \mathcal{R}\,, \tag{6.9}$$

where $\tilde{w}$ is some odd element of the Weyl group; then b can be replaced by $P_{\tilde{w}}b$ in the phase factor part of (6.5); using (6.8) and changing the summation variable over $\mathcal{W}$ to $w\tilde{w}$ shows that the contribution to χ is minus itself and therefore must be zero. A similar argument applies if $b = \overline{P}_{\tilde{w}} b \bmod \mathcal{R}$. The condition (6.9) implies that

$$\tilde{w}\, k^i\omega_i = k^i\omega_i + pN^i e_i\,. \tag{6.10}$$

Choosing $\tilde{w}$ to be a reflection in an arbitrary root $e_{m,n} =$ shows that (6.9) is certainly the case if

$$\sum_{i=m}^{n} k_i = 0 \quad \bmod p, \text{ for some } m, n\!: \; r \geq n \geq m > 0\,. \tag{6.11}$$

A similar exercise on $\overline{P}$ yields the conditions

$$\sum_{i=m}^{n} \ell_i = 0 \quad \bmod p', \text{ for some } m, n\!: \; r \geq n \geq m > 0\,. \tag{6.12}$$

It follows that only those bs for which

$$\sum_{i=m}^{n} b_i \neq 0 \quad \bmod p' \textit{ or } \bmod p, \text{ for any } m, n\!: \; r \geq n \geq m > 0\,. \tag{6.13}$$

contribute to the sum over $\mathcal{Q}$ in (6.5). In particular for the bs remaining $b^i, k^i, \ell^i \neq 0$ and so b, $k^i\omega_i$ and $\ell^i\omega_i$ all lie inside Weyl chambers (never on the boundaries) and can always be moved to the fundamental Weyl chamber $\mathcal{C}_0$ by the application of a Weyl transformation. Then if b lies in $\mathcal{C}_0 \cap \mathcal{Q}$

$$b \cdot \theta \leq \frac{pp'}{2}\, \theta \cdot \theta \tag{6.14}$$

i.e. $\sum_{i=1}^{r} b^i \leq pp'$; but equality is ruled out by (6.13) so b always lies inside $\mathcal{Q}$ (never on the boundary).

Now consider $k^i\omega_i$ in $\mathcal{C}_0$ and define the hyperplanes Π_M by

$$\Pi_M : \quad (x - \frac{1}{2}Mp\theta)\cdot\theta = 0, \quad M = 1, 2, \ldots\;. \tag{6.15}$$

Note that because of (6.11) $x = k^i\omega_i$ can never lie on the hyperplanes; supposing that x lies between Π_M and Π_{M+1} reflect it in Π_M to get

$$x' = w_\theta x + Mp\theta \,. \tag{6.16}$$

If x' lies outside C_0 a Weyl transformation will put it back so

$$Tx = w(w_\theta x + Mp\theta) \tag{6.17}$$

lies between Π_{M-1} and Π_M and in $\mathcal{C}_0$. Successive transformations will shift x to the region between Π_0 and Π_1 and in $\mathcal{C}_0$; in this region x satisfies

$$x \cdot \theta < p \,. \tag{6.18}$$

The effect of T on b is

$$\begin{aligned} b &\to p' w(w_\theta k^i\omega_i + Mp\theta) - p\ell^i\omega_i \\ &= P_{ww_\theta} b \mod \mathcal{R} \,. \end{aligned} \tag{6.19}$$

Similar manipulations on $\ell^i\omega_i$ lead to the conclusion that any b can be written in the form

$$\begin{aligned} b &= w' P_w b_0 \,, \\ b_0 &= p' k^i\omega_i - p\ell^i\omega_i \,, \end{aligned} \tag{6.20}$$

where k^i and ℓ^i are positive integers satisfying

$$\sum_{i=1}^{r} \ell^i < p', \quad \sum_{i=1}^{r} k^i < p \,. \tag{6.21}$$

The b_0s are the same as the set of allowed λs derived from fusion and discussed in Section 3. However we have not finished. Clearly some b_0s fall into $\mathcal{C}_0$; these satisfy

$$\begin{aligned} \theta \cdot b_0 = \sum_{i=1}^{r} b_0^i &= p' \sum_{i=1}^{r} k^i - p \sum_{i=1}^{r} \ell^i \\ &\le p'(p-1) - rp \,. \end{aligned} \tag{6.22}$$

Others do not fall into $\mathcal{C}_0$ but can be put there by Weyl transformation, $b_0' = wb_0$, so that

$$\theta \cdot b_0' = \pm \sum_{i=m}^{n} p' k^i - p\ell^i \quad \text{for some } m, n\text{: } r \ge n \ge m > 0 \,, \tag{6.23}$$

where we have used the fact that $w\theta$ is a root. Thus we get the bounds

$$\begin{aligned}\sum_{i=1}^{r} {b'_0}^{i} &\leq p'(p-1) - rp - (r-n+m-1)(p'-p)\,, \quad \text{if "+" in (6.23)},\\ &\leq p'(p-1) - rp - (n-m)(p'-p)\,, \quad \text{if "−" in (6.23)}, \qquad (6.24)\end{aligned}$$

and so the b_0s are equivalent, up to a Weyl transformation, to a sub-set of b_+s defined by

$$b_+^i \geq 1, \quad \sum_{i=1}^{r} b_+^i \leq p'(p-1) - rp\,, \qquad (6.25)$$

and the conditions (6.13). The transformations T are invertible so there are no bs which cannot be obtained by starting with a point outside the b_+ domain and applying T. However some points in the b_+ domain may correspond to either $k^i\omega_i$ or $\ell^i\omega_i$ lying outside (6.21). (The possibility of cancellation between the two terms in b ensures that it is never the situation that *both* lie outside.) Suppose that $k^i\omega_i$ lies outside then for some m, n

$$\begin{aligned}\sum_{m}^{n} b_+^i &= (p+K)p' - p(p'-L), \quad 0 < K,\ 0 < L < p\\ &= Kp' + Lp, \quad 0 < K < p,\ 0 < L < p'. \qquad (6.26)\end{aligned}$$

A similar argument deals with the case that $\ell^i\omega_i$ lies outside, except that this time we must take "−" in (6.23). Thus if

$$\sum_{m}^{n} b_+^i = Kp' + Lp, \quad 0 < K < p,\ 0 < L < p', \qquad (6.27)$$

then b_+ cannot be one of the b_0s. We conclude that all bs contributing to the sum over $\mathcal{Q}$ in (6.5) can be written

$$b = w' P_w \mu, \quad w', w \in \mathcal{W}, \qquad (6.28)$$

where $\mu = \mu^i\omega_i$ lies in the set $\mathcal{B}$ defined by

$$\begin{aligned}\mathcal{B}: \qquad & \mu^i \geq 1,\\ & \sum_{i=1}^{r} \mu^i \leq p'(p-1) - rp\,,\\ & \sum_{m}^{n} \mu^i \neq Kp' + Lp\,, \quad 0 < K < p,\ 0 < L < p',\\ & \qquad \text{for any } m, n\text{: } r \geq n \geq m > 0\,. \qquad (6.29)\end{aligned}$$

It is also convenient to define the self-conjugate subset

$$\mathcal{C}: \quad \mu \in \mathcal{B};\, \mu = -S\mu\,. \tag{6.30}$$

Each $\mu \in \mathcal{B}$ labels a primary field in the CFT and each distinct primary field has just one μ. For example, in the case $W_3(5,4)$ (the critical point of the 3-state Potts model) application of the rules (6.29) yields

$$\begin{aligned} I:&\ \mu = \omega_1 + \omega_2 \\ \sigma:&\ \mu = 2\omega_1 + \omega_2 \\ \sigma^\dagger:&\ \mu = \omega_1 + 2\omega_2 \\ \psi:&\ \mu = 6\omega_1 + \omega_2 \\ \psi^\dagger:&\ \mu = \omega_1 + 6\omega_2 \\ \epsilon:&\ \mu = 3\omega_1 + 3\omega_2\,, \end{aligned} \tag{6.31}$$

whereas $W_4(5,4)$ yields just one solution

$$\mu = \omega_1 + \omega_2 + \omega_3\,, \tag{6.32}$$

which is the identity operator, and so on.

The characters $\{\chi_\mu(q),\ \mu \in \mathcal{B}\}$ form a representation of the modular group. Combining (6.28) and (6.5) gives

$$\begin{aligned} \chi_\lambda(q) = &\frac{1}{(pp')^{\frac{1}{2}r}\eta(\tau')^r\sqrt{\det A}} \\ &\times \sum_{\substack{w,w',w''\in\mathcal{W}\\ N\in\mathbb{Z}^r\\ \mu\in\mathcal{B}}} \varepsilon_w q'^{|pp'N^i e_i + w'' P_{w'}\mu|^2/2pp'}\, e^{i2\pi w'' P_{w'}\mu\cdot P_w\lambda/pp'}, \end{aligned} \tag{6.33}$$

which can be simplified by changing variables to $\{w'', w', w'w\}$ and using (6.8) to

$$\begin{aligned} \chi_\lambda(q) = &\frac{1}{(pp')^{\frac{1}{2}r}\eta(\tau')^r\sqrt{\det A}} \\ &\times \sum_{\substack{w,w',w''\in\mathcal{W}\\ N\in\mathbb{Z}^r\\ \mu\in\mathcal{B}}} \varepsilon_w \varepsilon_{w'} q'^{|pp'N^i e_i + P_{w'}\mu|^2/2pp'}\, e^{i2\pi w''\mu\cdot P_w\lambda/pp'}, \end{aligned} \tag{6.34}$$

which is nothing but

$$\chi_\lambda(q) = \sum_{\mu\in\mathcal{B}} S_{\lambda\mu}\chi_\mu(q'), \tag{6.35}$$

where

$$S_{\lambda\mu} = \frac{1}{(pp')^{\frac{1}{2}r}\sqrt{\det A}} \sum_{w,w'\in\mathcal{W}} \varepsilon_w \, e^{i2\pi\mu\cdot w' P_w \lambda/pp'} . \tag{6.36}$$

Thus we see that the characters for the set of primary fields given by (6.29) are covariant under modular transformations. Note that S is symmetric, real if both $\lambda, \mu \in \mathcal{C}$, and that $S_{I\lambda} > 0$ by Cardy's argument [8]. One can easily check by explicit evaluation that (6.36) generates the correct modular S matrix for the three state Potts model. It is easy to check that S is unitary and that $S^2 = C$, the charge conjugation matrix; for example

$$S^2_{\lambda\rho} = \frac{1}{(pp')^r \det A} \sum_{\mu\in\mathcal{B}} \sum_{w,w'\in\mathcal{W}} \sum_{\bar{w},\bar{w}'\in\mathcal{W}} \varepsilon_w \varepsilon_{\bar{w}} \, e^{i2\pi\mu\cdot(w' P_w \lambda + \bar{w}' P_{\bar{w}} \rho)/pp'} . \tag{6.37}$$

Now change variables to $b = w' P_w \mu$ and reintroduce those bs which were excluded in (6.13) because they did not contribute to the sum; they don't contribute here either so the sum over w, w', μ can be replaced by the unrestricted sum over $\mathcal{Q}$ to give, after a change of variables in the remaining Weyl sum,

$$S^2_{\lambda\rho} = \frac{1}{(pp')^r \det A} \sum_{b\in\mathcal{Q}} \sum_{w,w'\in\mathcal{W}} \varepsilon_w \, e^{i2\pi b\cdot(\lambda + w' P_w \rho)/pp'} . \tag{6.38}$$

The sum over $\mathcal{Q}$ is now zero unless $\lambda + w' P_w \rho = 0$ which occurs only if $w = 1$, $w' = w_0$ and $\rho = -w_0\lambda$, ie ρ and λ are conjugate representations; in which case the sum just gives the volume of $\mathcal{Q}$ which cancels the denominator and leaves $S^2_{\lambda\rho} = 1$. A similar argument can be used to demonstrate unitarity.

We now return to the mixed amplitudes (5.12) which can be rewritten using the notation of this section as

$$\widetilde{\chi}_\lambda(q) = \frac{1}{\eta(\tau)^{r-2\lfloor\frac{r}{2}\rfloor}\eta(2\tau)^{\lfloor\frac{r}{2}\rfloor}} \sum_{\substack{w\in\widetilde{\mathcal{W}} \\ N\in\mathbb{Z}^{\lceil\frac{r}{2}\rceil}}} \varepsilon_w q^{|pp' N^i d_i + P_w \lambda|^2/2pp'} , \tag{6.39}$$

where $\lambda \in \mathcal{C}$. Using the basis (5.10) in which

$$\lambda = \frac{1}{2}\,\hat{\lambda}_i d_i \, , \quad \hat{\lambda}_i = \lambda \cdot d_i \, , \tag{6.40}$$

$\widetilde{\chi}_\lambda(q)$ takes the factorized form

$$\begin{aligned}\widetilde{\chi}_\lambda(q) &= \prod_{i=1}^{\frac{1}{2}r} \widehat{\chi}_{\hat{\lambda}_i}(q), \quad \text{if } r \text{ even}, \\ &= \chi^V_{\hat{\lambda}_{\frac{r+1}{2}}}(q) \prod_{i=1}^{\frac{1}{2}(r-1)} \widehat{\chi}_{\hat{\lambda}_i}(q), \quad \text{if } r \text{ odd},\end{aligned} \tag{6.41}$$

where

$$\begin{aligned}\widehat{\chi}_{\hat{\lambda}}(q) &= \frac{1}{\eta(2\tau)} \sum_{\substack{\varepsilon=\pm1 \\ N\in\mathbb{Z}}} \varepsilon q^{pp'(N+P_\varepsilon\hat{\lambda}/2pp')^2}, \\ \chi^V_{\hat{\lambda}}(q) &= \frac{1}{\eta(\tau)} \sum_{\substack{\varepsilon=\pm1 \\ N\in\mathbb{Z}}} \varepsilon q^{pp'(N+P_\varepsilon\hat{\lambda}/2pp')^2},\end{aligned} \tag{6.42}$$

and

$$P_\varepsilon = p' r_0 \varepsilon - p s_0 \,. \tag{6.43}$$

The behavior of these functions under modular transformation, which will be needed in the next section, is given by

$$\begin{aligned}\widehat{\chi}_{\hat{\lambda}}(q) &= \frac{2}{\sqrt{pp'}} \sum_{a=1}^{pp'-1} \sum_{\varepsilon=\pm1} \varepsilon \cos\left(\frac{\pi a P_\varepsilon \hat{\lambda}}{pp'}\right) G_a(q'), \\ G_a(q) &= \frac{1}{\eta(\tau/2)} \sum_{N\in\mathbb{Z}} q^{pp'(N+a/2pp')^2},\end{aligned} \tag{6.44}$$

and

$$\begin{aligned}\widehat{\chi}^V_{\hat{\lambda}}(q) &= \frac{\sqrt{2}}{\sqrt{pp'}} \sum_{a=1}^{pp'-1} \sum_{\varepsilon=\pm1} \varepsilon \cos\left(\frac{\pi a P_\varepsilon \hat{\lambda}}{pp'}\right) G^V_a(q'), \\ G^V_a(q) &= \frac{1}{\eta(\tau)} \sum_{N\in\mathbb{Z}} q^{pp'(N+a/2pp')^2}.\end{aligned} \tag{6.45}$$

In these formulae the sum over a can omit multiples of p' and p because the coefficient vanishes. Note that the set of functions appearing on the right hand is now not the same as on the left hand side and at this point there is no particular relationship between the indices $\hat{\lambda}$ and a. Using these formulae shows that

$$\widetilde{\chi}_\lambda(q) = \sum_{\psi\in\mathcal{P}} \Psi_{\lambda\psi} H_\psi(q'), \tag{6.46}$$

where $H_\psi(q')$ are some set of modular functions assembled from products of the Gs, and $\mathcal{P}$ is some domain not yet determined but which contains at least as many members as $\mathcal{C}$.

7. Physical boundary states

Physical boundary states are defined so that when cylinder amplitudes between them are expressed in terms of the annulus variable q' the result is a power series in q' in which every coefficient is a (positive) integer. Thus the annulus partition function with given boundary conditions is essentially formed from the contributions of physical degrees of freedom propagating round the annulus. The $W_3(5,4)$ (critical three state Potts) model is also a Virasoro minimal model which can be exploited to ease the calculation of the physical boundary states. In general these shortcuts are not available and we have to proceed rather differently.

First introduce the condensed notation for the basis states in the Hilbert space (5.6)

$$\begin{aligned} |\,\lambda\,\rangle\!\rangle &\equiv |\,\alpha = 2\alpha_0\rho - (pp')^{-\frac{1}{2}}\lambda, -I\,\rangle\!\rangle \\ |\,\hat{\lambda}\,\rangle\!\rangle &\equiv |\,\alpha = 2\alpha_0\rho - (pp')^{-\frac{1}{2}}\lambda, S\,\rangle\!\rangle. \end{aligned} \tag{7.1}$$

For every primary field there is an ordinary, W current conserving, physical boundary state constructed in the usual way [8],

$$|\widetilde{\lambda}\rangle = \sum_{\mu\in\mathcal{B}} \frac{S_{\lambda^*\mu}}{\sqrt{S_{I\mu}}} |\,\mu\,\rangle\!\rangle, \tag{7.2}$$

where λ^* denotes the conjugate to λ. In addition we expect there might be physical boundary states constructed from the W violating states

$$|\widetilde{A}\rangle = \sum_{\mu\in\mathcal{C}} b_{A\mu} |\,\hat{\mu}\,\rangle\!\rangle. \tag{7.3}$$

The annulus partition function with boundary conditions labelled by $\widetilde{\lambda}$ and

$\widetilde{A}$ is then

$$\begin{aligned} Z_{\widetilde{\lambda}\widetilde{A}} &= \langle\widetilde{\lambda}|\, q^{\frac{1}{2}(L_0+\overline{L}_0-\frac{c}{12})}\, |\widetilde{A}\rangle \\ &= \sum_{\mu\in\mathcal{C}} \frac{S_{\lambda^*\mu} b_{A\mu}}{\sqrt{S_{I\mu}}} \widetilde{\chi}_\mu(q) \\ &= \sum_{\mu\in\mathcal{C}} \sum_{\psi\in\mathcal{P}} \frac{S_{\lambda^*\mu} b_{A\mu}}{\sqrt{S_{I\mu}}}\, \Psi_{\mu\psi} H_\psi(q')\,. \end{aligned} \tag{7.4}$$

Provided that the $H_\psi(q')$ are linearly independent the coefficient of $H_\psi(q')$ should be an integer. The equations determining the boundary states are then

$$\sum_{\mu\in\mathcal{C}} \frac{S_{\lambda^*\mu} b_{A\mu}}{\sqrt{S_{I\mu}}}\, \Psi_{\mu\psi} = n^{\psi}_{\widetilde{\lambda}\widetilde{A}}\,. \tag{7.5}$$

From now on we will confine our attention to the $(p+1,p)$ models which are unitary so there should be no question that the physical boundary states exist and satisfy the criteria discussed above. $W_3(p+1,p)$ is the simplest case because there is only one basis vector, d_1 (5.10), and pp' is necessarily even. Then (6.44) can be written

$$\widetilde{\chi}_\lambda(q) = \frac{2}{\sqrt{pp'}} \sum_{\text{even } a=2}^{2pp'-2} \left(\cos\left(\frac{\pi(p'-p)a\hat{\lambda}}{2pp'}\right) - \cos\left(\frac{\pi(p'+p)a\hat{\lambda}}{2pp'}\right)\right) G_{a/2}(q')\,. \tag{7.6}$$

For the same reasons as in the previous section, there is no contribution if a is a multiple of p or p'; thus the sum runs over $a \in \mathcal{H}$, defined as the set of even numbers between 0 and $2pp'$ that are not an integer multiple of p or p'.

We will now show that the set

$$\mathcal{H}' = \{\mu\cdot d_1,\ (p+p')\mu\cdot d_1,\ 2pp'-\mu\cdot d_1,\ 2pp'-(p+p')\mu\cdot d_1;\ \mu\in\mathcal{C}\} \mod 2pp' \tag{7.7}$$

is the same as $\mathcal{H}$. The number of elements in $\mathcal{H}$ is

$$\begin{aligned} |\mathcal{H}| &= (p-1)^2 \quad p \text{ odd}, \\ &= p(p-2) \quad p \text{ even}. \end{aligned} \tag{7.8}$$

If $\mu \in \mathcal{C}$ then $N = \mu\cdot d_1$ is a positive even number such that

$$\begin{aligned} &N \leq p^2 - 2p - 1 \\ &N \neq np + mp', \quad p > n, m \geq 0\,. \end{aligned} \tag{7.9}$$

The second condition can be rewritten

$$N \neq np + m, \quad 0 \leq m \leq n < p \tag{7.10}$$

from which we can immediately see that allowed Ns are given by

$$N = np + m, \quad 0 \leq n < m < p, \; n \leq p-3. \tag{7.11}$$

Some further constraints, which depend on whether p is odd or even, are needed on m, n to ensure that N is even. These only play a role in calculating $|\mathcal{C}|$, the number of allowed Ns. Using (7.11) we find that

$$\begin{aligned} |\mathcal{C}| &= \frac{1}{4}(p-1)^2 \qquad p \text{ odd}, \\ &= \frac{1}{4}p(p-2) \qquad p \text{ even}. \end{aligned} \tag{7.12}$$

Thus $|\mathcal{H}| = |\mathcal{H}'|$ and of course $\mathcal{H}'$ does not contain any multiples of p or p' because of the definition of $\mathcal{C}$ (6.29,6.30). It remains to show that all the elements of $\mathcal{H}'$ are distinct.

Consider

$$\begin{aligned} \bar{N} = (p+p')N \bmod 2pp' &= p(2m-n) + m \\ &= \bar{n}p + \bar{m}\,. \end{aligned} \tag{7.13}$$

Note that $0 < \bar{m} < \bar{n}$ which means that $\bar{N}$ can never be one of the Ns. Now consider

$$\bar{\bar{N}} = 2pp' - N = p(2(p-m) + 2m - n + 1) + (p-m)\,. \tag{7.14}$$

Comparing this with (7.13) and noting that $2m - n + 1 > 0$ shows that $\bar{\bar{N}}$ can never be one of the $\bar{N}$s. This completes the proof that $\mathcal{H}' \equiv \mathcal{H}$.

Now we can replace the sum over $a \in \mathcal{H}$ in (7.6) by a sum over $a \in \mathcal{H}'$ which gives

$$\widetilde{\chi}_\lambda(q) = \sum_{\mu\in\mathcal{C}} \Psi_{\lambda\mu}\big(G_{\hat{\mu}/2}(q') + G_{pp'-\hat{\mu}/2}(q') - G_{(p'+p)\hat{\mu}/2}(q') - G_{pp'-(p'+p)\hat{\mu}/2}(q')\big)\,, \tag{7.15}$$

where

$$\Psi_{\lambda\mu} = \frac{4}{\sqrt{pp'}} \sin\left(\frac{\pi\hat{\mu}\hat{\lambda}}{2p}\right) \sin\left(\frac{\pi\hat{\mu}\hat{\lambda}}{2p'}\right). \tag{7.16}$$

The combination of G functions appearing here has a series expansion in q' with all positive coefficients and is linearly independent for different μ. Thus this is the form required; note that Ψ is in fact a square matrix in this case and it is straightforward to check that

$$\Psi_{\lambda\rho}\Psi_{\rho\mu} = \delta_{\lambda\mu}\,. \tag{7.17}$$

When p is small the conditions (7.5) can now be solved by brute force. For $p = 4$ they yield exactly the "free" and "new" boundary conditions found by Affleck et al [9]. To solve the constraints (7.5) one can proceed analogously to the calculation of the usual boundary states; suppose that $n^{\psi}_{\widetilde{0}\widetilde{A}} = \delta^{\psi}_{\widetilde{A}}$ and use the invertibility of Ψ to get

$$b_{A\mu} = \frac{\Psi_{\mu A}}{\sqrt{S_{I\mu}}}\,. \tag{7.18}$$

Substituting these back in (7.5) yields

$$\sum_{\mu\in\mathcal{C}} \frac{S_{\lambda^*\mu}\Psi_{\mu A}\Psi_{\mu\psi}}{S_{I\mu}} = n^{\psi}_{\widetilde{\lambda}\widetilde{A}}\,. \tag{7.19}$$

For $p = 5$, which has four self-conjugate fields with $\hat{\mu} = 2, 4, 8, 14$, and for $p = 6$, which has six with $\hat{\mu} = 2, 4, 8, 10, 16, 22$, we have checked explicitly that the expressions on the left hand side generate positive integers. Thus these models generate an analogue of the Verlinde formula for the symmetry violating sector (see also [19, 20] for other examples of this).

For rank 3 and above the situation is more complicated and even for the simplest rank 3 model, $W_4(6,5)$, we have not been able to construct consistent physical states in the W violating sector. The reason is that the number of independent $G_{\mathbf{a}}(q')$ appearing in the equivalent of (7.6) is now too large to be accounted for by the equivalent of $\mathcal{H}'$ (7.7). This strongly suggests that the basis of symmetry violating states may be incomplete. Further evidence for this is given in the next section.

8. *W* currents

In this section we will assume that the W fields can indeed be constructed according to the prescription of [21] and show that $|B(\alpha; S)\rangle$ necessarily violates conservation of *all* the currents with conformal dimension greater than 2. The first step in obtaining the Virasoro primary fields $\{W_K,\ K = 3, \dots N\}$ which, together with T, form the W algebra is to define the generating functional

$$(2i\alpha_0)^N \mathcal{D}_N = \prod_{K=1}^{N} (2i\alpha_0\partial_z + h_K \cdot \partial\phi(z)) :\,, \tag{8.1}$$

where, if the product were written out, the value of K increases from left to right. This can be evaluated [11, 12] to get

$$(2i\alpha_0)^N \mathcal{D}_N = (2i\alpha_0\partial)^N + \sum_{K=1}^{N} u_K[\phi(z)]\,(2i\alpha_0\partial)^{N-K}, \tag{8.2}$$

where the $u_K[\phi(z)]$ are fields of conformal dimension K (u_1 vanishes identically). Unfortunately the u_K, apart from $u_2(z) \equiv T(z)$, are not themselves primary fields. The true W_K are constructed from combinations of $\{u_K, \ldots, u_2, \partial\}$. In the present case it is sometimes more useful to think of them being assembled iteratively as combinations of $\{u_K, W_{K-1}\ldots, W_2, \partial\}$. Note that W_K is essentially just a normal ordered multinomial expression in $\partial\phi^i$ and its derivatives but that unfortunately precise expressions are unknown for $K > 5$ (for the completely known algebras see [22]). The following arguments will make use of the facts that a)

$$\mathcal{S}_{K>1}(x) \equiv \sum_{L_1>L_2>\ldots L_K} (h_{L_1}\cdot x)(h_{L_2}\cdot x)\ldots(h_{L_K}\cdot x) = \frac{(-1)^{K+1}}{K}\sum_{L=1}^{N}(h_L\cdot x)^K \tag{8.3}$$

and b), defining $\mathcal{S}_1(x) = 1$,

$$\sum_{k=1}^{\lfloor\frac{1}{2}K\rfloor} A_k \mathcal{S}_{K-k}(x)\mathcal{S}_k(x) \neq 0, \quad \text{unless } A_k = 0, \forall k, \tag{8.4}$$

(i.e. linear independence of the product functions).

Conservation of the W current at the boundary is given in terms of modes by

$$\left(W_{K\,n} - (-1)^K \overline{W}_{K\,-n}\right)|B(\alpha;\Lambda)\rangle = 0. \tag{8.5}$$

Since the W_K are primary, it is sufficient to check the $n = 0$ case so let

$$\left(W_{K\,0} - (-1)^K \overline{W}_{K\,0}\right)|B(\alpha;\Lambda)\rangle = C_\Lambda \Delta_0^K\,|\alpha,\overline{\alpha}\rangle, \tag{8.6}$$

and Δ_0^{KL} be the part of Δ_0^K containing exactly L factors of a_n or $\overline{a}_n$ with $n \neq 0$. Terms in W_K which do not contain ∂ but just products of primaries satisfy this condition automatically if the primaries do; if they do not then the violations cannot be cancelled between different products of primaries on account of (8.4) and W_K automatically violates the condition too. So, of the possible terms in W_K which contribute to Δ_0^{KK} only the ones coming from u_K have unknown properties at the boundary. Now note that

$$h_K \cdot \partial\phi\, C_\Lambda\, |\alpha,\overline{\alpha}\rangle = -iC_\Lambda \left(\frac{h_K \cdot \alpha}{z} + \sum_{n<0} \frac{h_K \cdot a_n}{z^{n+1}} + \sum_{n<0} \frac{h_K \cdot \Lambda \overline{a}_n}{z^{-n+1}} \right) |\alpha,\overline{\alpha}\rangle,$$
$$h_K \cdot \overline{\partial}\,\overline{\phi}\, C_\Lambda\, |\alpha,\overline{\alpha}\rangle = -iC_\Lambda \left(\frac{h_K \cdot \overline{\alpha}}{\overline{z}} + \sum_{n<0} \frac{h_K \cdot \overline{a}_n}{\overline{z}^{n+1}} + \sum_{n<0} \frac{h_K \cdot \Lambda a_n}{\overline{z}^{-n+1}} \right) |\alpha,\overline{\alpha}\rangle, \tag{8.7}$$

which implies

$$\Delta_0^{KK} = (-i)^K P_0 \left[\mathcal{S}_K(Q(z)) - (-1)^K \mathcal{S}_K \left(\Lambda Q \left(\frac{1}{\overline{z}} \right) \right) \right], \tag{8.8}$$

where

$$Q(z) = \sum_{n<0} \frac{a_n}{z^n} + \sum_{n<0} \frac{\Lambda \overline{a}_n}{z^{-n}} \tag{8.9}$$

and P_0 projects out terms $O(z^0)$. Evaluating (8.8) gives

$$\begin{aligned} \Delta_0^{KK} &= 0, \quad \text{if } \Lambda = -I, \\ &= 0, \quad \text{if } \Lambda = S \text{ and } K \text{ even}, \\ &\neq 0, \quad \text{if } \Lambda = S \text{ and } K \text{ odd}. \end{aligned} \tag{8.10}$$

Thus the states $|B(\alpha;S)\rangle$ violate W_K conservation for all odd K. To determine whether this is also the case for even $K > 2$ we have to examine the terms containing $K-1$ factors a_n with $n \neq 0$.

The operator product $T(w)u_K(z)$ contains a piece proportional to $(w-z)^{-3}$, and containing $K-1$ factors $a_n, n \neq 0$, which it is known can be cancelled by forming the combination [13]

$$u'_K = u_K - 2i\alpha_0 (\tfrac{N-K+1}{2})\, \partial u_{K-1}\,. \tag{8.11}$$

Now terms in W_K of the form $b_k \partial W_{K-k-1} W_k$, $k = 2\ldots$ can also generate such pieces in the OPE with T; but the complete cancellation (8.11) means that these contributions must cancel among themselves. Therefore, by the linear independence property (8.4), $b_k = 0$ and $\Delta_0^{K\,K-1}$ must come from u'_K. Both contributions to u'_K contain $\partial^2\phi$ which in terms of the modes satisfies

$$\partial^2\phi\, C_\Lambda = iC_\Lambda \left(-\frac{a_0}{z^2} - \frac{Q}{z^2} + \frac{1}{z}\, Q'(z) \right),$$
$$\overline{\partial}^2\overline{\phi}\, C_\Lambda = iC_\Lambda \left(-\frac{\overline{a}_0}{\overline{z}^2} - \frac{\Lambda Q}{\overline{z}^2} - \frac{1}{\overline{z}}\, \Lambda Q' \Big(\frac{1}{\overline{z}} \Big) \right). \tag{8.12}$$

This shows that for each holomorphic sector contribution to $\Delta_0^{K\,K-1}$ of the form $(h{\cdot}Q)^{K-2}h{\cdot}Q'$ there is one $(h{\cdot}\Lambda Q)^{K-2}h{\cdot}\Lambda Q'$ from the anti-holomorphic sector. Thus when $\Lambda = -I$ they cancel (remember we are now considering just even K). On the other hand if K is even and $\Lambda = S$ they add up. The remaining terms involve just Q and a_0 or $\overline{a}_0$ and explicit calculation for this combination yields

$$\Delta_0^{K\,K-1} = P_0 \sum_L \Bigg[(h_L \cdot Q(z))^{K-1} \left(h_L \cdot \alpha - 2\alpha_0 (\frac{1}{2}(N+1) - L) \right) - (-1)^K \left(h_L \cdot \Lambda Q\Big(\frac{1}{\overline{z}}\Big)\right)^{K-1} \left(h_L \cdot \overline{\alpha} - 2\alpha_0 \Big(\frac{1}{2}(N+1) - L\Big)\right)\Bigg] \tag{8.13}$$

which implies that

$$\begin{aligned} \Delta_0^{K\,K-1} &= 0, \quad \text{if } \Lambda = -I \text{ provided (5.4)}\ , \\ &\neq 0, \quad \text{if } \Lambda = S. \end{aligned} \tag{8.14}$$

This completes the proof that the states $|B(\alpha;S)\rangle$ violate W_K conservation for all $K > 2$. It also shows that the condition (5.4) is necessary, although not sufficient, for the W_K to be conserved by the states $|B(\alpha;I)\rangle$.

Now we see that for rank 2, where there is only one W charge, the boundary states either conserve it or break it. However for $r \geq 3$ our states conserve either all or none of the W charges. We do not have any states that partially break the W algebra down to a smaller one. These may be the missing states which prevented us from constructing the physical boundary states for $r \geq 3$.

9. Summary and Discussion

The coherent states $|B(\alpha;\Lambda)\rangle$ can be used to construct a basis for boundary states. These are specified by a field label α and a generator Λ of the group of outer automorphisms of the Dynkin diagram of A_r (the reader should note that this was derived, not assumed). Those corresponding to the identity element $\Lambda = -I$ conserve the entire W algebra while those corresponding to charge conjugation $\Lambda = S$ (inversion of the Dynkin diagram) maximally break the chiral algebra down to Virasoro. The states $|B(\alpha;\Lambda)\rangle$ lie in the Fock space; to get states in the Hilbert space $|\,\alpha;\Lambda\,\rangle\!\rangle$ it is necessary to take linear combination s of the $|B(\alpha;\Lambda)\rangle$ as specified by Felder's BRST construction. It is important to note that this can be done in such a way that cylinder amplitudes between the $|\,\alpha; -I\,\rangle\!\rangle$ reproduce the W characters. Then the form of the cylinder amplitudes (or characters) can be used together with the requirement of modular invariance to determine the minimal consistent set of field labels α; these then specify the primary fields of the minimal $W_r(p', p)$ models. The fact that this whole construction produces a self-consistent structure can be regarded as a powerful demonstration that Felder's construction is correct for arbitrary rank r.

With these basis states we can try to construct physical boundary states. In general this is a complicated exercise even if we confine attention to the unitary $W_r(p+1, p)$ models. It is straightforward to do it for the W conserving sector where the calculation just follows Cardy's usual procedure; however in the W breaking sector it seems that for $r \geq 3$ we do not have enough basis states to make a consistent construction of W violating physical boundary states. However for rank 2 the unitary $W_3(p+1, p)$ theories do have, in addition to the usual Cardy states, an extra W violating physical boundary state for each self-conjugate primary field. Each of these states corresponds to an extra physical boundary condition for which the annulus amplitude is an admissible partition function with positive definite Boltzman weights. There is a straightforward generalization of the Verlinde formula to the symmetry violating sector. In sum the results are essentially similar to the Potts case and to those found in some other examples of CFTs with extended chiral symmetries [20].

The situation for $r \geq 3$ is unclear and remains an open problem. Certainly the physical boundary states must be well defined at least for the unitary models. It would be strange if all the basis states we *do* know were not to appear in the physical boundary states; on this intuition we then expect W violating physical boundary states to exist. The question is whether there are in fact the necessary extra basis states which would allow this to happen.

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