QUARTIC ANHARMONIC MANY-BODY OSCILLATOR

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Two quantum quartic anharmonic many-body oscillators are introduced. One of them is the celebrated Calogero model (rational A_n model) modified by quartic anharmonic two-body interactions which support the same symmetry as the Calogero model. Another model is the three-body Wolfes model (rational G_2 model) with quartic anharmonic interaction added which has the same symmetry as the Wolfes model. Both models are studied in the framework of algebraic perturbation theory and by the variational method.

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This work is dedicated to the memory of Ian Kogan who died so young that is hard for me to imagine that he is not with us anymore. I knew him for about 30 years since the time when he appeared at ITEP Theory Division as a young, very brilliant student. Then for many years we were sitting in the next door offices at ITEP. Sometimes we talked on science being both intrigued by the transition from Quantum Mechanics to Quantum Field Theory. I am sure that Ian would be pleased to read the present article.

1. Introduction

Anharmonic oscillators play a crucially important role in contemporary physics since they model intrinsic anharmonic effects of the real world. The goal of the present work is to introduce and then to study a special type of quantum anharmonic oscillator – *many-body anharmonic oscillators*. One of them can be considered as an anharmonic perturbation of the celebrated many-body Calogero model (see [1]) or, in other words, the rational A_n model. Another is an anharmonic perturbation of the three body Wolfes model or, equivalently, the rational G_2 model (see [2]). The first system describes n interacting particles on a line with fixed ordering with pairwise interaction, while the second one corresponds to three identical interacting particles on a line with fixed ordering with two- and three-body interactions. It is rather natural to impose a requirement that these anharmonic systems should possess the same symmetry properties as the original Calogero or Wolfes models: (i) translation invariance, (ii) permutation invariance, (iii) reflection symmetry with respect to a change of the sign of all coordinates.

The one-dimensional anharmonic oscillator

$$\mathcal{H} \ = \ -\frac{d^2}{dx^2} \ + \ x^2 + \lambda x^4 \ , \qquad x \in R \, , \tag{1}$$

is perhaps one of the most celebrated and the most studied problems in quantum mechanics. A systematic study was carried out by Bender-Wu in 1969-1973 in their seminal papers [3]. Even this simplest anharmonic oscillator possesses exceptionally rich properties:

- divergent perturbation theory (PT) in λ [3,4], i.e.

$$E = \sum_k a_k \lambda^k \ , \qquad a_k \propto k\,! \ ,$$

- highly non-trivial but convergent PT in $1/\lambda$ (strong coupling expansion) [3,5], i.e. $E = \sum_k b_k \lambda^{1/3-2k/3}$,

- analytic structure in λ; all even (odd) eigenvalues are analytically related through square-root branch points, which accumulate to $\lambda = 0$ (see Fig.1) [3].

  Hence by studying one eigenstate the whole family of eigenstates is explored!

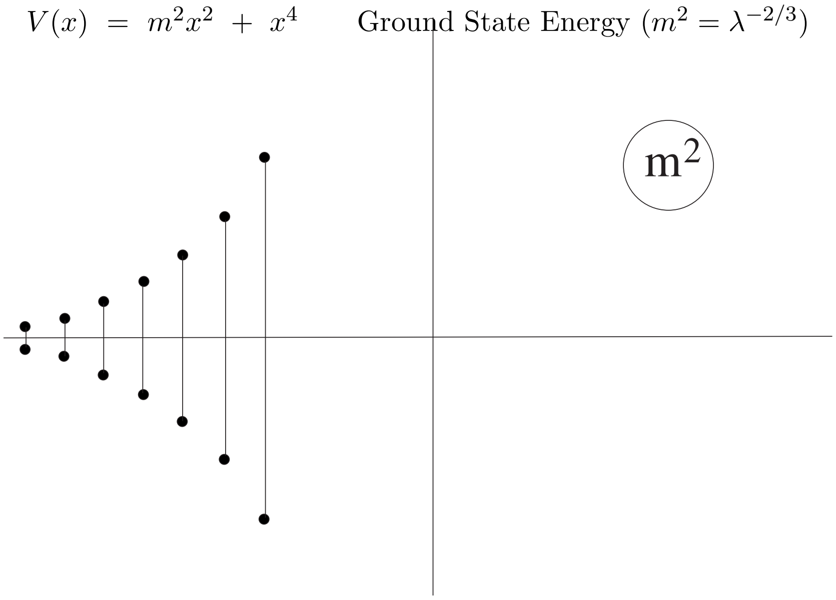

Figure 1. Structure of singularities in the inverse coupling constant $m^2 = \lambda^{-1/3}$ on the first sheet of the Riemann surface of the ground state energy. Bullets denote the square-root branch points connected by cuts (vertical lines)

So far very little (almost nothing) is known about eigenfunctions as functions of λ.

2. Anharmonic Calogero Model

The Hamiltonian of the A_{n-1} anharmonic oscillator takes the form

$$\mathcal{H}_{\rm A} = -\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial {x_i}^2} + g\sum_{i>j}^{n}\frac{1}{(x_i-x_j)^2} + \frac{\omega^2}{2}\sum_{i>j}^{n}(x_i-x_j)^2 + \frac{\lambda}{n+6}\sum_{i>j}^{n}(x_i-x_j)^4\,, \tag{2}$$

where $g > -\frac{1}{4}$, ω is the frequency, $\lambda \geq 0$ is the coupling constant with a factor $(n+6)$ which is introduced for a convenience and $n = 2, 3, \ldots$. This Hamiltonian describes a system of n identical particles situated on the straight line with pairwise interaction separated from each other by impenetrable barriers. The configuration space is

$$-\infty < x_1 \le x_2 \le \ldots \le x_n < \infty \ . \tag{3}$$

If the coupling constant $\lambda = 0$, the Hamiltonian (2) corresponds to the celebrated Calogero model and the domain (3) is nothing but the Weyl chamber of the A_{n-1} root system. The ground state eigenfunction of the Calogero model is

$$\Psi_0^{(c)}(x) = \Delta^{\nu}(x) e^{-\frac{\omega}{2n} X_2} \ , \quad g = \nu(\nu-1) \ , \tag{4}$$

where $\Delta(x) = \prod_{i<j} |x_i - x_j|$ is the Vandermonde determinant and $X_2 = \sum_{i>j}(x_i - x_j)^2$, when the ground state energy is $E_0^{(c)} = \omega(1+\nu n)$.

In order to deal with translation invariance of many-body systems we replace the Cartesian coordinates by the center-of-mass coordinate, $Y = \sum_{j=1}^{n} x_j$, and the translation-invariant relative coordinates – the Perelomov coordinates [6],

$$y_i \ = \ x_i - \frac{1}{n} Y \ , \quad i = 1, 2, \ldots, n \ , \tag{5}$$

which obey the constraint $\sum_{j=1}^{n} y_j = 0$, where x_i are the Cartesian coordinates. The coordinates (5) make sense as translation-invariant relative coordinates which measure a distance from the center of mass to a particle position. Since we consider a system of identical particles, permutation symmetry holds. In order to make manifest the permutation symmetry we introduce permutationally symmetric coordinates. The most convenient candidate is the invariants of the symmetric group. Eventually, we arrive at elementary symmetric polynomials of the arguments y (see Eq.(5)) as new coordinates

$$(x_1, x_2, \ldots x_n) \to \big(Y, \tau_k(x) = \sigma_k(y(x))|_{\ k=2,3,\ldots n}\big) \, . \tag{6}$$

Here,

$$\sigma_k(x) = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$$

are elementary symmetric polynomials. As an illustration let us present in explicit form the τ coordinates for $n = 2, 3$, putting $-y_n = y_1 + y_2 + \ldots + y_{n-1}$,

- $n = 2 \qquad \tau_2 = -y_1^2$,
- $n = 3 \qquad \tau_2 = -y_1^2 - y_1 y_2 - y_2^2 \ , \quad \tau_3 = -y_1 y_2 (y_1 + y_2)$.

It is easy to recognize that the τ coordinates are nothing but a particular form of the Weyl invariant polynomials of the lowest degrees in the A_{n-1} root space.

It can easily be shown that

$$\sum_{i>j}^{n}(x_i - x_j)^2 = -2n\tau_2\,, \tag{7}$$

$$\sum_{i>j}^{n}(x_i - x_j)^4 = 2(n+6)\tau_2^2 - 4n\tau_4\,. \tag{8}$$

These relations reveal a remarkable feature of the τ coordinates – although the left-hand-side depends on all x_i coordinates, the right-hand-side depends on a finite number of τ's. Making a gauge rotation of the Hamiltonian (2) with the Calogero ground state eigenfunction (4) as the gauge factor and re-writing the result in the τ coordinates (6), we arrive at a strikingly simple expression after separating out the center-of-mass coordinate Y,

$$\begin{aligned} h_{\rm A} &= 2(\Psi_0^{(c)})^{-1}\,(\mathcal{H}_{\rm A} - E_0^{(c)})\,\Psi_0^{(c)} \equiv h_{\rm Cal} + \lambda v_p \\ &= \sum_{i,j=2}^{n}\mathcal{A}_{ij}\,\frac{\partial^2}{\partial\tau_i\partial\tau_j} + \sum_{i=2}^{n}\mathcal{B}_i\,\frac{\partial}{\partial\tau_i} + 2\lambda\Big[\tau_2^2 - \frac{2n}{n+6}\,\tau_4\Big]\,, \end{aligned} \tag{9}$$

where

$$\begin{aligned} \mathcal{A}_{ij} &= \frac{(n-i+1)(1-j)}{n}\,\tau_{i-1}\,\tau_{j-1} + \sum_{l\geq \max(1,j-i)}(2l-j+i)\,\tau_{i+l-1}\,\tau_{j-l-1}\ , \\ \mathcal{B}_i &= \left(\frac{1}{n}+\nu\right)(n-i+2)(n-i+1)\,\tau_{i-2} + 2\omega\, i\,\tau_i\ , \\ v_p &= 2\,\tau_2^2 - \frac{4n}{n+6}\,\tau_4\ . \end{aligned} \tag{10}$$

Explicit formulae for the first few coefficient functions are

$$\begin{aligned} \mathcal{A}_{22} &= 2\,\tau_2\,, \quad \mathcal{B}_2 = 4\,\omega\,\tau_2 + (1+\nu n)(n-1)\,, \\ \mathcal{A}_{23} &= 3\,\tau_3\,, \quad \mathcal{B}_3 = 6\,\omega\,\tau_2\,, \\ \mathcal{A}_{24} &= 4\,\tau_4\,, \quad \mathcal{B}_4 = 8\,\omega\,\tau_4 + \frac{1}{n}\,(1+\nu n)(n-2)(n-3)\,\tau_2\,, \\ \mathcal{A}_{33} &= 4\,\tau_4 - 2\Big(1-\frac{2}{n}\Big)\,\tau_2^2\,, \quad \mathcal{A}_{34} = 5\,\tau_5 - 2\Big(1-\frac{3}{n}\Big)\,\tau_2\tau_3\,, \\ \mathcal{A}_{44} &= 6\,\tau_6 + 2\,\tau_2\tau_4 - 3\Big(1-\frac{3}{n}\Big)\,\tau_3^2\,. \end{aligned}$$

In [7] it was demonstrated that at $\lambda = 0$ the gauge-rotated Calogero Hamiltonian $h_{\rm Cal}$ (9) has infinitely many finite-dimensional invariant subspaces

$$\mathcal{P}_k \;=\; \langle {\tau_2}^{p_2}{\tau_3}^{p_3}\ldots{\tau_n}^{p_n} |\; 0 \le \Sigma p_i \le k\rangle\ , \quad k=0,1,2,\ldots\ . \tag{11}$$

These spaces can be embedded one into another,

$$\mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \ldots \subset \mathcal{P}_k \subset \ldots\ ,$$

thus forming an *infinite flag (filtration)* $\mathcal{P}$. Hence one can say that the operator $h_{\rm Cal}$ preserves the flag $\mathcal{P}$. Another property of $h_{\rm Cal}$ is the existence of a hidden gl_{n-1} algebra. The Hamiltonian $h_{\rm Cal}$ can be written as a second degree polynomial in generators of the gl_n algebra in the totally symmetric representation $(k,0,0,\ldots,0)$,

$$\begin{aligned}
\mathcal{J}_i^- &= \frac{\partial}{\partial \tau_i}, \qquad\qquad i=2,3\ldots n\ ,\\
\mathcal{J}_{ij}^0 &= \tau_i\,\frac{\partial}{\partial \tau_j}, \qquad\quad i,j=2,3\ldots n\ ,\\
\mathcal{J}^{\,0} &= \sum_{i=2}^{n}\tau_i\,\frac{\partial}{\partial \tau_i} - k\,,\\
\mathcal{J}_i^+ &= \tau_i\mathcal{J}^0 = x_i\Big(\sum_{j=2}^{n}\tau_j\,\frac{\partial}{\partial \tau_j} - k\Big), \quad i=2,3\ldots n\ ,
\end{aligned} \tag{12}$$

in such a way that the generators $\mathcal{J}_i^+$ do not appear. It is worth mentioning that for integer k, n^2 the generators (12) possess a common finite-dimensional invariant subspace $\mathcal{P}_k$. This is nothing but a finite-dimensional irreducible representation space of the algebra gl_k in the realization (12). Therefore the flag $\mathcal{P}$ is made out of irreducible finite-dimensional representation spaces of the algebra gl_n taken in the realization (12). It is worth emphasizing that the perturbation potential is itself an element of the representation spaces (11),

$$v_p \in \mathcal{P}_k\ , \quad k=2,3,\ldots\ . \tag{13}$$

2.1. *Perturbation theory (generalities)*

Now let us consider the spectral problem for the operator $h_{\rm A}$,

$$h_{\rm A}\phi = 2\,\epsilon\,\phi\ . \tag{14}$$

The spectral parameter ϵ is related to the energies E of (2) by

$$E \;=\; E_0^{(c)} + \epsilon \ ,$$

and ϕ is related to the eigenfunction of (2) through

$$\Psi(x) = \phi(x)\Psi_0^{(c)}(x) \ ,$$

where $\Psi_0^{(c)}(x)$ is the ground state eigenfunction of the Calogero Hamiltonian (2) at $\lambda = 0$. We develop perturbation theory for the equation (14) in powers of λ,

$$\phi = \sum_{k=0}^{\infty} \phi_k \lambda^k \ , \quad \epsilon = \sum_{k=0}^{\infty} \epsilon_k \lambda^k \ , \tag{15}$$

which is in fact the Dalgarno-Lewis form of perturbation theory [8]. It is easy to derive an equation to find the kth correction

$$(h_{\rm Cal} - 2\epsilon_0)\phi_k = 2\sum_{i=1}^{k} \epsilon_i\, \phi_{k-i} - v_p\, \phi_{k-1} \ . \tag{16}$$

Following the theorem from [9], as a consequence of the property (13) the perturbation theory (15) is algebraic – all perturbation corrections ϕ_k are polynomials in τ's of finite degree. Hence the construction of perturbation theory is a linear algebraic procedure.

2.2. *Perturbation theory (concrete results)*

2.2.1. *Ground state*

The ground state of the gauge-rotated Calogero Hamiltonian $h_{\rm Cal}$ is (see (9))

$$\phi_0 = 1 \ , \quad \epsilon_0 = 0 \ . \tag{17}$$

A simple analysis of equation (16) shows that the first eigenfunction correction $\phi_1 \in \mathcal{P}_2$, hence it should be a second degree polynomial in the τ's. After substitution of such an Ansatz into (16) simple calculations give

$$\epsilon_1 = \frac{n(n-1)(1+\nu n)[6-\nu(6-5n)]}{4(n+6)\omega^2} \ ,$$

$$\phi_1 = \frac{n}{2(n+6)\omega}\,\tau_4 - \frac{1}{4\omega}\,\tau_2^2 + \frac{n[6-\nu(6-5n)]}{4(n+6)\omega^2}\,\tau_2 \ . \tag{18}$$

It is worth mentioning that the correction ϕ_1 depends on two τ variables only, $\tau_{2,4}$.

A similar analysis of equation (16) shows that the second eigenfunction correction $\phi_2 \in \mathcal{P}_4$, hence it should be a fourth degree polynomial in the τ's. After substitution of such an Anzatz into (16) simple calculations give for the second correction

$$\epsilon_2 = -\frac{n(n-1)(1+\nu n)}{16(n+6)^2\omega^5}\big[150n + 36 + \nu(5n-6)(49n+6) \\ + \nu^2 n(101n^2 - 245n + 150)\big]\,, \tag{19}$$

$$(n+6)^2\phi_2 = \bigg[-\frac{n^2}{4\omega^3}\,\tau_6 + \frac{n^2}{4\omega^2}\,\tau_4^2 - \frac{n(n+6)}{4\omega^2}\,\tau_4\tau_2^2 \\ + \frac{n(7n+8-6\nu n+5\nu n^2)}{4\omega^3}\,\tau_4\tau_2 - \frac{n[19n+6+\nu n(14n-19)]}{8\omega^4}\,\tau_4 \\ + \frac{n(n-3)}{8\omega^3}\,\tau_3^2 + \frac{(n+6)^2}{16\omega^2}\,\tau_2^4 - \frac{(n+6)[4(5n+3)+3\nu n(5n-6)]}{24\omega^3}\,\tau_2^3 \\ + \frac{55n^2+120n+36+\nu n(74n^2+5n-114)+\nu^2 n^2(5n-6)^2}{16\omega^4}\,\tau_2^2 \\ - \frac{n[150n+36+\nu(49n+6)(5n-6)+\nu^2 n(101n^2-245n+150)]}{16\omega^5}\,\tau_2\bigg]\,.$$

It is worth mentioning that the correction ϕ_2 depends on four τ variables only, $\tau_{2,3,4,6}$.

A similar analysis of equation (16) shows that the third eigenfunction correction $\phi_3 \in \mathcal{P}_6$, hence it should be a sixth degree polynomial in the τ's. After substitution of such an Anzatz into (16) simple calculations give for the third energy correction

$$\epsilon_3 = \frac{n(n-1)(1+\nu n)}{32(n+6)^3\omega^8}\big[18\,(36+144n+215n^2) \\ -9\nu\,(72+504n+772n^2-1033n^3) \\ +n\nu^2(2592+6948n-16524n^2-7529n^3) \\ -n^2\nu^3(3870-9297n+7529n^2-2052n^3)\big]\,. \tag{20}$$

It is worth mentioning that the correction ϕ_3 depends on six τ variables only, $\tau_{2,3,4,5,6,8}$.

It can be shown that the kth correction to the eigenfunction $\phi_k \in \mathcal{P}_{2k}$ for $2k+2 \le n$; hence it should be a $2k$th degree polynomial in the τ's. It takes the form

$$\phi_k \;=\; Pol_{2k}(\tau_2, \tau_3, \ldots \tau_{2k+2}) \tag{21}$$

and depends on $2k$ of the τ variables only, $\tau_{2,3,4,\ldots,2k,2k+2}$. In general, only when $2k+2 \geq n$ does the kth correction begin to depend on all n of the τ variables. Hence the first corrections (which are important in practice) contain very few τ's independently of n.

The first three energy corrections $\epsilon_{1,2,3}$ have a quite non-trivial property – they vanish at non-physical values of $n = 0, 1, -1/\nu$. It seems quite natural to conjecture that the correction of arbitrary order will continue to have this property so that

$$\epsilon_k = -\frac{n(n-1)(1+\nu n)}{(n+6)^k \omega^{3k-1}} \, \tilde{\epsilon}_k(n,\nu) \ . \tag{22}$$

Most likely there exist some physical reasons explaining this property, but the present author is not aware of them.

It is worth mentioning that there is no doubt that the present perturbation theory (15) is divergent. The coefficients ϵ_k should grow factorially with k. However, it is not clear how to calculate the index of divergence.

If $g = 0$ in (2), the singular term in the potential disappears and the formulae for corrections simplify. This happens when $\nu = 0, 1$ (see (4)) for which

- at $\nu = 0$,

$$\epsilon_1 = \frac{3n(n-1)}{2(n+6)\omega^2} \ ,$$

$$\epsilon_2 = -\frac{3n(n-1)(25n+6)}{8(n+6)^2\omega^5} \ ,$$

$$\epsilon_3 = \frac{9n(n-1)(215n^2+144n+36)}{16(n+6)^3\omega^8} \ ,$$

- at $\nu = 1$,

$$\epsilon_1 = \frac{5n^2(n^2-1)}{4(n+6)\omega^2} \ ,$$

$$\epsilon_2 = -\frac{n^2(n^2-1)(101n^2+36)}{16(n+6)^2\omega^5} \ ,$$

$$\epsilon_3 = \frac{n^2(n^2-1)(324+1035n^2+1026n^4)}{16(n+6)^3\omega^8} \ .$$

It is interesting to point out that at $\nu = 1$ the numerators of $\epsilon_{1,2,3}$ depend on n^2. This could be a general feature of arbitrary corrections.

For the two-body case, $n = 2$, the problem is reduced to a standard one-dimensional anharmonic oscillator where $\nu = 0$ and $\nu = 1$ cases correspond to the ground state and the first excited state, respectively. Explicitly the corrections are,

- at $\nu = 0$,

$$\epsilon_1 = \frac{3}{8\omega^2}\ , \quad \epsilon_2 = -\frac{21}{32\omega^5}\ , \quad \epsilon_3 = \frac{333}{128\omega^8}\ ,$$

- at $\nu = 1$,

$$\epsilon_1 = \frac{15}{8\omega^2}\ , \quad \epsilon_2 = -\frac{165}{32\omega^5}\ , \quad \epsilon_3 = \frac{3915}{64\omega^8}\ ,$$

in agreement with the results of the calculation carried out in [3] and [10].

2.2.2. *First excited state*

The first excited state of the gauge-rotated Calogero Hamiltonian $h_{\rm Cal}$ (see (9)) is characterized by,

$$\phi_0 = \tau_2 + \frac{(n-1)(1+\nu n)}{4\omega}\ , \quad \epsilon_0 = 2\omega\ . \tag{23}$$

A simple analysis of equation (16) shows that the first eigenfunction correction $\phi_1 \in \mathcal{P}_3$, hence it should be a third degree polynomial in τ's. After substitution of such an Anzatz into (16) simple calculations give

$$\epsilon_1 = \frac{n[6+\nu(5n-6)][n+11+\nu n(n-1)]}{4(n+6)\omega^2}\ , \tag{24}$$

$$\begin{aligned}\phi_1 = {} & \frac{n}{2(n+6)\omega}\,\tau_4\tau_2 + \frac{n[n-9+\nu n(n-1)]}{8(n+6)\omega^2}\,\tau_4 - \frac{1}{4\omega}\,\tau_2^3 \\ & - \frac{n^2-27n-54+\nu n(n^2-15n+18)}{16(n+6)\omega^2}\,\tau_2^2 \\ & + \frac{n[6+\nu(5n-6)][n+11+\nu n(n-1)]}{16(n+6)\omega^3}\,\tau_2\ .\end{aligned}$$

It is worth mentioning that the correction ϕ_1 depends on two τ variables only, $\tau_{2,4}$.

A similar analysis of equation (16) shows that the second eigenfunction correction $\phi_2 \in \mathcal{P}_5$, hence it should be a fifth degree polynomial in τ's.

After substitution of such an Anzatz into (16) simple calculations give the following results for the second correction

$$\epsilon_2 = \frac{1}{32(n+6)^2\omega^5}\big[-3n^4 + 342n^3 + 6471n^2 + 5574n - 360 \\ - 2\nu n(3n^4 - 437n^3 - 4564n^2 + 3666n + 2916) \\ - \nu^2 n^2(3n^4 - 734n^3 - 2421n^2 + 9092n - 6324) \\ + 2\nu^3 n^3(n-1)(101n^2 - 245n + 150)\big] \ .$$

We will not present the explicit form of ϕ_2 due to its complexity. It is worth mentioning that the correction ϕ_2 depends on four τ variables only, $\tau_{2,3,4,6}$ as happens for the ground state (see (19)). Neither ϵ_1 nor ϵ_2 vanish simultaneously for some value of n. It can be shown that the kth eigenfunction correction has the property $\phi_k \in \mathcal{P}_{2k+1}$.

If $g = 0$ in (2), the singular term in the potential disappears and the formulae for corrections simplify. This happens when $\nu = 0, 1$ (see (4)) for which

- at $\nu = 0$, $\phi_0 = \tau_2 + (n-1)/4\omega$,

$$\epsilon_1 = \frac{3n(n+11)}{2(n+6)\omega^2} \ ,$$

$$\epsilon_2 = \frac{3(n^4 - 114n^3 - 2157n^2 - 1858n + 120)}{32(n+6)^2\omega^5} \ ,$$

- at $\nu = 1$, $\phi_0 = \tau_2 + (n^2-1)/4\omega$,

$$\epsilon_1 = \frac{5n^2(n^2+11)}{4(n+6)\omega^2} \ ,$$

$$\epsilon_2 = -\frac{199n^6 + 36n^5 + 4082n^4 + 78n^3 + 5463n^2 - 258n - 360}{32(n+6)^2\omega^5} \ .$$

For the two-body case, $n = 2$ the problem is reduced to a standard one-dimensional anharmonic oscillator where $\nu = 0$ and $\nu = 1$ cases correspond to the second and third excited states, respectively. Explicitly the corrections are,

- at $\nu = 0$,

$$\epsilon_1 = \frac{39}{8\omega^2} \ , \quad \epsilon_2 = -\frac{615}{32\omega^5} \ ,$$

- at $\nu = 1$,

$$\epsilon_1 = \frac{75}{8\omega^2}\ , \quad \epsilon_2 = -\frac{1575}{32\omega^5}\ ,$$

in agreement with the results of the calculation carried out in [3] and [10].

2.3. *Correlation functions and perturbation theory*

By purely algebraic means we calculated the first correction to the ground state energy (18). Making a comparison of this result with a formula for the first correction in the Rayleigh-Schroedinger perturbation theory we find [9] that we have calculated the expectation value

$$\epsilon_1 \ = \ \frac{\langle 0|v_p|0\rangle}{\langle 0|0\rangle} = \frac{n(n-1)(1+\nu n)[6-\nu(6-5n)]}{4(n+6)\omega^2}\ .$$

This expectation value is a rational function of the parameters ω, n, ν

$$\frac{\displaystyle\int_{D_c} \sum_{i<j}^{n} (y_i-y_j)^4 \prod_{i<j}^{n} |y_i-y_j|^{2\nu} e^{-\frac{\omega}{2n}\sum (y_i-y_j)^2} d^{n-1}y}{\displaystyle\int_{D_c} \prod_{i<j} |y_i-y_j|^{2\nu} e^{-\frac{\omega}{2n}\sum (y_i-y_j)^2} d^{n-1}y} =$$

$$= \ \frac{n(n-1)(1+\nu n)[6-\nu(6-5n)]}{4(n+6)\omega^2}\ ,$$

where $-y_n = \sum_{i=1}^{n-1} y_i$ and the domain of integration is the Weyl chamber. It is quite amazing that although each integral is a complicated combination of Euler Γ-functions, their ratio reduces to the rational function.

2.4. *Variational study*

We consider a strong coupling limit $\lambda \to \infty$ in (2), which is equivalent to putting $\omega = g = 0$. Following the recipe for choice of the trial functions (see e.g. [10]), the simplest trial function for the ground state can be written as

$$\Psi_{trial} \ = \ e^{-\alpha(-\tau_2)-\frac{2}{3}\beta(-\tau_2)^{3/2}-\gamma(a^2+\tau_3^2)^{1/2}-\delta(\tau_2(\frac{2n}{n+6}\tau_4-\tau_2^2))^{1/2}}\ , \qquad (25)$$

where $\alpha, \beta, \gamma, \delta$ and a are variational parameters. From dimensional arguments it seems clear that the ground state energy should be of the form

$$E_n \ = \ f(n)\lambda^{\frac{1}{3}}\ . \qquad (26)$$

For two- and three-body cases the result of calculations is

$$f(2) = 0.53042 \quad (\alpha = 0.837,\ \beta = 0.837,\ a = \gamma = \delta = 0)\ , \qquad (27)$$

$$f(3) = 1.17273 \quad (\alpha = 0.914,\ \beta = 0.845,\ a = \gamma = \delta = 0)\ . \tag{28}$$

For the two-body case one can make a comparison with the best numerical studies, $f(2)_{numerics} = 0.530362...$ (see e.g. [10]). Hence the simple trial function reproduces four significant digits in the energy.

3. Anharmonic Wolfes Model

Let us introduce the Hamiltonian which describes a system of three identical particles with two- and three-body interactions

$$\mathcal{H}_{\rm G} = \frac{1}{2}\sum_{k=1}^{3}\left[-\frac{\partial^2}{\partial x_k^2} + \omega^2 x_k^2\right] + g\sum_{k<l}^{3}\frac{1}{(x_k-x_l)^2} + 3g_1\sum_{k<l,\,k,l\neq m}^{3}\frac{1}{(x_k+x_l-2x_m)^2} + \frac{\lambda}{36}\sum_{k<l}^{3}(x_k-x_l)^4\ , \tag{29}$$

where ω is the parameter, $g = \nu(\nu-1) > -\frac{1}{4}$ and $g_1 = \mu(\mu-1) > -\frac{1}{4}$ are the coupling constants associated with the two-body and three-body interactions, respectively, $\lambda \geq 0$ is an anharmonic coupling constant and the factor $1/36$ is introduced for convenience. We call this system the G_2 *anharmonic oscillator.*

At $\lambda = 0$ the Hamiltonian (29) becomes the Hamiltonian of the rational G_2 model which was introduced for the first time by Wolfes [2] and later obtained in the Hamiltonian Reduction method [11, 12]. Its ground state is given by

$$\Psi_0^{(\rm r)}(x) = (\Delta_1^{(r)}(x))^{\nu}(\Delta_2^{(r)}(x))^{\mu}e^{-\frac{1}{2}\omega\sum_{k=1}^3 x_k^2}\ , \quad E_0 = \frac{3}{2}\,\omega(1+2\nu+2\mu)\ , \tag{30}$$

where $\Delta_1^{(r)}(x) = \prod_{i<j}^3 |x_i - x_j|$ and $\Delta_2^{(r)}(x) = \prod_{i<j;\ i,j\neq k}^3 |x_i + x_j - 2x_k|$.

An interesting observation is that all fourth order permutationally symmetric and translation invariant polynomials correspond to two body interactions because

$$(x_1+x_2-2x_3)^4 + (x_1+x_3-2x_2)^4 + (x_2+x_3-2x_1)^4 = \\ = 9\left[(x_1-x_2)^4 + (x_1-x_3)^4 + (x_2-x_3)^4\right]\ ,$$

and

$$(x_1-x_2)^2(x_1-x_3)^2 + (x_1-x_2)^2(x_2-x_3)^2 + (x_1-x_3)^2(x_2-x_3)^2 = \\ = 1/2\left[(x_1-x_2)^4 + (x_1-x_3)^4 + (x_2-x_3)^4\right]\ .$$

This leads to the important conclusion that a general fourth degree polynomial translation invariant potential reduces to two body interactions. Therefore, the Hamiltonian (29) describes the most general permutationally symmetric and translationally invariant anharmonic oscillator associated with the G_2 rational model with fourth order polynomial anharmonicity.

Let us make a gauge rotation of the Hamiltonian (29) with the ground state eigenfunction (30),

$$h_{G_2} = 2(\Psi_0^{(r)}(x))^{-1}(\mathcal{H}_{G_2} - E_0)\Psi_0^{(r)}(x) \ . \tag{31}$$

The result can be written in terms of two relative coordinates and the center-of-mass coordinate X.

Now let us take the Perelomov relative coordinates (5) and introduce new permutationally symmetric relative coordinates,

$$\lambda_1 = y_1^2 + y_2^2 + y_3^2 \ , \quad \lambda_2 = y_1^2 y_2^2 y_3^2 \ , \tag{32}$$

with the condition $-y_3 = y_1 + y_2$ (cf. (6)). Making in (31) a change of variables

$$(x_1, x_2, x_3) \to (Y, \lambda_1, \lambda_2)$$

and separating the center-of-mass motion (and then omitting it), the remaining part of the Hamiltonian (31) takes the form

$$\begin{aligned} h_{\mathrm{G}_2} &= -4\lambda_1 \partial^2_{\lambda_1\lambda_1} - 24\lambda_2 \partial^2_{\lambda_1\lambda_2} - 18\lambda_1^2\lambda_2 \partial^2_{\lambda_2\lambda_2} \\ &\quad + \{4\omega\lambda_1 - 4[1+3(\mu+\nu)]\}\,\partial_{\lambda_1} + \left[12\omega\lambda_2 - 9(1+2\nu)\lambda_1^2\right]\partial_{\lambda_2} \ + \lambda\lambda_1^2 \ . \end{aligned} \tag{33}$$

This is the *algebraic* form of the G_2 anharmonic model (cf. (9) at $n = 3$). This Hamiltonian possesses a remarkable property – among eigenfunctions there exists a family which depends on the variable λ_1 only (!). The ground state belongs to this family. In order to find the eigenfunctions depending on λ_1 only it is necessary to solve the spectral problem for the operator

$$\tilde{h}_G \ = \ -4\lambda_1 \partial^2_{\lambda_1\lambda_1} + \{4\omega\lambda_1 - 4[1+3(\mu+\nu)]\}\,\partial_{\lambda_1} + \lambda\lambda_1^2 \ . \tag{34}$$

By making a gauge rotation the operator (34) can be reduced to the two-body Hamiltonian

$$\mathcal{H} = \underbrace{-\frac{1}{2}\sum_{i=1}^{2}\frac{\partial^2}{\partial x_i^2} + \frac{\omega^2}{2}(x_1-x_2)^2 + \frac{[\,9\,(\mu+\nu)^2 - 1/4\,]}{(x_1-x_2)^2}}_{A_1-\text{rational model}} + \frac{\lambda}{8}(x_1-x_2)^4 \ .$$

Similarly to what was done for the A_n anharmonic many-body oscillator in Section 1.2, one can develop perturbation theory in powers of λ for the Hamiltonian (29) taken in the algebraic form (34).

4. Conclusion

We introduced an anharmonic perturbation of two completely-integrable and exactly-solvable systems, which are in fact anharmonic many-body oscillators. It is not clear that these systems remain integrable or whether the anharmonic terms break this feature. However, the calculation of perturbation corrections is not influenced by existence or non-existence of integrability. Perhaps, it is interesting for the A_n-anharmonic oscillator to study the limit $n \to \infty$ and a field-theoretic limit. Another interesting question is about the existence of the quasi-exactly-solvable anharmonic generalizations of the Calogero and Wolfes models other than those found in [13].

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