SPINNING STRINGS AND AdS/CFT DUALITY

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We review a special class of semiclassical string states in $AdS_5 \times S^5$ which have a regular expansion of their energy in integer powers of the ratio of the square of the string tension ('t Hooft coupling) and the square of the large angular momentum in S^5. They allow one to check AdS/CFT duality quantitatively for states in the non-supersymmetric sector and also help to uncover the role of integrable structures on both sides of the string theory – gauge theory duality.

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1. Introduction

Better understanding of the duality between type IIB superstring theory in $AdS_5 \times S^5$ and the planar limit of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory [1] and extending it to less supersymmetric cases may allow us to find simple string-theoretic descriptions of various dynamical aspects of gauge theories, from high-energy scattering to confinement. This AdS/CFT duality is usually viewed as an example of strong coupling – weak coupling duality: while the large N perturbative expansion in SYM theory assumes that the 't Hooft coupling $\lambda = g^2_{\rm YM} N$ is small, the string perturbative (inverse tension) expansion applies for $\sqrt{\lambda} = \frac{R^2}{\alpha'} \gg 1$. In general, observables (such as scaling dimensions, correlation functions, finite temperature free energy, etc.) depend on λ through non-trivial functions $f(\lambda)$ of the couplings, with the perturbative SYM and string theories describing opposite asymptotic regions. For special "protected" BPS observables the dependence on λ may become trivial due to supersymmetry and then can be directly reproduced on the two sides of the duality [2].

Checking duality beyond the BPS cases remains a challenge. The Green–Schwarz superstring action in $AdS_5 \times S^5$ appears to have a complicated structure [3], so finding, e.g., its full spectrum exactly in λ seems hard at present. One may hope to bypass the computation of non-trivial functions $f(\lambda)$ by considering special limits involving other parameters or quantum numbers besides λ. Specific progress can be made by concentrating on a particular (and basic) class of observables which should be related according to AdS/CFT: a spectrum of energies of single-string states (in global $AdS_5 \times S^5$ coordinates) and scaling dimensions of the corresponding single-trace gauge-invariant local operators in SYM theory. These may carry quantum numbers such as $SO(2,4) \times SO(6)$ spins of string states or powers of scalar fields and covariant derivatives in the SYM operators. A remarkable recent development initiated in [4] (which in turn was inspired by [5,6]) is based on the idea that for a special subset of string/SYM states parametrized by large quantum numbers [7,8] there may be new interesting limits in which certain quantum corrections may be suppressed. One may then be able to check the AdS/CFT correspondence for such special non-supersymmetric states by comparing the corresponding string energies with the perturbative gauge theory scaling dimensions.

In the BMN case [4] (see [9] for reviews) one concentrates on a particular "semiclassical" [8] sector of near-BPS states represented by small closed strings with center of mass moving along a large circle of S^5 with angular momentum $J \gg 1$. The SYM operators are of the type $\mathrm{tr}(Z^J ...)$ where

$Z = \Phi_5 + i\Phi_6$, Φ_M are $SO(6)$ scalars and dots stand for a small number of other SYM fields or covariant derivatives. By considering the limit $J \to \infty$, $\frac{\lambda}{J^2}$ =fixed one is able to establish a precise correspondence between the energies of the string states and the scaling dimensions of the corresponding SYM operators [4, 10] (for a complete list of references see [9]). The reason why this is possible can be understood by interpreting this sector of states as "semiclassical" states [8] corresponding to quadratic fluctuations near a point-like string running along a geodesic in S^5 with angular velocity $w = \frac{J}{\sqrt{\lambda}}$. One is then able to argue [11, 12] that higher than one-loop string sigma model corrections to the leading ("quadratic" or "plane-wave") string energies are suppressed in the limit $J \to \infty$ with w held fixed.

One may hope to apply similar reasoning to other, far from BPS, semiclassical sectors of string states. For example, considering a string rotating with large spin S in AdS_5 one discovers [8] a new *qualitative* test of AdS/CFT: the agreement between the dependence of the string energy E on large spin S and the spin dependence of the anomalous dimension of twist 2 gauge-invariant SYM operators: $E = S + f(\lambda) \ln S + ...$ Here $f(\lambda) = b_0\sqrt{\lambda} + b_1 + \frac{b_2}{\sqrt{\lambda}} + ...$ on the perturbative string side and $f(\lambda) = a_1\lambda + a_2\lambda^2 + ...$ on the perturbative gauge theory side [8, 11, 13]. According to the AdS/CFT duality the two expansions must represent different asymptotics of the same function. Checking this in a precise manner is obviously hard since that would require first finding all terms in the respective perturbative series and then resumming them.

For other semiclassical string states one might expect to find similar "interpolation in λ" patterns, which again preclude a direct *quantitative* comparison with perturbative SYM theory. Remarkably, as was noticed in [14, 15], there are exceptions: for certain multispin string states (with at least one large S^5 spin component J) the classical energy has a *regular* expansion in $\frac{\lambda}{J^2}$ while quantum superstring sigma model corrections are suppressed in the limit $J \to \infty$, $\frac{\lambda}{J^2}$ =fixed. It was proposed [14] that for such states one can carry out the precise test of the AdS/CFT duality in a non-BPS sector by comparing the $\frac{\lambda}{J^2} \ll 1$ expansion of the *classical* string energy with the corresponding *quantum* anomalous dimensions in perturbative SYM theory.

This was indeed successfully accomplished in a series of recent papers [16–21]. The main technical problem – how to find the eigenvalues of the anomalous dimension matrix for "long" (large J) scalar operators – was solved (at the one-loop level) using the interpretation of the anomalous dimension matrix as an integrable spin-chain Hamiltonian [22, 23].* This allowed one to find the one-loop anomalous dimensions by applying Bethe ansatz techniques. The leading order $\frac{\lambda}{J}$ terms in the energies of particular string solutions were then reproduced as one-loop anomalous dimensions on the SYM side by choosing particular Bethe root distributions in the "thermodynamic" limit of "long" ($J \to \infty$) operators. There is some evidence [19] that the correspondence extends, as one of course expects, to the next $\frac{\lambda^2}{J^3}$ order, but checking this explicitly and going beyond the two-loop level remains an important open problem.

Our aim here will be to review a class of such classical string solutions in $AdS_5 \times S^5$ [11, 14, 15, 17, 18, 34] whose energy E has a regular expansion in integer powers of λ (i.e. the square of the effective string tension) divided by the square of the total S^5 spin J, and for which quantum sigma model corrections to E should be suppressed in the $J \to \infty$ limit.

Let us first make some general comments on the structure of this semi-classical expansion for the string energy. The form of a classical solution cannot depend on the value of the string tension, i.e. on $\sqrt{\lambda}$, which appears as a factor in front the string action $I = \frac{\sqrt{\lambda}}{4\pi} \int d^2\xi \ G_{\mu\nu}(x)\partial_a X^\mu \partial^a X^\nu + ...$ Thus the classical energy can be written as $E = \sqrt{\lambda}\ \mathcal{E}(w)$, where w stands for all constant parameters that enter the classical solution. These param-

* The integrable spin chain connection was uncovered and extensively studied previously in the context of QCD. In particular, the Regge-like asymptotic behavior of scattering amplitudes was described by evolution equations that were related to the $SL(2, C)$ Heisenberg spin chain [24]. More importantly for the present discussion, the one-loop anomalous dimensions of certain (quasi-partonic) composite operators were related to the energies of the $SL(2, R)$ XXX Heisenberg spin chain [25, 26]. Similar relations hold in other asymptotically free gauge theories, in particular, supersymmetric theories [27, 28]. The role of conformal symmetry in QCD and these integrability relations were reviewed in [29]. More recent work relating integrability of light-cone QCD operators to gauge/string duality appeared in [30–32]. In $\mathcal{N}$=4 SYM theory viewed as a particular gauge theory with adjoint matter the above QCD-inspired work implies that the (one-loop, large N) anomalous dimension matrix for the minimal-twist operators (such as $\mathrm{tr}(\bar\Phi D^S \Phi) + ...$, $D = D_0 + D_3$) should be the same as the Hamiltonian of the $SL(2, R)$ XXX spin chain. Independently, it was observed in [22] that the one-loop planar anomalous dimension matrix in the pure-scalar sector of operators $\mathrm{tr}(\Phi_{M_1} ... \Phi_{M_J})$ can be interpreted as the Hamiltonian of an integrable $SO(6)$ spin chain. Ref. [23] generalized these facts to all superconformal operators to claim that the complete one-loop planar dilatation operator of $\mathcal{N}$=4 SYM is equivalent to the Hamiltonian of an integrable $SU(2, 2|4)$ (super) spin chain. More recent work [33] addressed the same problem using the original (light-cone operator) QCD approach, i.e. considering the sub-sector of supermultiplets of quasipartonic operators ($\mathrm{tr}(D^{s_1}\Phi_{M_1} ... D^{s_n}\Phi_{M_n}) + ...$, etc.) with the conclusion that in this case the one-loop dilatation operator coincides with the Hamiltonian of an $SL(2|4)$ spin chain. The relation between the approaches of [23] and [33] and also whether the $SL(2)$ integrability in the twist 2 sector may be somehow related by supersymmetry to $SO(6)$ integrability in the pure-scalar sector seems worth clarifying further.

eters should be fixed in the standard sigma model loop ($\alpha' \sim \frac{1}{\sqrt{\lambda}}$) expansion. The classical values of the integrals of motion such as the S^5 and AdS_5 angular momentum components are also proportional to the string tension, e.g., $J = \sqrt{\lambda}\ \mathcal{J}(w)$ (they take integer values in the full quantum theory). Expressed in terms of these integrals the classical energy is $E = \sqrt{\lambda}\ \mathcal{E}(\mathcal{J}) = \sqrt{\lambda}\ \mathcal{E}(\frac{J}{\sqrt{\lambda}})$.

In the limit of large values of semiclassical parameters and the corresponding quantum charges the classical energy of a string solution in any $AdS_p \times S^m$ space goes as *linear* function of J, i.e. $E = J + ...$ This linear behavior [35] (seen explicitly on examples of particular solutions [8,11,36–38]) is different from the flat-space Regge one $E \sim \sqrt{J}$ and is a consequence of the constant curvature of AdS space. This is consistent with AdS/CFT duality: one expects that the large J expression for the full dimension of the corresponding gauge theory operator should start with its canonical dimension.

We would like to identify a class of special classical string solutions in $AdS_5 \times S^5$ whose energy has a particular dependence on conserved charges that allows for a direct comparison with anomalous dimensions on the perturbative SYM side. While such extended string solutions turn out to have *several* conserved global charges, here for notational simplicity we shall keep track of just one of them – the total S^5 angular momentum $J = \sqrt{\lambda}\ \mathcal{J}$. For the solutions we will be interested in the classical energy $E = \sqrt{\lambda}\ \mathcal{E}$ should have the following expansion in large classical parameter $\mathcal{J} \gg 1$

$$\mathcal{E} = \mathcal{J}\Big(1 + \frac{c_1}{\mathcal{J}^2} + \frac{c_2}{\mathcal{J}^4} + ...\Big), \tag{1.1}$$

i.e. $\frac{\mathcal{E}}{\mathcal{J}}$ should have an expansion in *even* inverse powers of $\mathcal{J}$. The coefficients c_i may be functions of ratios of conserved charges that are finite in the large-charge limit. Equivalently, for $\frac{1}{\mathcal{J}} = \frac{\sqrt{\lambda}}{J} \ll 1$

$$E = \ J\ \Big(1 + \frac{c_1\lambda}{J^2} + \frac{c_2\lambda^2}{J^4} + ...\Big) = J + \frac{c_1\lambda}{J} + \frac{c_2\lambda^2}{J^3} + ...\ , \tag{1.2}$$

which formally looks like an expansion in positive integer powers of λ. Rotating string solutions with this property were indeed found in [11,14,17,18,34] and will be reviewed below.

Furthermore, let us assume that in such cases the string sigma model loop corrections to the energy which in general can be computed in the standard inverse tension expansion

$$E_{\rm tot} = \sqrt{\lambda}\, \mathcal{E}(\mathcal{J}) + \mathcal{E}_1(\mathcal{J}) + \frac{1}{\sqrt{\lambda}}\, \mathcal{E}_2(\mathcal{J}) + \frac{1}{(\sqrt{\lambda}\,)^2}\, \mathcal{E}_3(\mathcal{J}) + ... = E + \sum_{n=1}^{\infty} E_n \tag{1.3}$$

should have the following specific form of their expansion in $\mathcal{J} \gg 1$

$$\mathcal{E}_n(\mathcal{J}) = \frac{d_{n1}}{\mathcal{J}^{n+1}} + \frac{d_{n2}}{\mathcal{J}^{n+3}} + ... \ , \qquad n = 1, 2, ... \ . \tag{1.4}$$

This behavior was verified in [15] for $n = 1$ on a particular example of a solution [14] satisfying (1.2). Equation (1.4) implies that the n-loop term in the quantum-corrected energy (1.3) will be given, for $\frac{\lambda}{J^2} \ll 1$, by

$$E_n = \frac{1}{(\sqrt{\lambda}\,)^{n-1}}\, \mathcal{E}_n(\mathcal{J}) = \frac{d_{n1}\lambda}{J^{n+1}} + \frac{d_{n2}\lambda^2}{J^{n+3}} + ... \ . \tag{1.5}$$

In general, the energy $E_{\rm tot} = E_{\rm tot}(J, \lambda)$ should be some function of J and the string tension but if the above assumptions (1.2) and (1.5) are true, it will be given by the following *double expansion* in $\frac{\lambda}{J^2}$ *and* $\frac{1}{J}$:

$$E_{\rm tot} = J \left[1 + \sum_{k=1}^{\infty} (\frac{\lambda}{J^2})^k \left(c_k + \sum_{n=1}^{\infty} \frac{d_{nk}}{J^n} \right) \right] . \tag{1.6}$$

Then if we first take the limit of $J \gg 1$ for fixed $\frac{\lambda}{J^2}$, all quantum sigma model corrections will be suppressed and the full energy $E_{\rm tot}$ will be given just by its *classical* part E (1.2).

In the BMN case [4], where one expands [8,11] near a point-like BPS string state, the ground-state energy is not renormalized, i.e. $E_{\rm tot} = E = J$, but the double expansion similar to the one for $E_{\rm tot}/J$ in (1.6), namely, $E_{\rm fluct} = J + \sum_{k=0}^{\infty} (\frac{\lambda}{J^2})^k \big(h_k + \sum_{n=1}^{\infty} \frac{f_{nk}}{J^n} \big)$, applies to energies of string fluctuations near the geodesic, i.e. to energies of excited string modes [11, 12, 39, 40]. In the limit $J \to \infty$ their energies are then determined by the quadratic ("one-loop" or "plane-wave") approximation.

The general conditions for the validity of the expansions (1.2) and (1.5) remain to be clarified (some observations made in recent papers [41, 42] relating the large $\mathcal{J}$ limit to an ultra-relativistic limit may turn out to be useful for that). In particular, the "regularity" of the expansion of the energy in λ (1.2) may apply not only to multi-spin rotating but also to S^5 pulsating [21, 38] solutions.

In order to test AdS/CFT duality one should reproduce the same expression for the (quantum-corrected) AdS_5 string energy $E_{\rm tot}(\lambda, J)$ (1.6) as the exact scaling dimension $\Delta(\lambda, J)$ of the corresponding SYM operator, i.e.

as a particular eigenvalue of the dilatation operator having the same global charges (i.e. belonging to the same $SO(2,4) \times SO(6)$ representation as the string state). Given that (1.6) looks like an expansion in the 't Hooft coupling λ it is natural to expect that the perturbative ($\lambda \ll 1$) expansion for $\Delta(\lambda, J)$ can be organized in the following way

$$\Delta(\lambda, J) = J + \sum_{k=1}^{\infty} q_k(J)\ \lambda^k\ , \tag{1.7}$$

where the functions $q_k(J)$ should have the following form for $J \gg 1$

$$q_k(J) = \frac{1}{J^{2k-1}}\Big(a_k + \frac{a_{k1}}{J} + ...\Big)\ . \tag{1.8}$$

Assuming that this is indeed the case and taking the $J \to \infty$ limit, one could then directly compare the classical part of the energy in (1.6) expanded in $\frac{\lambda}{J^2}$ with the sum of the leading ($J \gg 1$) terms at each order of expansion of Δ in powers of λ. The AdS/CFT correspondence implies then that the two expressions should coincide, i.e. that $c_k = a_k$. The *classical* string energy should thus represent the leading $J \to \infty$ term in the *quantum* SYM scaling dimension.

In particular, the coefficient c_1 of the first subleading (order λ) term in the classical string energy (1.2) should match the coefficient a_1 in the one-loop SYM term in (1.7), (1.8). This was indeed verified on specific examples in [16–19, 21]. There is also a numerical evidence [19] that this matching extends to the λ^2 term, i.e. $c_2 = a_2$.

The $J \to \infty$ behavior (1.8) of the one-loop correction to the anomalous dimensions was checked using the spin chain relation and the Bethe ansatz for particular large R-charge or large spin operators [16, 19, 21]. The general proof of (1.8) which should follow from a higher-loop structure of the dilatation operator [44, 45] and should be heavily based on the superconformal symmetry of the $\mathcal{N} = 4$ SYM theory remains to be given.

Let us now summarize the contents of the following sections. In section 2 we shall write down the bosonic part of the superstring action in $AdS_5 \times S^5$ and the corresponding integrals of motion as a preparation for a discussion of classical finite energy closed string solutions which carry several $SO(2,4)\times SO(6)$ spin components. In section 3 we shall consider the special case of the $SO(6)$ invariant sigma model (embedded into string theory by adding time direction from AdS_5) and briefly review its integrability (local and non-local conserved charges, etc.).

Then in section 4 we shall concentrate on a particular class of semiclassical string states rotating in S^5 with three angular momenta J_i and show that for the rotating string ansatz the $R_t \times S^5$ sigma model reduces to a well-known one-dimensional integrable system – the Neumann-Rosochatius (NR) system. Its special case is the $n = 3$ Neumann system describing an oscillator on a 2-sphere. This relation allows one to classify the corresponding rotating string solutions, which, as in flat space, can be of folded or circular type.

In section 5 we shall study a simple special class of circular rotating string solutions on S^5 whose energy has a regular large-spin expansion as in (1.2). We shall also determine (in section 5.3) the spectrum of quadratic fluctuations near these circular solutions pointing out some analogies with the point-like (BMN) case. In section 5.4 we shall consider the one-loop string sigma model correction to the energy of a particular solution (with two equal spins); this one-loop correction indeed turns out to be suppressed in the large spin limit, in agreement with (1.6).

The discussion of sections 4 and 5 will be generalized in section 6 to the case of states represented by semiclassical strings rotating in both S^5 and AdS_5 and thus carrying 3+2 spin quantum numbers. They are again described by a generalized NR integrable system. While the energy of strings rotating only in AdS_5 is non-analytic in λ (section 6.1), the expansion (1.2) is true for circular strings having large S^5 spin components.

Similar conclusions apply to other multi-spin solutions of the NR system representing folded and circular strings with more complicated ("inhomogeneous") dependence on the string coordinate σ. In particular, we consider a class of two-spin solutions for which the Neumann system degenerates to a sine-Gordon one and, as a result, the solutions are expressed in terms of the elliptic functions (section 7). The classical energy can then be found as a solution of two parametric equations involving elliptic integrals and has again a regular expansion as in (1.2).

Section 8 will contain a summary of some open problems and possible generalizations, including a brief discussion of pulsating string solutions.

2. Closed superstrings in $AdS_5 \times S^5$: classical solutions

Superstrings in $AdS_5 \times S^5$ can be described by a Green–Schwarz action [3] which defines a consistent perturbation theory near each semiclassical string configuration, e.g., a point-like massless geodesic in a light-cone type gauge as in [11, 46] or extended string configurations as in [11, 14, 15, 47].

The bosonic part of the action in the conformal gauge is the sum of the two coset-space sigma models ($AdS_5 = SO(2,4)/SO(1,4)$ and

$S^5 = SO(6)/SO(5)$)

$$I = -\frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \ \left[G^{(AdS_5)}_{mn}(x) \partial_a x^m \partial^a x^n \ + \ G^{(S^5)}_{pq}(y) \partial_a y^p \partial^a y^q \right] \ . \qquad (2.1)$$

The effective string tension $T_{\rm eff} = \frac{\sqrt{\lambda}}{2\pi} = \frac{R^2}{2\pi\alpha'}$ is related to the 't Hooft coupling $\lambda = g^2_{\rm YM} N$ on the SYM side of the string/gauge theory duality [1]. The AdS_5 and S^5 parts of the action are "coupled" at the classical level through the conformal gauge constraints.

The classical conformal invariance of this sigma model is preserved at the quantum level after addition of fermions with coupling to the metric and R-R 5-form background [3]. There are quadratic and quartic fermionic terms in the action (in a particular gauge). The quadratic part of the fermionic Lagrangian can be written as (see, e.g., [3, 11, 47])

$$L_F = i \, (\eta^{ab} \delta^{IJ} - \epsilon^{ab} s^{IJ}) \, \bar{\vartheta}^I \varrho_a D_b \vartheta^J \ , \qquad \varrho_a \equiv \Gamma_A e^A_a \ , \qquad (2.2)$$

where $I, J = 1, 2$, s^{IJ} =diag(1,-1), and $\varrho_a \equiv \Gamma_A E^A_\mu \partial_a X^\mu$ are the projections of the 10-d Dirac matrices. Here X^α are the string coordinates (given functions of τ and σ for a particular classical solution) corresponding to the AdS_5 ($\mu = 0, 1, 2, 3, 4$) and S^5 ($\mu = 5, 6, 7, 8, 9$) factors. The covariant derivative D_a can be put into the form

$$D_a \vartheta^I = (\delta^{IJ} {\rm D}_a - \frac{i}{2} \, \epsilon^{IJ} \Gamma_* \varrho_a) \vartheta^J \ , \qquad \Gamma_* \equiv i \Gamma_{01234} \ , \ \ \Gamma^2_* = 1 \ , \qquad (2.3)$$

where ${\rm D}_a = \partial_a + \frac{1}{4} \, \omega^{AB}_a \Gamma_{AB}$, $\omega^{AB}_a \equiv \partial_a X^\alpha \omega^{AB}_\alpha$ and the "mass term" originates from the R-R 5-form coupling.

Here we will be interested mostly in the classical bosonic finite-energy solutions for closed strings in $AdS_5 \times S^5$ space and ignore the fermions. To study these bosonic solutions it is useful to rewrite the action (2.1) in the form

$$I = \sqrt{\lambda} \ \int d\tau \int_0^{2\pi} \frac{d\sigma}{2\pi} \ (L_{AdS} + L_S) \ , \qquad (2.4)$$

where

$$L_{AdS} = -\frac{1}{2} \, \eta^{PQ} \partial_a Y_P \partial^a Y_Q + \frac{1}{2} \, \tilde{\Lambda} (\eta^{PQ} Y_P Y_Q + 1) \, , \qquad (2.5)$$

$$L_S = -\frac{1}{2} \, \partial_a X_M \partial^a X_M + \frac{1}{2} \, \Lambda (X_M X_M - 1) \, . \qquad (2.6)$$

We use $(-+)$ signature on the world sheet and X_M, $M = 1, \ldots, 6$ and Y_P, $P = 0, \ldots, 5$ are the embedding coordinates of R^6 with the Euclidean metric δ_{MN} in L_S and with $\eta_{PQ} = (-1, +1, +1, +1, +1, -1)$ in L_{AdS}, respectively.

Λ and $\tilde{\Lambda}$ are the Lagrange multiplier functions of τ and σ. The action (2.4) is to be supplemented with the usual conformal gauge constraints expressing the vanishing of the total 2-d energy-momentum tensor

$$\eta^{PQ}(\dot{Y}_P\dot{Y}_Q + Y'_PY'_Q) + \dot{X}_M\dot{X}_M + X'_MX'_M = 0 \ , \tag{2.7}$$

$$\eta^{PQ}\dot{Y}_P{Y'}_Q + \dot{X}_MX'_M = 0 \ , \tag{2.8}$$

where

$$\eta^{PQ}Y_PY_Q = -1 \ , \qquad X_MX_M = 1 \ . \tag{2.9}$$

We shall assume that the world sheet is a cylinder, i.e. impose the closed string periodicity conditions

$$Y_P(\sigma + 2\pi) = Y_P(\sigma) \ , \qquad X_M(\sigma + 2\pi) = X_M(\sigma) \ . \tag{2.10}$$

The classical equations that follow from (2.4) can be written as

$$\partial^a\partial_aY_P - \tilde{\Lambda}Y_P = 0 \ , \quad \tilde{\Lambda} = \eta^{PQ}\partial^aY_P\partial_aY_Q \ , \quad \eta^{PQ}Y_PY_Q = -1 \ , \tag{2.11}$$

$$\partial^a\partial_aX_M + \Lambda X_M = 0 \ , \quad \Lambda = \partial^aX_M\partial_aX_M \ , \quad X_MX_M = 1 \ . \tag{2.12}$$

The action is invariant under the $O(2,4)$ and $O(6)$ global symmetries with the corresponding conserved (on-shell) charges being

$$S_{PQ} = \sqrt{\lambda}\int_0^{2\pi}\frac{d\sigma}{2\pi}\ (Y_P\dot{Y}_Q - Y_Q\dot{Y}_P) \ , \tag{2.13}$$

$$J_{MN} = \sqrt{\lambda}\int_0^{2\pi}\frac{d\sigma}{2\pi}\ (X_M\dot{X}_N - X_N\dot{X}_M) \ . \tag{2.14}$$

We are interested in finding "spinning" solutions that have non-zero values of these charges. The physical target-space interpretation of the solutions depends on a particular choice of coordinates (that solve (2.9)) in AdS_5 and S^5. One natural ("global coordinate") choice is

$$\begin{aligned}
&\mathrm{Y}_1 \equiv Y_1 + iY_2 = \sinh\rho\ \sin\theta\ e^{i\phi_1}, \quad \mathrm{Y}_2 \equiv Y_3 + iY_4 = \sinh\rho\ \cos\theta\ e^{i\phi_2}, \\
&\mathrm{Y}_0 \equiv Y_5 + iY_0 = \cosh\rho\ e^{it}, \qquad\qquad \mathrm{X}_3 \equiv X_5 + iX_6 = \cos\gamma\ e^{i\varphi_3} \ , \\
&\mathrm{X}_1 \equiv X_1 + iX_2 = \sin\gamma\ \cos\psi\ e^{i\varphi_1}, \quad \mathrm{X}_2 \equiv X_3 + iX_4 = \sin\gamma\ \sin\psi\ e^{i\varphi_2}.
\end{aligned} \tag{2.15}$$

Then there is an obvious choice of the 3+3 Cartan generators of $SO(2,4)\times SO(6)$ corresponding to the 3+3 linear isometries of the

$AdS_5 \times S^5$ metric

$$(ds^2)_{AdS_5} = d\rho^2 - \cosh^2\rho\ dt^2 + \sinh^2\rho\ (d\theta^2 + \cos^2\theta\ d\phi_1^2 + \sin^2\theta\ d\phi_2^2), \quad (2.16)$$

$$(ds^2)_{S^5} = d\gamma^2 + \cos^2\gamma\ d\varphi_3^2 + \sin^2\gamma\ (d\psi^2 + \cos^2\psi\ d\varphi_1^2 + \sin^2\psi\ d\varphi_2^2)\,, \quad (2.17)$$

i.e. to the translations in AdS_5 time t, in the two angles ϕ_a and in three angles φ_i of S^5:

$$S_0 \equiv S_{50} \equiv E = \sqrt{\lambda}\ \mathcal{E}\,, \quad S_1 \equiv S_{12} = \sqrt{\lambda}\ \mathcal{S}_1\,, \quad S_2 \equiv S_{34} = \sqrt{\lambda}\ \mathcal{S}_2\,, \quad (2.18)$$

$$J_1 \equiv J_{12} = \sqrt{\lambda}\ \mathcal{J}_1\,, \qquad J_2 \equiv J_{34} = \sqrt{\lambda}\ \mathcal{J}_2\,, \quad J_3 \equiv J_{56} = \sqrt{\lambda}\ \mathcal{J}_3\,. \quad (2.19)$$

We will be interested in classical solutions that have finite values of the target-space energy E as well as of the spins S_a, J_i. For a solution to have a consistent semiclassical approximation, i.e. to correspond to an eigenstate of the Hamiltonian which carries the corresponding quantum numbers (and thus being associated to a particular SYM operator with definite scaling dimension) all other non-Cartan (i.e. non-commuting) components of the symmetry generators (2.13), (2.14) should vanish [11].

In the above $R^{2,4}$ embedding representation of AdS_5 the charges of the isometry group $SO(2,4)$ can be related to the boundary SYM theory conformal group generators as follows ($\mu, \nu = 0,1,2,3$):

$$S_{\mu\nu} = M_{\mu\nu}\,, \quad S_{\mu 4} = \frac{1}{2}(K_\mu - P_\mu)\,, \quad S_{\mu 5} = \frac{1}{2}(K_\mu + P_\mu)\,, \quad S_{54} = D\,. \quad (2.20)$$

One can identify the standard spin with $S_1 = S_{12} = M_{12}$, the second (conformal) spin with $S_2 = S_{34} = \frac{1}{2}(K_3 - P_3)$, and finally the conformal energy with the rotation generator in the 05 plane, i.e. with the global AdS_5 energy, $E = S_{05} = \frac{1}{2}(K_0 + P_0)$.*

* The energy of a string state in global AdS_5 space with boundary $R \times S^3$ should be equal to the energy of the corresponding SYM state on $R \times S^3$ (which can be mapped conformally to R^4). Through radial quantization this state may be associated with a local operator that creates it. At the same time, the AdS_5 energy or conformal Hamiltonian generates an $SO(2)$ subgroup of $SO(2,4)$ while the dilatation operator (whose eigenvalues are scaling dimensions) generates an $SO(1,1)$ subgroup of the conformal group. Their eigenvalues happen to be the same since the two representations (the unitary one classified by $SO(4) \times SO(2)$ and the one classified by $SO(4) \times SO(1,1)$) are related by a global $SO(2,4)$ similarity transformation (see, e.g., [43]). Alternatively, after the Euclidean continuation $Y_0 \to iY_{0E}$ (to allow for the mapping from $R \times S^3$ to R^4) one may exchange Y_{0E} with Y_4 which exchanges the generator $S_{54} = D$ with $S_{05} = \frac{1}{2}(P_0 + K_0) = E$. For all the solutions discussed below $S_{50} = E \neq 0$ while $S_{54} = D = 0$. One could, in principle, apply a similar $Y_{0E} \to Y_4$ transformation to string solutions, getting equivalent ones (but more complicated-looking, with the radial direction of AdS_5 depending on τ) that would have non-zero values of the $SO(1,1)$ generator S_{54}.

Let us first consider point-like string solutions, for which $Y_P = Y_P(\tau)$, $X_M = X_M(\tau)$, i.e. massless (cf. (2.7)) geodesics in $AdS_5 \times S^5$. As follows from the second-order equations in (2.11), (2.12), in this case $\Lambda =$ const, $\tilde{\Lambda} =$ const, i.e. Y_P and X_M are given by trigonometric functions. The constraint (2.7) implies that the two frequencies are related: $\Lambda = -\tilde{\Lambda} > 0$. Then a generic massless geodesic in $AdS_5 \times S^5$ can be shown to be one of the two "irreducible" types (up to a global $SO(2,4) \times SO(6)$ transformation): (i) a massless geodesic that stays entirely within AdS_5; (ii) a geodesic that runs along the time direction in AdS_5 and wraps a big circle of S^5. In the latter case the angular motion in S^5 provides an effective ("Kaluza-Klein") mass to a particle in AdS_5, i.e. the corresponding geodesic in AdS_5 is a massive one. Then we can choose the coordinates so that

$$Y_5 + iY_0 = e^{i\kappa\tau}, \quad X_5 + iX_6 = e^{iw\tau}, \quad \kappa = w = \sqrt{\Lambda}, \quad Y_{1,2,3,4} = X_{1,2,3,4} = 0\,. \tag{2.21}$$

Here the only non-vanishing integrals of motion are $E = J_3 = \sqrt{\lambda}\ \kappa$, representing the energy and $SO(6)$ spin of this BPS state, corresponding to $\mathrm{tr}Z^{J_3}$ operator in SYM theory. More generally, one may choose any massless geodesic in AdS_5 for which then $\eta^{PQ}\dot{Y}_P\dot{Y}_Q = -w^2$. The massless limit $w \to 0$ corresponds to $J_3 \to 0$, i.e. the resulting state should be representing a vacuum in string theory or a unit operator in SYM theory [4].

The former case, i.e. the "massless" $w \to 0$ limit, is actually subtle: naively, a massless geodesic in AdS_5 does not represent a semiclassical string state in the sense defined above. Indeed, for a point-like string moving inside AdS_5 we have $\eta^{PQ}\dot{Y}_P\dot{Y}_Q = 0$, i.e. $\ddot{Y}_P = 0$. Thus in terms of the embedding coordinates the massless geodesic is a straight line

$$Y_P(\tau) = A_P + B_P\tau\ , \qquad \eta^{PQ}B_PB_Q = \eta^{PQ}A_PB_Q = 0\ , \quad \eta^{PQ}A_PA_Q = -1\ . \tag{2.22}$$

Then the $SO(2,4)$ angular momentum tensor (2.13) is $S_{PQ} = \sqrt{\lambda}\ (A_PB_Q - A_QB_P)$ and can be shown to always have non-vanishing non-Cartan components. Indeed, by applying an $SO(2,4)$ rotation we may put the constant vectors A_P and B_P in a canonical form: $A_P = (0,0,0,0,0,1)$, $B_P = (p,0,0,p,0,0)$, i.e.

$$Y_5 + iY_0 = 1 + ip\ \tau\ , \qquad Y_3 = p\ \tau\ , \qquad Y_{1,2,4} = 0\ . \tag{2.23}$$

Here p is an arbitrary parameter and $S_{50} = S_{53} = \sqrt{\lambda}\ p$. An alternative choice of the parameters (related to the above one by an $SO(2,4)$ rotation with parameter u) gives $Y_5 + iY_0 = \frac{1+u^2}{2u} + i\frac{p}{u}\tau$, $Y_1 + iY_2 = \frac{1-u^2}{2u} + i\frac{p}{u}\tau$,

$Y_{3,4} = 0$.† It corresponds to the massless geodesic running parallel to the $R^{1,3}$ boundary in the Poincare coordinates where $(ds^2)_{AdS_5} = \frac{1}{z^2}(dx^m dx_m + dz^2)$: $x_0 = x_3 = p\tau,\ z = u = \text{const}$ (see also [12]). An expansion near this geodesic is used to define the light-cone gauge in [46], i.e. it should represent a light-cone vacuum state.

Below we would like to study non-trivial (σ-dependent) solitonic solutions of classical closed string equations in $AdS_5 \times S^5$ that have finite 2-d energy and carry finite space-time energy and spins, i.e. 1+2 plus 3 commuting conserved charges of the $O(2,4) \times O(6)$ isometry group. The conformal gauge constraints will then imply a relation between the energy and the spins ($a = 1, 2;\ i = 1, 2, 3$)

$$\mathcal{E} = \mathcal{E}(\mathcal{S}_a, \mathcal{J}_i; k_p) , \qquad \text{i.e.} \qquad E = \sqrt{\lambda}\, \mathcal{E}(\frac{S_a}{\sqrt{\lambda}}, \frac{J_i}{\sqrt{\lambda}}; k_p) , \tag{2.24}$$

where k_p are "topological" numbers determining the particular type (e.g., shape) of the rotating solutions. We will be interested in solutions that have a regular dependence of E on on λ in the large spin limit as in (1.2). A necessary condition for that appears to be to have large total angular momentum in the S^5 direction. That applies to both rotating [14] and oscillating [38] solutions. Note also that rotating solutions in S^5 (but not in AdS_5) have a "nearly-BPS" interpretation [41] in the formal $\lambda \to 0$ limit.

In general, coset space sigma models are known to be integrable [48, 49]. To make this formal integrability property explicit and useful one needs to specify a class of solutions by choosing a special ansatz for string coordinates. Before discussing particular rotating strings in S^5 and AdS_5 let us first make some general comments on the corresponding classical sigma model and its conserved charges.

3. $R_t \times S^5$ sigma model: classical integrability and conserved charges

Let us consider the classical S^5 sigma model embedded into string theory by adding an extra time direction R_t. This may be viewed as a special case of the $AdS_5 \times S^5$ sigma model where the string is placed at the center $\rho = 0$ of AdS_5 while moving in S^5.

Introducing $\xi^{\pm} = \frac{1}{2}(\tau \pm \sigma)$ and $\partial_{\pm} = \partial_\tau \pm \partial_\sigma$ the corresponding equations

† In this case in addition to the Cartan components $E = S_{50} = \sqrt{\lambda}\, \frac{1+u^2}{2u} p$ and $S_{12} = -\sqrt{\lambda}\, \frac{1-u^2}{2u} p$ we also have nonvanishing S_{01} and S_{25}.

of motion and conformal gauge constraints can be written as (cf. (2.12))

$$\partial_+\partial_- X_M + (\partial_+ X_N \partial_- X_N) X_M = 0\,, \qquad X_M X_M = 1\,, \tag{3.1}$$

$$\partial_+ X_M \partial_+ X_M = (\partial_+ t)^2\,, \qquad \partial_- X_M \partial_- X_M = (\partial_- t)^2\,, \tag{3.2}$$

where t satisfies $\partial_+\partial_- t = 0$, which have general solution ($\kappa = \text{const}$)

$$t = \kappa\tau + h_+(\tau+\sigma) + h_-(\tau-\sigma)\ . \tag{3.3}$$

The equations (3.1), (3.2) are invariant under 2-d conformal transformations, $\xi^\pm \to F_\pm(\xi^\pm)$, so given a solution $X_M(\xi^+,\xi^-)$ one can find another one as $\tilde X_M(\xi^+,\xi^-) = X_M(F_+(\xi^+), F_-(\xi^-))$. One can also use this residual conformal symmetry to make the components of the stress tensor $\partial_+ X_M \partial_+ X_M$ and $\partial_- X_M \partial_- X_M$ equal to a constant, or, which is equivalent in the present case, to gauge away $h_\pm$ in (3.3), putting t in the form $t = \kappa\tau$. When only 3 of X_M's are non-zero (as in the case of the $O(3)$ invariant sigma model) one can show [48] that (3.1) reduces to the 2-d sine-Gordon equation

$$\partial_+\partial_-\alpha + \sin\alpha = 0\ , \qquad \cos\alpha = \partial_+ X_M \partial_- X_M\ . \tag{3.4}$$

Similar reduced systems can be derived also from other $O(n)$ invariant sigma models [50].*

The above equations (3.1) admit various special solutions. One is the "flat-space" or "chiral" solution (for which the Lagrange multiplier Λ in (2.12) vanishes): $X_M = f^+_M(\xi^+)$ or $X_M = f^-_M(\xi^-)$ for particular values of M. In contrast to the flat-space case, a linear combination of such solutions is no longer a solution, so one may thus say that (3.1) describes scattering of left-moving and right-moving light-like energy lumps [48]. For chiral X_M to satisfy the string theory constraints (3.2) we need to make a special choice of $h_\pm$ in t.

Let us now review various types of local and non-local conserved currents in this sigma model (see, e.g., [50, 53]). One can define a first-order linear system (Lax pair) [48] whose consistency is equivalent to the equations (3.1):

$$\partial_+ R^{(\ell)} = (1-\ell^{-1}) j_+ R^{(\ell)}\,, \qquad \partial_- R^{(\ell)} = (1-\ell) j_- R^{(\ell)}\,, \tag{3.5}$$
$$R^{(\ell)}(R^{(\ell)})^T = (R^{(\ell)})^T R^{(\ell)} = I\,,$$

where $R^{(\ell)}$ is an $so(6)$ matrix and

$$(j_a)_{MN} = 2(X_M \partial_a X_N - X_N \partial_a X_M)\ . \tag{3.6}$$

* A relation to the sine-Gordon system appeared previously in the context of strings moving in constant curvature spaces in [51, 52].

One can then construct a new solution from a given one as $X_M^{(\ell)} = R_{MN}^{(\ell)} X_N$. Solving (3.5) by the inverse scattering method is subtle due to complications related to the choice of boundary conditions [49,50] (e.g., on an infinite line, for $j_a \to 0$ at spatial infinity the solution $R^{(\ell)}$ does not have plane-wave behavior). Still, (3.5) may be used as a basis for analyzing the integrability properties of the sigma model .

One approach is to look at non-abelian (non-commuting) non-local conserved charges related to Yangians [54]. At the same time, it is important also to study an infinite family of commuting local conserved charges whose existence is a manifestation of integrability of the corresponding equations of motion. These may be constructed using the Bäcklund transformation. If X_M is a given "trial" solution of (3.1), let us define its Bäcklund transform $X_M^{(\gamma)}$ as another solution satisfying [20,53],

$$\partial_+(X_M^{(\gamma)} + X_M) = \frac{1}{2}(1+\gamma^{-2}) X_N^{(\gamma)} \partial_+ X_N \ (X_M^{(\gamma)} - X_M) , \qquad (3.7)$$

$$\partial_-(X_M^{(\gamma)} - X_M) = -\frac{1}{2}(1+\gamma^{2}) X_N^{(\gamma)} \partial_- X_N \ (X_M^{(\gamma)} + X_M) , \qquad (3.8)$$

$$X_M^{(\gamma)} X_M^{(\gamma)} = 1 , \quad X_M X_M = 1 , \quad X_M^{(\gamma)} X_M = \frac{1-\gamma^2}{1+\gamma^2} , \quad X_M^{(0)} = X_M . \qquad (3.9)$$

Here γ is a spectral parameter. One can write the solution of the equations (3.8) as an expansion in γ

$$X_M^{(\gamma)} = X_M + \sum_{k=1}^{\infty} X_{(k)M} \gamma^k , \quad X_{(1)M} = \frac{2\partial_+ X_M}{\sqrt{\partial_+ X_N \partial_+ X_N}} , \ \dots . \qquad (3.10)$$

One can define the generating function of *local* commuting conserved *scalar* charges associated with the original solution X_M by [20,53]

$$Q(\gamma) = \frac{1}{2}\gamma \int_0^{2\pi} \frac{d\sigma}{2\pi} \ X_M^{(\gamma)} (\partial_+ X_M + \gamma^2 \partial_- X_M) = \sum_{k=2}^{\infty} Q_k \gamma^k , \qquad (3.11)$$

$$Q_2 = \frac{1}{2}\int_0^{2\pi} \frac{d\sigma}{2\pi} \ X_{(1)M} \partial_+ X_M = \int_0^{2\pi} \frac{d\sigma}{2\pi} \ \sqrt{\partial_+ X_N \partial_+ X_N} = \int_0^{2\pi} \frac{d\sigma}{2\pi} \ \partial_+ t , \qquad (3.12)$$

$$Q_k = \frac{1}{2}\int_0^{2\pi} \frac{d\sigma}{2\pi} \ \left(X_{(k-1)M} \partial_+ X_M + X_{(k-3)M} \partial_+ X_M \right) , \quad k \geq 3 . \qquad (3.13)$$

Here in (3.12) we used the constraint (3.2). Then Q_2 can be interpreted as the space-time energy: since the general solution for t is given by (3.3), we conclude that $Q_2 = \kappa = \mathcal{E}$. For values of these charges on specific solutions see [20].

One can also define an infinite number of conserved *non-local* $so(6)$ Lie algebra valued (i.e. *matrix*) currents and associated charges as for any principal or coset sigma model [48, 55]. Let us follow [55] and replace X_M by an orthogonal $O(6)$ (or unitary SU(4)) matrix

$$g = e^{i\pi P} = 1 - 2P \ , \tag{3.14}$$

where $P_{MN} \equiv X_M X_N$ is a projector since $X_M X_M = 1$. Then $g = g^{-1} = 1 - 2P$ and

$$\text{tr}\ (\partial^a g \partial_a g^{-1}) = 8\, \partial^a X_M \partial_a X_M \ , \qquad j_a \equiv g^{-1}\partial_a g = j_a(X) \ , \qquad \partial^a j_a = 0 \ , \tag{3.15}$$

where the conservation of $j_a(X)$ given by (3.6) follows from the equations of motion (2.12), (3.1) for X_M. Defining $D_a = \partial_a + j_a$ we get $[\partial_a, D_a] = 0$. Then starting with the conserved current j_a one can construct an infinite sequence of conserved non-local currents j_a^n using the following iterative procedure. Given a conserved current $j_a^{(n)}$ we define a matrix function $\chi^{(n)}$ and use it to construct the next conserved current

$$j_a^{(n)} = \epsilon_{ab}\partial^b \chi^{(n)} \ , \qquad j_a^{(n+1)} = D_a \chi^{(n)} \ , \qquad j_a^{(1)} \equiv j_a \ , \quad \chi^{(0)} = 1 \ . \tag{3.16}$$

This leads to an infinite set of conserved charges

$$\mathcal{Q}^{(n)} = \int_0^{2\pi} d\sigma\ j_\tau^{(n)}(\tau, \sigma) \ . \tag{3.17}$$

For example,

$$\mathcal{Q}^{(1)}_{MN} = \int_0^{2\pi} d\sigma\ j_{\tau MN}(\tau, \sigma) = 2 \int_0^{2\pi} d\sigma\ (X_M \partial_\tau X_N - X_N \partial_\tau X_M) \ , \tag{3.18}$$

is proportional to the $O(6)$ angular momentum J_{MN} (2.14), and

$$\partial_\sigma \chi^{(1)} = j_\tau \ , \qquad \chi^{(1)}_{MN}(\tau, \sigma) = \int d\sigma'\ j_{\tau MN}(\tau, \sigma') \ , \tag{3.19}$$

$$\mathcal{Q}^{(2)}_{MN} = \int_0^{2\pi} d\sigma'\ [\partial_\tau + j_\tau(\tau, \sigma')]_{MK} \chi^{(1)}_{KN}(\tau, \sigma') \ . \tag{3.20}$$

These relations can be consistently defined on an infinite spatial (σ) line but not on a circle which is what we need for the closed string case: for X_M periodic, j_a (3.6) is also periodic, but its integral in (3.19) may not be, and thus $\mathcal{Q}^{(2)}$ may not be well-defined (see also related comments in [56, 57]). There are, however, particular classes of solutions (such as the circular solutions discussed below) for which these charges may be well-defined.

Let us mention also that as in other sigma models with a current satisfying $j_a = g^{-1}\partial_a g,\ \partial^a j_a = 0$ we can construct a set of chiral currents – symmetric higher spin 2-d local currents which are scalars under $O(6)$ [58, 59]

$$T_{+\ldots+} = \text{tr}\ j_+^n\ , \quad \partial_- T_{+\ldots+} = 0\ , \quad T_{-\ldots-} = \text{tr}\ j_-^n\ , \quad \partial_+ T_{-\ldots-} = 0\ . \quad (3.21)$$

The special case of spin 2 currents T_{++} and T_{--} are the components of the sigma model stress tensor proportional to $\partial_\pm X_M \partial_\pm X_M$. There are also other examples of local chiral currents built out of totally symmetric invariant tensors associated with the corresponding Lie algebra [59].

Similar non-local and local currents can be defined [56] also for the full $AdS_5 \times S^5$ supercoset string sigma model of [3].†

To conclude, the above sigma model admits an infinite set of conserved charges which is usually interpreted as implying its integrability [48]. The use of this integrability of a 2-d system for classifying finite energy solutions on a 2-d cylinder is not immediately clear however. In one dimension a system is integrable if it has the same number of commuting integrals of motion as the number of its degrees of freedom. In 2-d where one has an infinite number of degrees of freedom it is usually assumed that having infinite set of conserved quantities implies integrability. A more practical definition of integrability could be a prescription of how to construct explicitly a generic solution with the required properties. Formal solution-generating techniques (see, e.g., [62]) are not guaranteed a priori to produce finite energy solutions (cf. [63]).

As we shall see below, one can understand the integrability of the $R_t \times S^5$ sigma model in a very explicit way by reducing it [18, 34] on a special "rotating string" ansatz [14] to a well-known 1-d integrable system, the Neumann system [62, 64], or its generalization, the Neumann-Rosochatius (NR) system [65, 66].

4. Reduction of $R_t \times S^5$ sigma-model to 1-d Neumann system

4.1. *Rotating string ansatz*

Let us consider a string located at the center of the spatial part of AdS_5 with time coordinate being proportional to the worldsheet time, i.e. with

† That the $AdS_5 \times S^5$ superstring, being a conformal extension of a bosonic coset sigma model , should be integrable was suggested previously in [46, 60] (see also [61]).

the AdS_5 coordinates in (2.15) given by

$$Y_5 + iY_0 = e^{it} \ , \qquad Y_1, ..., Y_4 = 0 \ , \qquad t = \kappa\tau \ . \tag{4.1}$$

The general case when the string can be extended and rotate in both AdS_5 and S^5 will be discussed below in section 6. The S^5 metric (2.17) has three commuting translational isometries in φ_i which give rise to three global commuting integrals of motion J_i (2.19) , so to get solutions with non-zero J_i it is natural to choose the following "rotating string" ansatz for the S^5 coordinates X_M in (2.15) [14, 18, 34]

$$\mathrm{X}_1 \equiv X_1 + iX_2 = z_1(\sigma)\ e^{iw_1\tau} \ , \qquad \mathrm{X}_2 \equiv X_3 + iX_4 = z_2(\sigma)\ e^{iw_2\tau} \ ,$$
$$\mathrm{X}_3 \equiv X_5 + iX_6 = z_3(\sigma)\ e^{iw_3\tau} \ . \tag{4.2}$$

Here w_i may be interpreted as frequencies of rotation in the three orthogonal planes. The functions z_i may be real [18] or, in general, complex [34] and should satisfy $X_M X_M = 1$, i.e.

$$z_i = r_i e^{i\alpha_i} \ , \qquad \sum_{i=1}^{3} r_i^2 = 1 \ . \tag{4.3}$$

Thus the shape of a rotating string is not changed in time (i.e. the string is rigid) and it always belongs to a 2-sphere.

The closed string periodicity condition (2.10) implies

$$r_i(\sigma + 2\pi) = r_i(\sigma) \, , \qquad \alpha_i(\sigma + 2\pi) = \alpha_i + 2\pi m_i \, , \qquad m_i = 0, \pm 1, \pm 2, ... \ . \tag{4.4}$$

Comparing (4.2) to (2.15) we conclude that the S^5 angles φ_i may depend on both τ and σ,

$$\varphi_i = w_i\tau + \alpha_i(\sigma) \, , \tag{4.5}$$

with the integers m_i in (4.4) thus playing the role of "winding numbers" in the Cartan directions φ_i.

The space-time energy E of the string in (2.18) and the spins (2.19) forming a Cartan subalgebra of $O(6)$ are given by

$$E = \sqrt{\lambda}\ \mathcal{E} \ , \qquad \mathcal{E} = \kappa \ , \tag{4.6}$$

$$J_i \equiv \sqrt{\lambda}\ \mathcal{J}_i \ , \qquad \mathcal{J}_i = w_i \int_0^{2\pi} \frac{d\sigma}{2\pi}\ r_i^2(\sigma) \, , \qquad \frac{\mathcal{J}_i}{w_i} = 1 \ . \tag{4.7}$$

All other components of the conserved angular momentum tensor J_{MN} (2.14) vanish automatically if all w_i are different [18], but their vanishing should be checked if two of the three frequencies are equal.

4.2. *Integrals of motion and constraints*

In general, starting with

$$\mathrm{X}_i(\tau,\sigma) = r_i(\tau,\sigma)\ e^{i\varphi(\tau,\sigma)}\ , \tag{4.8}$$

one finds that the Lagrangian (2.6) reduces to

$$L_S = \frac{1}{2}\sum_{i=1}^{3}\left[\dot r_i^2 - r_i'^2 + r_i^2(\dot\varphi_i^2 - \varphi_i'^2)\right] + \frac{1}{2}\Lambda\left(\sum_{i=1}^{3} r_i^2 - 1\right). \tag{4.9}$$

One can then check that the "rotating string" ansatz (4.2), i.e.

$$r_i = r_i(\sigma)\ , \qquad \varphi_i = w_i\tau + \alpha_i(\sigma) \tag{4.10}$$

is indeed consistent with the equations of motion following from (4.9). Note that because of the formal $\tau \leftrightarrow \sigma$ symmetry of the 2-d equations of motion another special solution is given by the "pulsating string" ansatz: $r_i = r_i(\tau),\ \varphi_i = m_i\sigma + \beta_i(\tau)$, where m_i are now integer winding numbers. Then $r_i(\tau)$ is a solution of a similar Neumann system discussed below (see also section 8).

Substituting (4.2) into (2.6) we get the following effective 1-d "mechanical" system for a particle on a 5-d sphere

$$\mathrm{L} = \frac{1}{2}\sum_{i=1}^{3}(z_i' z_i'^* - w_i^2 z_i z_i^*) - \frac{1}{2}\Lambda\left(\sum_{i=1}^{3} z_i z_i^* - 1\right), \tag{4.11}$$

with σ playing the role of time (we changed the sign of L). If we set $z_k = x_k + ix_{k+3}$ $(k = 1,2,3)$, this is recognized as a special case of the well-known integrable system – the standard $n = 6$ *Neumann* model [62, 64, 65] describing a harmonic oscillator on a 5-sphere

$$\mathrm{L}_N = \frac{1}{2}\sum_{M=1}^{6}(x_M'^2 - w_M^2 x_M^2) - \frac{1}{2}\Lambda\left(\sum_{M=1}^{6} x_M^2 - 1\right). \tag{4.12}$$

Here three of the six frequencies are equal to the other three, $w_{k+3} = w_k$, $k = 1,2,3$. This implies integrability of the model (4.11) and determines its integrals of motion. Indeed, the Neumann system (4.11) has the following six commuting integrals of motion (see, e.g., [64, 65]):

$$F_M = x_M^2 + \sum_{M\neq N}^{6}\frac{(x_M x_N' - x_N x_M')^2}{w_M^2 - w_N^2}\ , \qquad \sum_{M=1}^{6} F_M = 1\ . \tag{4.13}$$

Since in the present case 3 of the 6 frequencies are equal one needs to consider 3 non-singular combinations of F_M which then give the 3 integrals of (4.11):

$$I_i = F_i + F_{i+3} \ , \qquad i = 1, 2, 3 \ , \qquad \sum_{i=1}^{3} I_i = 1 \ . \tag{4.14}$$

More explicitly, (4.11) can be written as

$$\mathrm{L} = \frac{1}{2} \sum_{i=1}^{3} (r_i'^2 + r_i^2 \alpha_i'^2 - w_i^2 r_i^2) - \frac{1}{2} \Lambda \left(\sum_{i=1}^{3} r_i^2 - 1 \right) , \tag{4.15}$$

implying that

$$\alpha_i' = \frac{v_i}{r_i^2} \ , \qquad v_i = \text{const} \ , \tag{4.16}$$

where v_i are three integrals of motion, which complement the two independent integrals in (4.14). Eliminating α_i' from (4.15) (changing the sign of the corresponding term to reproduce the same equations for r_i) we then get the following effective Lagrangian for the radial coordinates

$$\mathrm{L} = \frac{1}{2} \sum_{i=1}^{3} \left(r_i'^2 - w_i^2 r_i^2 - \frac{v_i^2}{r_i^2} \right) - \frac{1}{2} \Lambda \left(\sum_{i=1}^{3} r_i^2 - 1 \right) . \tag{4.17}$$

When the new integration constants v_i vanish, i.e. α_i are constant, we go back to the case of the $n = 3$ Neumann model studied in [18]. For non-zero v_i [34] the Lagrangian (4.17) describes the so called *Neumann-Rosochatius* (NR) integrable system (see, e.g., [66]). As explained above, its integrability follows from the fact that it is a special case of the $n = 6$ Neumann system, with the integrals of motion (4.14) taking the following explicit form:

$$I_i = r_i^2 + \sum_{j \neq i}^{3} \frac{1}{w_i^2 - w_j^2} \left[(r_i r_j' - r_j r_i')^2 + \frac{v_i^2}{r_i^2} r_j^2 + \frac{v_j^2}{r_j^2} r_i^2 \right] . \tag{4.18}$$

This gives two additional (besides v_i) independent integrals of motion.

The conformal gauge constraints (2.7), (2.8) or (3.2) now become

$$\kappa^2 = \sum_{i=1}^{3} \left(r_i'^2 + w_i^2 r_i^2 + \frac{v_i^2}{r_i^2} \right) , \tag{4.19}$$

$$\sum_{i=1}^{3} w_i v_i = 0 \ . \tag{4.20}$$

As a consequence of (4.20) only two of the three integrals of motion v_i are independent of w_i. Note also that (4.19), (4.20) imply $\kappa^2 = \sum_{i=1}^3 \left[r_i'^2 + (w_i r_i \pm \frac{v_i}{r_i})^2\right]$, so that the space-time energy E or κ is minimized if $r_i = \text{const}$ and $(w_i r_i \pm \frac{v_i}{r_i})^2$ take minimal value, i.e. if $w_i^2 r_i - \frac{v_i^2}{r_i^3} = 0$. Since all three w_i/v_i cannot be positive, this does not mean that κ should vanish. We shall return to the discussion of such solutions below at the end of section 5.1.

To summarize, we are interested in finding periodic finite-energy solitonic solutions of the $O(6)$ sigma model defined on a 2-cylinder that carry three global charges J_i. As discussed in [18] (see also below), the periodicity condition (4.4) on r_i implies that the two integrals of motion b_a (two appropriate independent combinations of I_i in (4.18)) can be traded for two integers n_a labelling different types of solutions. Imposing the periodicity condition (4.4) on α_i gives, in view of (4.16), the following constraint:

$$v_i \int_0^{2\pi} \frac{d\sigma}{r_i^2(\sigma)} = 2\pi m_i \ . \tag{4.21}$$

It implies that v_i should be expressible in terms of the integers m_i, frequencies w_i and the "radial" integrals b_a or n_a (note also that since the integral in (4.21) is of a positive function, $m_i = 0$ implies $v_i = 0$). As a result, the moduli space of solutions will thus be parametrized by $(w_1, w_2, w_3; n_1, n_2; m_1, m_2, m_3)$. The constraint (4.20) will give one relation between these 3+2+3 parameters. As a consequence, trading w_i for the angular momenta using (4.7), the energy of the solutions as determined by (4.6) and the conformal gauge constraint (4.19) will be a function of the $SO(6)$ spins and the "topological" numbers n_a and m_i (cf. (2.24))

$$\mathcal{E} = \mathcal{E}(\mathcal{J}_i; n_a, m_i) \ , \qquad \text{i.e.} \qquad E = \sqrt{\lambda}\, \mathcal{E}\left(\frac{J_i}{\sqrt{\lambda}}; n_a, m_i\right) . \tag{4.22}$$

The constraint (4.20) will provide one additional relation between J_i and n_a, m_i. Our aim will be to study the relation (4.22) for various types of solutions and in various limits.

4.3. *Special case of $n = 3$ Neumann system*

In the special case of $v_i = 0$ (when the angles α_i are constant, i.e. φ_i in (4.10) depend only on τ) the NR system (4.17) reduces to the $n = 3$ Neumann system with the two independent integrals in (4.18) and only one non-trivial constraint (4.19), which expresses the fact that κ is related to the

1-d Hamiltonian of the Neumann system,

$$H = \frac{1}{2}\sum_{i=1}^{3}(r_i'^2 + w_i^2 r_i^2) = \frac{1}{2}\,\kappa^2\ . \tag{4.23}$$

Note that this Hamiltonian is related to the 3 integrals of motion in (4.18) by $H = \frac{1}{2}\sum_{i=1}^{3} w_i^2 I_i$. Any two of these three integrals are enough to integrate this dynamical system. In order to find the relevant closed string solutions we need also to impose the periodicity condition (4.4), i.e. we are interested in the subsector of periodic solutions of the Neumann model.

Since r_i belong to a 2-sphere (4.3), the corresponding equations can be expressed in terms of the two S^2 angles, namely, γ and ψ in (2.15). However, in the general case it is convenient to use another parametrization of S^2 – to replace r_i by the two "ellipsoidal" coordinates ζ_a which are the roots of the equation $\sum_{i=1}^{3}\frac{r_i^2}{\zeta - w_i^2} = 0$:

$$r_1 = \sqrt{\frac{(\zeta_1 - w_1^2)(\zeta_2 - w_1^2)}{w_{21}^2 w_{31}^2}}\ , \qquad r_2 = \sqrt{\frac{(w_2^2 - \zeta_1)(\zeta_2 - w_2^2)}{w_{21}^2 w_{32}^2}}\ , \tag{4.24}$$

$$r_3 = \sqrt{\frac{(w_3^2 - \zeta_1)(w_3^2 - \zeta_2)}{w_{31}^2 w_{32}^2}}\ , \qquad w_{ij}^2 \equiv w_i^2 - w_j^2\ . \tag{4.25}$$

Expressing the integrals of motion (4.13) in terms of ζ_a one finds a system of two 1-st order equations

$$\Big(\frac{d\zeta_1}{d\sigma}\Big)^2 = -4\,\frac{P(\zeta_1)}{(\zeta_2 - \zeta_1)^2}\,, \qquad \Big(\frac{d\zeta_2}{d\sigma}\Big)^2 = -4\,\frac{P(\zeta_2)}{(\zeta_2 - \zeta_1)^2}\,. \tag{4.26}$$

The function $P(\zeta)$ is the following 5-th order polynomial

$$P(\zeta) = (\zeta - w_1^2)(\zeta - w_2^2)(\zeta - w_3^2)(\zeta - b_1)(\zeta - b_2)\ . \tag{4.27}$$

The parameters b_1, b_2 here are the two constants of motion which can be expressed in terms of the integrals I_i in (4.18) by solving

$$\begin{aligned} b_1 + b_2 &= (w_2^2 + w_3^2)I_1 + (w_1^2 + w_3^2)I_2 + (w_1^2 + w_2^2)I_3\,, \\ b_1 b_2 &= w_2^2 w_3^2 I_1 + w_1^2 w_3^2 I_2 + w_1^2 w_2^2 I_3\,. \end{aligned} \tag{4.28}$$

The Neumann system's Hamiltonian (4.23) is then $H = \frac{1}{2}\big(w_1^2 + w_2^2 + w_3^2 - b_1 - b_2\big) = \frac{1}{2}\kappa^2$. The polynomial $P(\zeta)$ in (4.27) can be interpreted as defining a hyperelliptic curve of genus 2 defined by the equation $s^2 + P(\zeta) = 0$, with s and ζ being two complex coordinates of C^2. The formal solution of the system (4.26) is then given in terms of the related θ-functions [18, 67].

Thus, the most general three-spin string solutions are naturally associated with special genus 2 hyperelliptic curves [18]. The simpler two-spin case (e.g., $w_3 = 0$) is associated with an elliptic curve and the corresponding relation between the energy and the spins then involves elliptic functions (see [11, 17, 19]). Elliptic integrals appear also in the one-spin case [8, 36].

The system (4.26) allows one to achieve the full separation of the variables: dividing one equation in (4.26) by the other one can integrate, e.g., ζ_2 in terms of ζ_1 and then obtain a closed equation for ζ_1 as a function of σ. In finding solutions we need also to take into account the periodicity conditions on r_i now viewed as conditions on ζ_1, ζ_2. The spins $\mathcal{J}_i$ in (4.7) expressed in terms of ζ_1, ζ_2 satisfy [18]

$$\sum_{i=1}^{3} w_i(w_i - \mathcal{J}_i) = \int_0^{2\pi} \frac{d\sigma}{2\pi} \, (\zeta_1 + \zeta_2) \ , \tag{4.29}$$

$$\sum_{i=1}^{3} \frac{\mathcal{J}_i}{w_i^3} = \frac{1}{w_1^2 w_2^2 w_3^2} \int_0^{2\pi} \frac{d\sigma}{2\pi} \, \zeta_1 \zeta_2 \ , \qquad \sum_{i=1}^{3} \frac{\mathcal{J}_i}{w_i} = 1 \ . \tag{4.30}$$

To find the energy (4.6) as a function of the spins $\mathcal{J}_i$ we need to express the frequencies w_i and the Neumann integrals of motion or b_a in (4.28) in terms of $\mathcal{J}_i$. After finding a periodic solution of (4.26), this reduces to the problem of computing the two independent integrals on the r.h.s. of (4.29) and (4.30).

Let us briefly mention that the case of the NR system (4.17) with $v_i \neq 0$ can be treated similarly [34]. We can again introduce the ellipsoidal coordinates (ζ_1, ζ_2), and expressing the integrals of motion (4.18) in terms of ζ_a we end up with the same system (4.26) where now

$$\begin{aligned} P(\zeta) &= (\zeta - b_1)(\zeta - b_2)(\zeta - w_1^2)(\zeta - w_2^2)(\zeta - w_3^2) + v_1^2(\zeta - w_2^2)^2(\zeta - w_3^2)^2 \\ &+ \ v_2^2(\zeta - w_1^2)^2(\zeta - w_3^2)^2 + v_3^2(\zeta - w_1^2)^2(\zeta - w_2^2)^2 \ . \end{aligned} \tag{4.31}$$

The Hamiltonian of the NR system reduces to $H = \frac{1}{2}\big[\sum_{i=1}^3 (w_i^2 + v_i^2) - b_1 - b_2\big]$. As in the pure Neumann case, $P(\zeta)$ is the fifth order polynomial which again defines a hyperelliptic curve $s^2 + P(\zeta) = 0$. The general solution of equations (4.26) can be again given in terms of theta-functions associated with the Jacobian of the hyperelliptic curve. An example of a solution is provided by the $v_3 = 0$ case where $\zeta = w_3^2$ is a root of $P(\zeta)$ and then the NR system can be solved in terms of the elliptic functions [34].

4.4. *Types of solutions and rotating strings in flat space*

Let us consider for simplicity the case with $v_i = 0$ described by the $n = 3$ Neumann system (general solutions of the NR system have similar structure). The five parameters w_i (or $\mathcal{J}_i$) and b_a of the solutions of the Neumann system may be viewed as coordinates on the moduli space of periodic finite-energy solitons. The values of b_a will not be arbitrary: such solutions will be classified by two integer "winding number" parameters n_a which will be related to w_i and b_a through the periodicity condition. In general, there will be several different solutions for given values of $\mathcal{J}_i$, i.e. the energy of the string E will be a function not only of $\mathcal{J}_i$ but also of n_a: there will be a discrete series of solutions with energies starting from some minimal value.

Depending on the values of these parameters (i.e. location in the moduli space) one may find different geometric types (or shapes) of the resulting rotating string solutions. The shape of the string does not change with time and the string may be *"folded"* (with topology of an interval) or *"circular"* (with topology of a circle). A folded string may then be *"straight"* as in the one- and two-spin examples considered in [8] and [17], or *"bent"* (at one or several points) as in the general three-spin case [18]. A "circular" string may have the form of a round circle as in the two-spin and three-spin solutions of [14,15] or may have a more general "bent circle" shape as in the three-spin solutions in [18].

It is useful to review how these different string shapes appear in the case of a closed string rotating in flat $R^{1,5}$ Minkowski space. In the conformal gauge, string coordinates are then given by solutions of free 2-d wave equation, i.e. by combinations of $e^{in(\tau\pm\sigma)}$, subject to the standard constraints (3.2). For a closed string rotating in the two orthogonal spatial 2-planes and moving along the 5-th spatial direction we find (cf. (4.2); here $t = \kappa\tau$ as in (4.1))

$$X_1 + iX_2 = \ r_1(\sigma)\ e^{iw_1\tau}\ , \qquad X_3 + iX_4 = \ r_2(\sigma)\ e^{iw_2\tau}\ , \qquad X_5 = p_5\tau\ , \tag{4.32}$$

$$w_1 = n_1\ , \quad w_2 = n_2\ , \quad r_1 = a_1 \sin(n_1\sigma)\ , \qquad r_2 = a_2 \sin[n_2(\sigma + \sigma_0)]\ . \tag{4.33}$$

Here σ_0 is an arbitrary integration constant, and n_a are arbitrary integers. The conformal gauge constraint implies that $\kappa^2 = p_5^2 + n_1^2 a_1^2 + n_2^2 a_2^2$. Then the energy, the two spins and the 5-th component of the linear momentum are (here the tension parameter is $\sqrt{\lambda} \ \to \frac{1}{\alpha'}$)

$$E = \frac{\kappa}{\alpha'}\ , \qquad J_1 = \frac{n_1 a_1^2}{2\alpha'}\ , \qquad J_2 = \frac{n_2 a_2^2}{2\alpha'}\ , \qquad P_5 = \frac{p_5}{\alpha'}\ , \tag{4.34}$$

i.e.

$$E = \sqrt{P_5^2 + \frac{2}{\alpha'}(n_1 J_1 + n_2 J_2)} \ . \tag{4.35}$$

To get the two-spin states on the leading Regge trajectory (having minimal energy for given values of the *two* non-zero spins) one is to choose $n_1 = n_2 = 1$. The shape of the string depends on the values of σ_0 and n_1, n_2. If $\frac{\sigma_0}{\pi}$ is irrational then the string always has a "circular" (loop-like) shape. In general, the "circular" string will not be lying in one plane, i.e. will have one or several bends. For rational values of $\frac{\sigma_0}{\pi}$ the string can be either circular or folded, depending on the values of n_1, n_2.

Let $\sigma_0 = 0$. If $n_1 = n_2$ the string is folded and straight, i.e. have no bends. Indeed, then $X_1 + iX_2$ is proportional to $X_3 + iX_4$ and thus one may put the string in a single 2-plane by a global O(4) rotation. If both n_1 and n_2 are either even or odd and different then the string is folded and has several bends (in the 13 and 24 planes). For example, if $n_2 = 3n_1$ then the folded string is wound n_1 times and has two bends (for $a_1 = a_2$ we have $r_2 = r_1(3 - 4r_1^2)$). Next, let us choose $\sigma_0 = \frac{\pi}{2n_2}$. Then for $n_1 = n_2$ the string is an ellipsoid, becoming a round circle in the special case of $a_1 = a_2$ (i.e. $\mathcal{J}_1 = \mathcal{J}_2$) [14]. The string is also circular if n_1 is even and n_2 is odd. If, however, n_1 is odd and n_2 is even the string is folded, e.g., if $n_2 = 2n_1$ then the folded string is wound n_1 times and has a single bend at one point.

The structure of spinning string soliton solutions in curved S^5 case is analogous. The equations of motion of the Neumann system are linearized on the Jacobian of the hyperelliptic curve. The image of the string in the Jacobian whose real connected part is identified with the Liouville torus can wind around two non-trivial cycles with the winding numbers n_1 and n_2 respectively [18]. The size and the shape of the Liouville torus are governed by the moduli (w_i, b_a). Specifying the winding numbers n_1, n_2, two of the five parameters (w_i, b_a) are then uniquely determined by the periodicity conditions. The actual rigid shape of the physical string lying on the two-sphere will depend on the numbers n_1, n_2 and on the remaining moduli parameters (relative values of b_a and w_i^2): it may be of (bent) folded type or of (deformed) circular type. Various examples of folded and circular three-spin string solutions and their energies were discussed in [14,18]. In most three-spin cases finding an explicit relation for the energy (4.22) is complicated, but one can always develop the large $\mathcal{J}_i$ perturbation theory [18]. We shall discuss some examples of such solutions below.

Finally, let us note that while the Neumann or NR 1-d systems have a small finite number of commuting integrals, there are infinitely many commuting conserved charges in the original 2-d sigma model and the corresponding integrable spin chain on the SYM side. These are expressed in terms of the NR integrals in the present case, see [20] for details.

5. Simplest circular solutions in $R_t \times S^5$: $\Lambda = \text{const}$

A simple special class of solutions of the system (4.11) or (4.17) is found by demanding that the Lagrange multiplier Λ in (2.12) is constant, i.e. $\dot X_M^2 - X_M'^2 = \text{const}$. In this case the radii r_i turn out to be constant (and $n_a = 0$, i.e. there are no bends). This represents an interesting new class of circular three-spin solutions [34] which includes as a special case the circular solution of [14] where two out of three spins are equal.

5.1. *Constant radii solution*

Let us start with the Lagrangian (4.11) written in terms of three complex coordinates z_i. Then the equations of motion are

$$z_i'' + m_i^2 z_i = 0 \ , \qquad m_i^2 \equiv w_i^2 + \Lambda \ , \qquad \sum_{i=1}^3 |z_i|^2 = 1 \ , \tag{5.1}$$

$$\Lambda = \sum_{i=1}^3 (|z_i'|^2 - w_i^2 |z_i|^2) \ . \tag{5.2}$$

Equation (5.1) can be easily integrated if one assumes that $\Lambda = \text{const}$,

$$z_i = a_i e^{i m_i \sigma} + b_i e^{-i m_i \sigma} \ , \tag{5.3}$$

where a_i, b_i are complex coefficients. The periodicity condition $z_i(\sigma + 2\pi) = z_i(\sigma)$ implies that m_i must be integer. It is easy to show [34] that modulo the global $SU(3) \in SO(6)$ invariance the solution of (5.3) that satisfies both $\Lambda = \text{const}$ and $\sum_{i=1}^3 |z_i|^2 = 1$ should have $b_i = 0$ (*or* $a_i = 0$), i.e. should look like (m_i may be positive or negative and a_i may be made real by $U(1)$ rotations)

$$z_i = a_i e^{i m_i \sigma} \ , \qquad \sum_{i=1}^3 a_i^2 = 1 \ . \tag{5.4}$$

It may seem that one may get a new solution if two of the windings m_i are equal while the third is zero, i.e. $z_1 = a \cos m\sigma,\ z_2 = a \sin m\sigma,\ z_3 = \sqrt{1 - a^2}$ (which is, in fact, the circular solution of [14]), but this configuration can be transformed into the form (5.4) by a global $SU(2)$ rotation.

One can also rederive (5.4) by starting with (4.17), (4.16). The potential $w_i r_i^2 + \frac{v_i^2}{r_i^2}$ in (4.17) has a minimum, and that suggests that r_i =const may be a solution. The equations of motion that follow from (4.17)

$$r_i'' = -w_i^2 r_i + \frac{v_i^2}{r_i^3} - \Lambda r_i \ , \tag{5.5}$$

$$\Lambda = \sum_{j=1}^{3} \left(r_j'^2 - w_j^2 r_j^2 + \frac{v_j^2}{r_j^2} \right) \ , \qquad \sum_{j=1}^{3} r_j^2 = 1 \tag{5.6}$$

are indeed solved by

$$r_i(\sigma) = a_i = \text{const} \ , \qquad w_i^2 - \frac{v_i^2}{a_i^4} = \nu^2 = \text{const} \ , \qquad \Lambda = -\nu^2 \ , \tag{5.7}$$

where ν is an arbitrary constant (which may be positive or negative). Equation (5.7) then implies $a_i^2 = \frac{|v_i|}{\sqrt{w_i^2 - \nu^2}}$, $\alpha_i' = \frac{v_i}{a_i^2} = \frac{v_i}{|v_i|}\sqrt{w_i^2 - \nu^2} \equiv m_i$, i.e. $\alpha_i = \alpha_{0i} + m_i \sigma$, where m_i must be integer to satisfy the periodicity condition (4.4) and α_{0i} may be set to zero by independent $SO(2)$ rotations. Then

$$w_i^2 = m_i^2 + \nu^2 \ , \qquad v_i = a_i^2 m_i \ , \qquad \sum_{i=1}^{3} a_i^2 = 1 \ . \tag{5.8}$$

The constraints (4.19), (4.20) give $\kappa^2 = 2\sum_{i=1}^3 a_i^2 w_i^2 - \nu^2$, and $\sum_{i=1}^3 a_i^2 w_i m_i = 0$. As a result, we get the following relations for the energy and spins [34] (cf. (4.6), (4.7))

$$\mathcal{E}^2 = 2 \sum_{i=1}^{3} \sqrt{m_i^2 + \nu^2} \ \mathcal{J}_i - \nu^2 \ , \tag{5.9}$$

$$\sum_{i=1}^{3} \frac{\mathcal{J}_i}{\sqrt{m_i^2 + \nu^2}} = 1 \ , \tag{5.10}$$

$$\sum_{i=1}^{3} m_i \mathcal{J}_i = 0 \ . \tag{5.11}$$

We shall assume for definiteness that all w_i and thus all $\mathcal{J}_i$ are non-negative. Then (5.11) implies that one of the three m_i's must have the opposite sign to the other two. One can check directly that the only non-vanishing components of the $SO(6)$ angular momentum tensor J_{MN} (2.14) on this solution are indeed the Cartan ones J_i in (2.19).

The special case of $\nu^2 = 0$ (or $\Lambda = 0$) corresponds to a solution for the string in flat space which can be embedded into S^5 by choosing the free radial parameters of a circular string to satisfy the condition $\sum_{i=1}^3 a_i^2 = 1$. Indeed, as follows from (5.8) for $\nu^2 = 0$ we find that all frequencies must be integer $w_i = |m_i|$. We may choose, e.g., $m_1 < 0,\ m_2 > 0,\ m_3 > 0$, so that the solution is a combination of the left and right moving waves in different directions (we use complex combinations of the coordinates in (4.2), cf. (4.32), (4.33))

$$\mathrm{X}_1 = a_1 e^{im_1(\sigma-\tau)}, \quad \mathrm{X}_2 = a_2 e^{im_2(\sigma+\tau)}, \quad \mathrm{X}_3 = a_3 e^{im_3(\sigma+\tau)}, \quad \sum_{i=1}^{3} a_i^2 = 1 \ . \tag{5.12}$$

Here we get from (5.9)–(5.11)

$$\mathcal{E}^2 = 2\sum_{i=1}^{3} |m_i|\mathcal{J}_i \ , \qquad \sum_{i=1}^{3} \frac{\mathcal{J}_i}{|m_i|} = 1 \ , \qquad \sum_{i=1}^{3} m_i\mathcal{J}_i = 0 \ . \tag{5.13}$$

This corresponds to a very special point in the moduli space of solutions. For fixed m_i, we get two constraints on $\mathcal{J}_i$, and the energy is given by the standard flat-space Regge relation (cf. (4.35)). Then $|m_1|\mathcal{J}_1 = m_2\mathcal{J}_2 + m_3\mathcal{J}_3$ (where $\mathcal{J}_2$ and $\mathcal{J}_3$ are related via $\sum_{i=1}^3 \frac{\mathcal{J}_i}{|m_i|} = 1$) and thus $\mathcal{E}^2 = 4|m_1|\mathcal{J}_1$. The energy of this "flat" solution thus does *not* have a regular (1.2) expansion in integer powers of $\frac{1}{\mathcal{J}^2} = \frac{\lambda}{J^2}$, $\mathcal{J} = \sum_{i=1}^3 \mathcal{J}_i$. This will no longer be so in the genuinely "curved" $\nu \neq 0$ case where we will have indeed a regular expansion for the energy in $\frac{1}{\mathcal{J}^2}$, as in the case of the circular solution of [14]. This then opens up a possibility of direct comparison with perturbative anomalous dimensions in SYM theory.

5.2. *Energy as a function of spins*

In general, to express $\mathcal{E}$ in terms of $\mathcal{J}_i$ and m_i one first solves the condition (5.10) in terms of ν, determining ν as a function of $\mathcal{J}_i$ and m_i and then substitutes the result into (5.9). The condition (5.11) may be imposed at the very end, implying that for given spins $\mathcal{J}_i$ the solution exists only for a special choice of the integers m_i. Expanding in large total spin $\mathcal{J} = \sum_{i=1}^3 \mathcal{J}_i$ as in [14,18] one finds [34] that $\nu^2 = \mathcal{J}^2 - \sum_{i=1}^3 m_i^2 \frac{\mathcal{J}_i}{\mathcal{J}} + \ldots$ and thus

$$\mathcal{E} = \mathcal{J} + \frac{1}{2\mathcal{J}} \sum_{i=1}^{3} m_i^2 \frac{\mathcal{J}_i}{\mathcal{J}} + \ldots \ . \tag{5.14}$$

Thus, as in other examples in [14, 17, 18], here the energy admits a regular expansion in $\frac{1}{\mathcal{J}^2}=\frac{\lambda}{J^2}$ as in (1.2)

$$E = J\Big(1 + \frac{\lambda}{2J^2}\sum_{i=1}^{3} m_i^2 \frac{J_i}{J} + ...\Big) = J + \frac{\lambda}{2J}\sum_{i=1}^{3} m_i^2 \frac{J_i}{J} + ... \ , \tag{5.15}$$

where m_i should satisfy the constraint $\sum_{i=1}^{3} m_i J_i = 0$.

Let us now look at some special cases. In the one-spin case $(0, 0, J_3)$, i.e. $\mathcal{J}_1 = \mathcal{J}_2 = 0$, $a_1 = a_2 = 0$, we have $w_3^2 = \nu^2$, i.e. $m_3 = 0$ and $J_3 = w_3$, and then $\mathcal{E} = \mathcal{J}_3$. This is simply the point-like BMN geodesic case: there is no σ-dependence.

In the two-spin case $(J_1, J_1, 0)$, i.e. $\mathcal{J}_3 = 0$, $a_3 = 0$, the equation (5.10) for ν^2 becomes a quartic equation. Its simple explicit solution is found in the equal-spin case when $\mathcal{J}_1 = \mathcal{J}_2$, i.e. when

$$a_1 = a_2 = \frac{1}{\sqrt{2}} \ , \qquad m_2 = -m_1 \equiv m > 0 \ , \tag{5.16}$$

so that

$$\mathcal{E} = \sqrt{\mathcal{J}^2 + m^2} \ , \qquad \mathcal{J} \equiv \mathcal{J}_1 + \mathcal{J}_2 = 2\mathcal{J}_2 \ , \tag{5.17}$$

i.e.

$$E = J\sqrt{1 + m^2\frac{\lambda}{J^2}} \ . \tag{5.18}$$

We get

$$\mathrm{X}_1 = \frac{1}{\sqrt{2}}\, e^{iw\tau - im\sigma} \ , \qquad \mathrm{X}_2 = \frac{1}{\sqrt{2}}\, e^{iw\tau + im\sigma} \ , \qquad w = \sqrt{\nu^2 + m^2} \ . \tag{5.19}$$

This solution is thus equivalent to the circular two-spin solution of [14] – it is related to it by an $SO(4)$ rotation: $\mathrm{X}_1' = \frac{1}{\sqrt{2}}(\mathrm{X}_1 + \mathrm{X}_2)$, $\mathrm{X}_2' = \frac{1}{\sqrt{2}}(-\mathrm{X}_1 + \mathrm{X}_2)$. In the general case of two unequal spins we can again solve (5.10) in the limit of large $\mathcal{J}_1, \mathcal{J}_2$ (for fixed m_1, m_2), getting the special case of (5.14) with $m_1\mathcal{J}_1 + m_2\mathcal{J}_2 = 0$, $\mathcal{J}_3 = 0$.

Another special case is (J_2, J_2, J_3) when two out of three non-vanishing spins are equal, e.g., $\mathcal{J}_1 = \mathcal{J}_2$. Setting

$$m_3 = 0 \ , \quad m_1 = -m_2 = m \ , \quad a_3 = a < 1 \ , \quad a_1 = a_2 = \sqrt{1 - a^2} \ , \tag{5.20}$$

$$\mathcal{J}_3 = a^2\nu \ , \qquad \mathcal{J}_1 = \mathcal{J}_2 = \frac{1}{2}(1 - a^2)\sqrt{m^2 + \nu^2} \ , \tag{5.21}$$

we thus find from (5.14)

$$\mathcal{E} = \mathcal{J} + \frac{m^2 \mathcal{J}_2}{\mathcal{J}^2} + ... , \quad \text{i.e.} \quad E = J + \frac{m^2 \lambda J_2}{J^2} + ... , \qquad J = 2J_2 + J_3 \ . \quad (5.22)$$

This solution is equivalent to the circular three-spin solution with two equal spins in [14, 15] (the two backgrounds are related by a global rotation in X_2, X_3 directions converting $e^{im\sigma}$ into $\cos m\sigma$ and $\sin m\sigma$). The corresponding operator in the gauge theory tr $(X^{J_1} Y^{J_1} Z^{J_3}) + ...$ (belonging to the SO(6) representation with Dynkin indices $[J_2 + J_3, 0, J_2 - J_3]$ for $J_2 > J_3$) which has the one-loop anomalous dimension equal to (5.22) does indeed exist as was found in [21].

More generally, we may consider a three-spin solution (J_2, J_2, J_3) with $m_3 \neq 0$, so that $(m_1 + m_2)J_2 + m_3 J_3 = 0$. Then (5.15) gives

$$\mathcal{E} = \mathcal{J} + \frac{\mathcal{J}_2}{2\mathcal{J}^2} \Big[m_1^2 + m_2^2 + (m_1 + m_2)^2 \, \frac{\mathcal{J}_2}{\mathcal{J}_3} \Big] + ... , \qquad (5.23)$$

which generalizes (5.22) to the case when $m_1 + m_2 \neq 0$. The energy is minimal in the latter case. This suggests that the band of such states in the same representation $[J_2 + J_3, 0, J_2 - J_3]$ (if $|\frac{m_3}{m_1 + m_2}| > 1$) but with higher energy than (5.22) should also be found on the SYM side.

To summarize, the constant-radii solutions of the NR system represent a simple generalization of the circular two-spin and three-spin solutions of [14] which have regular expansion of the energy in powers of $\frac{\lambda}{J^2}$. Therefore, it should be possible to match, as in [16, 18, 19, 21], the coefficient of the $O(\lambda)$ term in (5.14) with the SYM anomalous dimensions determined by the Hamiltonian of the integrable $SU(2, 2|4)$ spin chain [22, 23] in the corresponding three-spin subsector of states.

5.3. *Quadratic fluctuations near circular solutions*

The remarkable simplicity of the circular solutions discussed above makes it easy to find the quadratic fluctuation action and to compute the corresponding spectrum of string fluctuations. This in turn allows one to analyze the stability of the solution and to find the string one-loop correction to the ground-state energy, in the same way as was done in [15] for a particular three-spin circular solution with two equal spins (5.21). In spite of the σ-dependence of the solution, the quadratic action turns out to have constant coefficients, just like in the BMN case [4,6] when one expands near the point-like geodesic in S^5 [8,11]. Sending $J \to \infty$ for fixed $\frac{\lambda}{J^2} \ll 1$ suppresses higher loop corrections to masses of excited string states. As a result, as in the "plane-wave" BMN case, the string fluctuation spectrum can be found *exactly.*

To illustrate this, we shall consider the bosonic part of the quadratic fluctuation action following [34]. The fermionic part of the spectrum can be easily found in the same way as was done (in a special case (5.21)) in a [15]. In contrast to the BMN case, here we are expanding near a non-supersymmetric solution, and the resulting world-sheet string action (in the static or light-cone type gauge) will not have a world-sheet supersymmetry. There remains an interesting question if a "nearly-BPS" property of similar rotating string solutions in the $\lambda \to 0$ limit observed in [41] imposes certain constraints on the world-sheet action.

It is straightforward to find the quadratic fluctuation Lagrangian by expanding near the solution (5.4) or (5.7)–(5.8) following [15, 34]. Using 3 complex combinations of coordinates in (4.2) and expanding ($\mathrm{X}_i \to \mathrm{X}_i + \tilde{\mathrm{X}}_i$) the sigma model action (2.5) near the classical solution (5.4),

$$\mathrm{X}_i = a_i e^{iw_i\tau + im_i\sigma} , \qquad w_i^2 = \sqrt{m_i^2 + \nu^2} , \tag{5.24}$$

$$\sum_{i=1}^{3} a_i^2 = 1 , \qquad \sum_{i=1}^{3} a_i^2 w_i m_i = 0 , \tag{5.25}$$

we find the following Lagrangian for the quadratic fluctuations (see [15])

$$\tilde{L} = -\frac{1}{2}\,\partial_a \tilde{\mathrm{X}}_i\,\partial^a \tilde{\mathrm{X}}_i^* + \frac{1}{2}\,\Lambda \tilde{\mathrm{X}}_i\,\tilde{\mathrm{X}}_i^* , \tag{5.26}$$

where $\Lambda = -\nu^2$ (see (5.7)) and $\tilde{\mathrm{X}}_i$ are subject to the constraint*

$$\sum_{i=1}^{3} (\mathrm{X}_i \tilde{\mathrm{X}}_i^* + \mathrm{X}_i^* \tilde{\mathrm{X}}_i) = 0 . \tag{5.27}$$

To solve this constraint we set

$$\tilde{\mathrm{X}}_i = e^{iw_i\tau + im_i\sigma}(g_i + if_i) , \tag{5.28}$$

* The imposition of the conformal gauge constraints on the fluctuations is not necessary in order to determine the non-trivial part of the fluctuation spectrum [14, 15] (solving the constraints in terms of fluctuation of t leads to equivalent results [15]). In addition to S^5 fluctuations there are also AdS_5 fluctuations: one massless and four massive ones with mass κ coming from the classical value of the Lagrange multiplier $\tilde{\Lambda}$ [14, 15].

where g_i and f_i are real functions of τ and σ. Then (5.27) reduces to

$$\sum_{i=1}^{3} a_i g_i = 0 \ . \tag{5.29}$$

Using (5.28) the Lagrangian (5.26) becomes (after integrating by parts, cf. [15])

$$\tilde{L} = \sum_{i=1}^{3} \left[\frac{1}{2} \left(\dot{f}_i^2 + \dot{g}_i^2 - f_i'^2 - g_i'^2 \right) - 2 w_i f_i \dot{g}_i + 2 m_i f_i g_i' \right] \ . \tag{5.30}$$

To solve the linear relation (5.29) we may apply a global $O(3)$ rotation to g_i, $\bar{g}_i = M_{ij}(a) g_j$, which preserves the kinetic terms in (5.30) and transforms $\sum_{i=1}^{3} a_i g_i$ into $\bar{g}_1$; then we may set the latter to zero in the resulting Lagrangian (5.30). Equivalently, we may solve (5.29) directly for g_1 and substitute it into (5.30). The result (after diagonalization) is a special case of the following 2-d Lagrangian (summation over p, q is assumed)

$$L = \frac{1}{2} \dot{x}_p^2 - \frac{1}{2} x_p'^2 + F_{pq} x_p \dot{x}_q - H_{pq} x_p x_q' \ , \tag{5.31}$$

where $x_p = (f_1, f_2, f_3, g_2, g_3)$ and F_{pq} and H_{pq} are constant antisymmetric matrices depending on a_i, w_i, m_i. Equation (5.31) can be written also as (ignoring total derivative)

$$L = \frac{1}{2} (\dot{x}_p + F_{pq} x_q)^2 - \frac{1}{2} (x_p' + H_{pq} x_q)^2 - (F_{pq} F_{qk} - H_{pq} H_{qk}) x_p x_k \ , \tag{5.32}$$

i.e. it represents a massive scalar 2-d theory coupled to a constant 2-d gauge field (which can be "rotated away" at the expense of making the mass term τ and σ dependent). The Lagrangian (5.31) can be also interpreted as a *light-cone gauge* ($u = \tau$) Lagrangian for the bosonic string sigma model $L = -(\eta^{ab} g_{mn} + \epsilon^{ab} B_{mn}) \partial_a x^m \partial_b x^n$ in a (in general, non-conformal) *plane-wave* background with the following metric and antisymmetric 2-form field

$$ds^2 = 2 du dv + 2 F_{pq} x_p dx_q du + dx_p dx_p \ , \qquad B_2 = 2 H_{pq} x_p dx_q \wedge du \ . \tag{5.33}$$

By analogy with the BMN case, one may say that the geometry "seen" in the large $\mathcal{J}$ limit by the circular rotating string is a generalized plane-wave background. The resulting quadratic string excitation spectrum for such an action can be found in a more or less explicit way (as in [69]).

For example, let us consider the two-spin case where $m_3 = 0$ and

$$a_1^2 + a_2^2 = 1 \ , \quad a_3 = 0 \ , \quad a_1^2 m_1 w_1 + a_2^2 m_2 w_2 = 0 \ , \qquad w_1^2 - m_1^2 = w_2^2 - m_2^2 = \nu^2 \ . \tag{5.34}$$

We shall assume that $w_i > 0,\ m_1 < 0,\ m_2 > 0$. In this case f_3, g_3 decouple (they have mass ν, cf. (5.26)) and we get the following Lagrangian for the remaining three x_s-fluctuations f_1, f_2 and (rescaled) g_2

$$\tilde{L} = \frac{1}{2}(\dot{f}_1^2 + \dot{f}_2^2 + \dot{g}_2^2 - f_1'^2 - f_2'^2 - g_2'^2) \\ + 2\,(a_2 w_1 f_1 - a_1 w_2 f_2)\,\dot{g}_2 - 2\,(a_2 m_1 f_1 - a_1 m_2 f_2)\,g_2' \ . \qquad (5.35)$$

To find the spectrum of characteristic frequencies corresponding to this action we note that since f_i and g_i must be periodic in σ one can expand the solution of the quadratic fluctuation equations in modes

$$x_s = \sum_{n=-\infty}^{\infty} \sum_{k=1}^{8} A_{sn}^{(k)} e^{i(\Omega_{n,k}\tau \,+\, n\sigma)} \ , \qquad (5.36)$$

where k labels different frequencies for a given value of n (we shall suppress the index k below). Plugging this into the classical equations that follow from (5.35) one finds the following equation for the four non-trivial characteristic frequencies (it expresses the vanishing of the determinant of the characteristic matrix)

$$(\Omega^2 - n^2)^2 - 4a_2^2(w_1\Omega - m_1 n)^2 - 4a_1^2(w_2\Omega - m_2 n)^2 = 0 \ . \qquad (5.37)$$

The stability condition is that all four roots should be real. The solutions are obviously real for $n = 0$ so an instability may appear only for $n = \pm 1, ...$ In the special case of the equal-spin circular solution of [14], i.e. (5.16), (5.17), we find $(\Omega^2 - n^2)^2 - 4w^2\Omega^2 - 4m^2n^2 = 0$, i.e. [14]

$$\Omega_\pm^2 = n^2 + 2\nu^2 + 2m^2 \pm 2\sqrt{(\nu^2 + m^2)^2 + n^2(\nu^2 + 2m^2)} \ , \qquad (5.38)$$

which implies instability when $n^2 - 4m^2 < 0$, i.e. for $n = \pm 1, ..., \pm(2m - 1)$ [15]. This instability is present also for generic two-spin solutions with $a_1 \neq a_2,\ m_1 \neq -m_2$.

In spite of the instability it is useful to work out the spectrum of frequencies in the limit of large spins (i.e. large ν, cf. (5.21)) since the resulting energies may be compared to SYM theory. The large ν expansion of (5.38) gives (for the lower-energy modes)

$$\Omega_- = \pm\frac{1}{2\nu}\, n\sqrt{n^2 - 4m^2} + O\left(\frac{1}{\nu^3}\right) \ , \qquad (5.39)$$

and so the contribution to the energy of a rotating string from (a pair of) such modes is (here $\kappa^2 = \nu^2 + 2m^2$, $J = J_1 + J_2 = \sqrt{\lambda}\ \sqrt{\nu^2 + m^2}$)

$$\Delta E_n = \frac{2|\Omega_-|}{\kappa} = \frac{1}{\nu^2}\, n\sqrt{n^2 - 4m^2} + O\Big(\frac{1}{\nu^4}\Big) = \frac{\lambda}{J^2} n\sqrt{n^2 - 4m^2} + O\Big(\frac{\lambda^2}{J^4}\Big) \ . \tag{5.40}$$

This expression was indeed reproduced [16] on the SYM side (for $m = 1$) as the anomalous dimension of excited string states corresponding to a particular Bethe root distribution of a Heisenberg spin chain related to the dilatation operator in the two R-charge sector.

In the general (m_1, m_2) case, there are modes that have $\Omega \sim \frac{1}{\nu}$ and modes for which $\Omega^2 \to 4\nu^2$ at large ν (see [15]). Expanding (5.37) at large ν assuming $\Omega = O(\frac{1}{\nu})$ we find the following generalization of (5.39)

$$\Omega_- = \frac{1}{2\nu} n\Big[2a_2^2 m_1 + 2a_1^2 m_2 \pm \sqrt{n^2 - 4a_1^2 a_2^2 (m_1 - m_2)^2}\Big] + O\Big(\frac{1}{\nu^3}\Big) \ , \tag{5.41}$$

where $a_1^2 + a_2^2 = 1$. Equation (5.41) reduces to (5.39) in the equal-spin case when $a_1^2 = a_2^2 = \frac{1}{2}$, $m_1 = -m_2$. Recalling that we have the constraint $m_1 J_1 + m_2 J_2 = 0$ where $J_i = a_i^2\sqrt{m_i^2 + \nu^2}$, one concludes that there exist unstable modes with $n^2 < 4|m_1 m_2|$ [34]. Again, one should be able to reproduce the analog of (5.40) in the case of (5.41) on the gauge theory side.

It is straightforward to find the generalization of (5.35), (5.37) to the three-spin case, i.e. when a_3 is non-zero. The resulting spectrum is similar to the spectrum in the $(J_1, J_2 = J_3)$ case in [15]. The generalization of the equation. (5.37) to the three-spin case is [34]

$$\begin{aligned}(\Omega^2 - n^2)^4 - (\Omega^2 - n^2)^2\big[(a_2^2 + a_3^2)\tilde\Omega_1^2 + (a_2^2 + a_3^2)\tilde\Omega_2^2 + (a_1^2 + a_2^2)\tilde\Omega_3^2\big] \\ +\ a_3^2\tilde\Omega_1^2\tilde\Omega_2^2 + a_2^2\tilde\Omega_1^2\tilde\Omega_3^2 + a_1^2\tilde\Omega_2^2\tilde\Omega_3^2 = 0 \ ,\end{aligned} \tag{5.42}$$

where $\tilde\Omega_i \equiv 2(w_i\Omega - m_i n)$ and a_i and w_i can be expressed in terms of ν and m_i using (5.24). Setting $\tilde\Omega_3 = 0$, $a_3 = 0$ leads us back to (5.37). Equation (5.42) gives eight characteristic frequencies, four of which scale as $\frac{1}{\nu}\bar\Omega$ in the large ν (large $\mathcal{J}$) limit. In general, there is a range of parameters for which the solution is stable [15, 34], i.e. $\bar\Omega$'s are real.

For example, for the choice of the parameters in (5.20) when two of the spins are equal, we find [15] ($\Omega \to \frac{1}{\nu}\bar\Omega$)

$$\bar\Omega^2 = \frac{1}{4} n^2\Big[n^2 + 2(3a^2 - 1)m^2 \pm 2m\sqrt{(3a^2 - 1)^2 m^2 + 4a^2(n^2 - m^2)}\Big] \ . \tag{5.43}$$

Note that the limit $m = 0$ corresponds to the point-particle (BMN) case when $\Omega = \sqrt{\nu^2 + n^2}$. The condition of stability, i.e. $\Omega^2 \ge 0$ is obtained by demanding that $(p^2-4)(p^2-4a^2) \geq 0$ and $(3a^2-1)^2+4a^2(p^2-1) \geq 0$, where $p \equiv \frac{n}{m}$. For $m=1$ the stability condition is satisfied if $a^2 \geq \frac{1}{4}$ [15]. Similar stability conditions on a (or $\cos\gamma_0$ in the notation of [15]) are found for other values of m [15, 34].

5.4. *One-loop string correction to the classical energy*

As was shown in [15], for the stable three-spin solution (5.20) one can compute the one-loop correction to the classical energy (5.22) by summing over all (bosonic and fermionic) fluctuation frequencies. As in the static gauge, here $t=\kappa\tau$ and so the space-time energy and the 2-d energy (sum of $\frac{1}{2}\omega_n$ for all oscillator frequencies) are related by [11, 15] $E=\frac{1}{\kappa}\mathrm{E}_{2-\mathrm{d}}$. Thus the one-loop correction is given by the standard sum of the oscillator frequencies

$$E = \frac{1}{\kappa}\,\mathrm{E}_{2-d} = \frac{1}{2\kappa}\left(\sum_{n\in Z}\omega_n^B - \sum_{r\in Z+\frac{1}{2}}\omega_r^F\right) , \tag{5.44}$$

where $\omega_n^B = \sum_{k=1}^8 \Omega_{n,k}^B$ and $\omega_r^F = \sum_{k=1}^8 \Omega_{r,k}^F$ and the index k labels the characteristic frequencies. Here we need also to include contributions of AdS_5 fluctuations with masses equal to κ [15]. As expected on the basis of conformal invariance of the $AdS_5\times S^5$ string theory, this expression is found to be UV finite [15]. The one-loop correction vanishes in the "point-particle" limit when $m=0$ in (5.20), in agreement with the non-renormalization of the energy of the corresponding BPS state dual to a gauge theory operator with protected conformal dimension [4].

As was found in [15], the leading term in E_1 in the large $\kappa\approx\nu\approx\mathcal{J}$ limit is given for $m=1$ by

$$E_1 = \frac{1}{\kappa^2}\,d_1 + O\left(\frac{1}{\kappa^3}\right) , \tag{5.45}$$

$$d_1 = -\frac{1}{2}\left[5a^2+4-\sqrt{3(4a^2-1)}-4\sqrt{3a^2+1}\right] . \tag{5.46}$$

We are interested in the limit when $J_i\to\infty$ with $\frac{J_i}{J}$ held fixed (here $J=\sum_{i=1}^3 J_i = J_1+2J_2$). Since at large κ we have $\frac{1}{\kappa^2}=\frac{\lambda}{J^2}+...$, and (see (5.21)) $a=a(\frac{J_2}{J})\approx 1-\frac{J_2}{J}\geq\frac{1}{2}$ we get (cf. (1.5))

$$E_1 = \frac{\lambda}{J^2}\,d_1\left(\frac{J_2}{J}\right) + ... \tag{5.47}$$

For $J\gg J_2$ we find $d_1(\frac{J_2}{J})\approx 1-\frac{7J_2}{J}$. Combining this with the classical result for the energy (5.22), we obtain [15]

$$E = J + \frac{\lambda}{J^2}\Big[J_2 + d_1\Big(\frac{J_2}{J}\Big) + ...\Big] + ... = J + \frac{\lambda}{J}\Big[\frac{J_2}{J} + \frac{1}{J}\,d_1\Big(\frac{J_2}{J}\Big) + ...\Big] + ... \ , \quad (5.48)$$

i.e.

$$E = J + \frac{\lambda}{J}\Big[\frac{J_2}{J} + \frac{1}{J}\Big[1 - \frac{7J_2}{J} + O\Big(\frac{J_2^2}{J^2}\Big)\Big] + O\Big(\frac{1}{J^2}\Big)\Big] + O\Big(\frac{\lambda^2}{J^3}\Big) \ . \quad (5.49)$$

We conclude that the leading order J term in the classical energy is not modified by the one-loop correction, and that the one-loop contribution to the first classical correction term $\frac{\lambda}{J}$ is subleading in the $\frac{1}{J}$ expansion, in agreement with (1.6).

It is natural to conjecture that all higher-loop sigma model superstring corrections are also subleading at large J. As in the BMN case (see [11] and sect. 3.2 in [12]), the underlying reasons for this should be that (i) the 2-d energy of this 2-d UV finite QFT on a compact space (cylinder) should admit a regular inverse-mass expansion, and (ii) the space-time supersymmetry of the superstring action "spontaneously" broken by the solution should imply some kind of "asymptotic supersymmetry". That would mean that in the limit when J is sent to infinity for fixed $\frac{\lambda}{J^2}$ the classical expression for the ground-state energy (5.15), (5.22) *and* the energies of excited string states obtained from quadratic fluctuations are *exact*, just like in the BMN case.

Similar conclusions should apply for all multispin string solutions that have energy admitting a regular expansion in $\frac{\lambda}{J^2}$ as in (1.2). If there is indeed a general relation between the regularity of the classical expression of the energy and the suppression of quantum corrections to it in the $J \to \infty$ limit, this remains to be understood.

As discussed in section 1, it should be then be possible to compare the classical energy with the SYM anomalous dimension also computed in the limit of large J. Such a comparison was indeed successfully performed for the two-spin circular [16, 18, 19] and folded [16, 17, 19] string solutions and the three-spin circular solution of [14] with two equal spins [21].

Another interesting open problem is to compare the string one-loop $\frac{1}{J}$ correction to the leading $\frac{\lambda}{J}$ term in (5.49) with the corresponding $\frac{1}{J}$ correction to the thermodynamic limit of the Bethe ansatz expressions for the anomalous dimension [21] on the gauge theory side.

6. Rotating strings in $AdS_5 \times S^5$

Let us now generalize the discussion of section 4 to the case when the string can rotate in both AdS_5 and S^5. For that we need to supplement the S^5

rotating string ansatz (4.2) by a similar AdS_5 one [14,18,34]:

$$\mathrm{Y}_0 \equiv Y_5 + iY_0 = \mathrm{z}_0(\sigma)e^{i\omega_0\tau} \ ,$$

$$\mathrm{Y}_1 \equiv Y_1 + iY_2 = \mathrm{z}_1(\sigma)e^{i\omega_1\tau} \ , \qquad \mathrm{Y}_2 \equiv Y_3 + iY_4 = \mathrm{z}_2(\sigma)e^{i\omega_2\tau} \ . \tag{6.1}$$

Here the functions $\mathrm{z}_r = (\mathrm{z}_0, \mathrm{z}_1, \mathrm{z}_2)$ are in general complex and, because of the condition $\eta_{MN}Y^M Y^N = -1$, their real radial parts lie on a hyperboloid ($\eta_{rs} = (-1, 1, 1)$, cf. (4.3))

$$\mathrm{z}_r = \mathrm{r}_r e^{i\beta_r} \ , \qquad \eta^{rs}\mathrm{r}_r\mathrm{r}_s \equiv -\mathrm{r}_0^2 + \mathrm{r}_1^2 + \mathrm{r}_2^2 = -1 \ . \tag{6.2}$$

In sections 4 and 5 we had $\mathrm{r}_0 = 1, \ \mathrm{r}_1 = \mathrm{r}_2 = 0, \ \beta_r = 0$. To satisfy the closed string periodicity conditions we need, as in (4.4),

$$\mathrm{r}_r(\sigma + 2\pi) = \mathrm{r}_r(\sigma) \ , \qquad \beta_r(\sigma + 2\pi) = \beta_r(\sigma) + 2\pi k_r \ , \tag{6.3}$$

where k_r are integers. Comparing (6.1) to (2.15) we conclude that the AdS_5 time t and the angular coordinates ϕ_1, ϕ_2 are related to β_r by

$$t = \omega_0\tau + \beta_0(\sigma) \ , \qquad \phi_1 = \omega_1\tau + \beta_1(\sigma) \ , \qquad \phi_2 = \omega_2\tau + \beta_2(\sigma) \ . \tag{6.4}$$

We shall require the time coordinate t to be single-valued, i.e. ignore windings in the time direction and will also rename ω_0 as κ, i.e.

$$k_0 = 0 \ , \qquad \omega_0 \equiv \kappa \ . \tag{6.5}$$

The three $O(2,4)$ Cartan generators (spins) in (2.18) here are ($S_0 = E, \ \omega_r = (\omega_0, \omega_1, \omega_2)$)

$$S_r = \sqrt{\lambda}\omega_r \int_0^{2\pi} \frac{d\sigma}{2\pi} \ \mathrm{r}_r^2(\sigma) \equiv \sqrt{\lambda} \ \mathcal{S}_r \ . \tag{6.6}$$

In view of (6.2), they satisfy the relation

$$\sum_{s,r} \eta^{sr}\frac{\mathcal{S}_r}{\omega_s} = -1 \ , \quad \text{i.e.} \quad \frac{\mathcal{E}}{\kappa} - \frac{\mathcal{S}_1}{\omega_1} - \frac{\mathcal{S}_2}{\omega_2} = 1 \ . \tag{6.7}$$

Substituting the above rotational ansatz into the AdS_5 Lagrangian (and changing overall sign) we find the analog of the 1-d Lagrangian (4.11) in the S^5 case [34] (we assume a sum over repeated indices r, s)

$$\tilde{\mathrm{L}} = \frac{1}{2}\eta^{rs}(\mathrm{z}_r'\mathrm{z}_s^{*\prime} - \omega_r^2\mathrm{z}_r\mathrm{z}_s^*) - \frac{1}{2}\tilde{\Lambda}(\eta^{rs}\mathrm{z}_r\mathrm{z}_s^* + 1) \ . \tag{6.8}$$

Like its S^5 counterpart (4.11), this 1-d Lagrangian is a special case of an $n = 6$ Neumann system, now with signature $(-++++-)$, and thus represents again an integrable system, being related to a special euclidean-signature

Neumann model by an analytic continuation. The reduction of the total $AdS_5 \times S^5$ Lagrangian on the rotation ansatz is then given by the sum of (4.11) and (6.8). ¿From (6.8) we find as in (4.16)

$$\beta'_r = \frac{u_r}{\mathrm{r}_r^2} \ , \qquad\qquad u_r = \mathrm{const} \ , \tag{6.9}$$

so that the effective Lagrangian for the radial coordinates becomes

$$\tilde{\mathrm{L}} = \frac{1}{2}\,\eta^{rs}\Big(\mathrm{r}'_r\mathrm{r}'_s - \omega_r^2\mathrm{r}_s\mathrm{r}_s - \frac{u_r u_s}{\mathrm{r}_r\mathrm{r}_s}\Big) - \frac{1}{2}\,\tilde{\Lambda}\,(\eta^{rs}\mathrm{r}_r\mathrm{r}_s + 1) \ . \tag{6.10}$$

Thus (6.10) describes a Neumann-Rosochatius integrable system with indefinite signature, i.e. with δ_{ij} replaced by η_{rs} (cf. (4.17)).

We should also require the periodicity condition analogous to (4.21): $u_r\int_0^{2\pi}\frac{d\sigma}{\mathrm{r}_r^2(\sigma)} = 2\pi k_r$. Then k_0 implies that we should set $u_0 = 0$ as a consequence of single-valuedness of the AdS_5 time.

While the equations for r_i and r_r following from (4.15) and (6.10) are decoupled, the variables of the two NR systems are mixed in the conformal gauge constraints (2.7), (2.8) which now take the form (generalizing (4.19), (4.20) where we had $\mathrm{r}_0 = 1,\ u_r = 0,\ \mathrm{r}_a = 0$)

$$\mathrm{r}_0'^2 + \kappa^2\mathrm{r}_0^2 = \sum_{a=1}^{2}\Big(\mathrm{r}_a'^2 + \omega_a^2\mathrm{r}_a^2 + \frac{u_a^2}{\mathrm{r}_a^2}\Big) + \sum_{i=1}^{3}\Big(r_i'^2 + w_i^2 r_i^2 + \frac{v_i^2}{r_i^2}\Big) \ , \tag{6.11}$$

$$\sum_{a=1}^{2}\omega_a u_a + \sum_{i=1}^{3} w_i v_i = 0 \ . \tag{6.12}$$

Here $\mathrm{r}_0^2 - \sum_{a=1}^2 \mathrm{r}_a^2 = 1$, and $\sum_{i=1}^3 r_i^2 = 1$ and we used that $u_0 = 0$. One can then repeat the discussion of sections 4.2, 4.3 and 4.4 in the present case, classifying general solutions of the resulting NR system. One again finds folded and circular solutions, and the two-spin folded solution exists only if the string is bent [18].

6.1. *Simple circular strings in* AdS_5

Let us first assume that the string is not rotating in S^5 (i.e. $w_i, v_i = 0,\ r_i =$ const) and consider the AdS_5 analog of the simplest circular solution of section 5 by demanding $\tilde{\Lambda} =$ const. The discussion is exactly the same as in section 5 with a few signs reversed. As in section 5.1, finding solutions with $\tilde{\Lambda} =$ const turns out to be equivalent to looking for constant radii ($\mathrm{r}_r =$ const) solutions. Then (cf. (5.7), (5.8))

$$\mathrm{r}_r = \mathrm{a}_r = \mathrm{const} \ , \qquad \beta_a = k_a\sigma \ , \qquad k_0 = 0 \ , \qquad u_0 = 0 \ , \qquad u_a = \mathrm{a}_a^2 k_a \ , \tag{6.13}$$

$$\omega_0^2 \equiv \kappa^2 = \tilde{\Lambda} \ , \qquad \omega_a^2 = k_a^2 + \kappa^2 \ , \qquad a = 1, 2 \ . \tag{6.14}$$

The energy as a function of spins is then obtained by solving the system that follows from the definition of the charges (6.6) and the constraints (6.11), (6.12) with κ as a parameter (cf. (5.10)–(5.11))

$$\frac{\mathcal{E}}{\kappa} - \frac{\mathcal{S}_1}{\sqrt{k_1^2 + \kappa^2}} - \frac{\mathcal{S}_2}{\sqrt{k_2^2 + \kappa^2}} = 1 \ , \tag{6.15}$$

$$\kappa\mathcal{E} - \frac{1}{2}\kappa^2 = \sqrt{k_1^2 + \kappa^2}\ \mathcal{S}_1 + \sqrt{k_2^2 + \kappa^2}\ \mathcal{S}_2 \ , \qquad k_1\mathcal{S}_1 + k_2\mathcal{S}_2 = 0 \ . \tag{6.16}$$

This implies $\frac{\mathcal{S}_1 k_1^2}{\sqrt{k_1^2+\kappa^2}} + \frac{\mathcal{S}_2 k_2^2}{\sqrt{k_2^2+\kappa^2}} = \frac{1}{2}\kappa^2$. Considering the limit of large spins $\mathcal{S}_i \gg 1$, with k_a being fixed we conclude that $\kappa = (2k_1^2\mathcal{S}_1 + 2k_2^2\mathcal{S}_2)^{1/3} + ...$ and then

$$\mathcal{E} = \mathcal{S}_1 + \mathcal{S}_2 + \frac{3}{4}\,(2k_1^2\mathcal{S}_1 + 2k_2^2\mathcal{S}_2)^{1/3} + ... \ , \tag{6.17}$$

or, in view of $k_1\mathcal{S}_1 = -k_2\mathcal{S}_2$ (treating S_1, S_2 and k_1 as independent)

$$\mathcal{E} = \mathcal{S} + \frac{3}{4}\Big(2k_1^2\mathcal{S}\,\frac{\mathcal{S}_1}{\mathcal{S}_2}\Big)^{1/3} + ... \ , \qquad \mathcal{S} \equiv \mathcal{S}_1 + \mathcal{S}_2 \ . \tag{6.18}$$

Using (6.6) this can be rewritten as

$$E = S + \frac{3}{4}\,(\lambda S)^{1/3}\Big(2k_1^2\,\frac{S_1}{S_2}\Big)^{1/3} + ... \ . \tag{6.19}$$

The case of $k_1 = -k_2 = k$ when the two spins are equal $\mathcal{S}_1 = \mathcal{S}_2 = \frac{1}{2}\mathcal{S}$ is that of the the circular solution found in [14] for which we get

$$E = S + \frac{3}{4}\,(2k^2\lambda S)^{1/3} + ... \ . \tag{6.20}$$

As was shown in [14], this $k_1 = -k_2$ solution is stable only for small enough $\mathcal{S}$ (namely, $\mathcal{S} \leq \frac{5}{8}\sqrt{\frac{7}{2}}$ for $k = 1$).

The "non-perturbative" scaling of the subleading term in (6.19) with λ precludes a direct comparison of the above energies to the anomalous dimensions of the corresponding [14] SYM operators which (in euclidean version) have the following structure [14] $\mathrm{tr}(\bar{\Phi}(D_1 + iD_2)^{S_1}(D_3 + iD_4)^{S_2}\Phi) + ...$. This is unfortunate, since such operators are of more "realistic" type similar to the ones relevant for high-energy scattering in non-supersymmetric gauge theories – they contain many covariant derivatives instead of many scalars and thus may appear in less supersymmetric gauge theories without adjoint scalars.

It turns out that one needs a large J spin in S^5 directions to have a regular (1.2) expansion of the energy. Indeed, the situation changes when we consider "hybrid" solutions where the circular string rotates in both AdS_5 and S_5 directions.

6.2. *Constant radii circular strings in* $AdS_5 \times S^5$

It is straightforward to combine the solutions of sections 5.1 and 6.1 to write down the most general circular constant-radii solution in $AdS_5 \times S^5$ [34]. It is parametrized by the frequencies ($a = 1, 2;\ i = 1, 2, 3$)

$$\omega_0 = \kappa \ , \quad \omega_a^2 = k_a^2 + \kappa^2 \ , \quad w_i^2 = m_i^2 + \nu^2 \ , \quad \kappa^2 = \tilde{\Lambda} \ , \quad \nu^2 = -\Lambda \ , \tag{6.21}$$

related to the energy $\mathcal{E}$ and 2+3 spins $\mathcal{S}_a$ and $\mathcal{J}_i$ and topological numbers k_a and m_i. These will be related by (5.10) and (6.7) as well as by the conformal gauge constraints (6.11) and (6.12). Explicitly, we get the following generalization of both (5.9)–(5.11) and (6.15), (6.16)

$$\sum_{i=1}^{3} \frac{\mathcal{J}_i}{\sqrt{m_i^2 + \nu^2}} = 1 \ , \qquad \frac{\mathcal{E}}{\kappa} - \sum_{a=1}^{2} \frac{\mathcal{S}_a}{\sqrt{k_a^2 + \kappa^2}} = 1 \ , \tag{6.22}$$

$$2\kappa\mathcal{E} - 2\sum_{a=1}^{2} \sqrt{k_a^2 + \kappa^2}\ \mathcal{S}_a - \kappa^2 = 2\sum_{i=1}^{3} \sqrt{m_i^2 + \nu^2}\ \mathcal{J}_i - \nu^2 \ , \tag{6.23}$$

$$\sum_{a=1}^{2} k_a \mathcal{S}_a + \sum_{i=1}^{3} m_i \mathcal{J}_i = 0 \ . \tag{6.24}$$

Here κ and ν (or the two Lagrange multipliers in (6.21)) are parameters that need to be solved for in order to find $\mathcal{E}$ as a function of the spins $\mathcal{S}_a, J_i$ and windings k_a, m_i. The solution exists only for such integers k_a and m_i that satisfy (6.24).

If all spins are of the same order and large $\mathcal{S}_a \sim \mathcal{J}_i \gg 1$ we find

$$\kappa = \mathcal{J} + \frac{1}{2\mathcal{J}^2}\Big(\sum_{i=1}^{3} m_i^2 \mathcal{J}_i + 2\sum_{a=1}^{2} k_a^2 \mathcal{S}_a\Big) + O\Big(\frac{1}{\mathcal{J}^2}\Big) \ , \qquad \mathcal{J} \equiv \sum_{i=1}^{3} \mathcal{J}_i \ ,$$

$$\nu = \mathcal{J} - \frac{1}{2\mathcal{J}^2} \sum_{i=1}^{3} m_i^2 \mathcal{J}_i + O\Big(\frac{1}{\mathcal{J}^2}\Big) \ , \tag{6.25}$$

and thus ($\mathcal{S} \equiv \sum_{a=1}^2 \mathcal{S}_a$)

$$\mathcal{E} = \mathcal{J} + \mathcal{S} + \frac{1}{2\mathcal{J}^2}\Big(\sum_{i=1}^3 m_i^2 \mathcal{J}_i + \sum_{a=1}^2 k_a^2 \mathcal{S}_a \Big) + O\Big(\frac{1}{\mathcal{J}^3}\Big) , \tag{6.26}$$

or [34]

$$E = J + S + \frac{\lambda}{2J^2}\Big(\sum_{i=1}^3 m_i^2 J_i + \sum_{a=1}^2 k_a^2 S_a \Big) + O\Big(\frac{\lambda^2}{J^3}\Big) . \tag{6.27}$$

This expression is a direct generalization of (5.15) in the $\mathcal{S}_a = 0$ case. The energy is minimal if m_i^2 and k_a^2 have minimal possible values (0 or 1). We may also look at a different limit when $\mathcal{J} \gg \mathcal{S} \gg 1$. In this case we get "BMN-type" (single J rotation) asymptotics with the leading term still given by (6.27), i.e. $\Delta E \sim \frac{\lambda}{2J^2} S$.

As an example, let us consider the simplest hybrid solution when only one of each types of the spin is non-zero, i.e. $\mathcal{J}_1 = \mathcal{J},\ \mathcal{S}_1 = \mathcal{S},\ \mathcal{S}_2 = \mathcal{J}_2 = \mathcal{J}_3 = 0$. Then $\mathrm{r}_0^2 - \mathrm{r}_1^2 = 1,\ \mathrm{r}_3 = 0$ and $r_1 = 1,\ r_2 = r_3 = 0$, i.e. (cf. (2.15))

$$\mathrm{Y}_0 = \cosh\rho_0\ e^{i\kappa\tau} , \qquad \mathrm{Y}_1 = \sinh\rho_0\ e^{i\omega\tau + ik\sigma} , \quad \mathrm{X}_1 = e^{iw\tau + im\sigma} , \tag{6.28}$$

where $\mathrm{r}_0 = \cosh\rho_0$ determines the fixed radial coordinate in AdS_5 at which the string is located while it is spread and rotating in ϕ_1 (it is positioned at $\theta = \frac{\pi}{2}$ and $\phi_2 = 0$ in S^3 of AdS_5). Also, the string is a rotating circle along φ_1 in S^5 located at $\varphi_2 = \varphi_3 = 0,\ \gamma = \frac{\pi}{2},\ \psi = 0$. Its energy for $\mathcal{J} \sim \mathcal{S} \gg 1$ is then

$$E = J + S + \frac{\lambda}{2J^2}(m^2 J + k^2 S) + ... \tag{6.29}$$

One can easily analyze the fluctuations near this solution as was done in in section 5.3 [34]. We find one massless and four massive (mass ν) fluctuations in S^5 directions; in addition to two massive (mass κ) decoupled AdS_5 fluctuations there are also three coupled ones with a Lagrangian similar to (5.35). Then the equation (5.37) for the characteristic frequencies becomes

$$(\Omega^2 - n^2)^2 + 4\mathrm{r}_1^2(\kappa\Omega)^2 - 4\mathrm{r}_0^2(\omega_1\Omega - kn)^2 = 0 , \tag{6.30}$$

and one concludes that this (S, J) solution is always stable. Indeed, setting $\mathrm{r}_0 = a,\ \mathrm{r}_1 = \sqrt{a^2 - 1}$ we get

$$\Omega_- = \frac{1}{2\kappa} n \Big[2a^2 k \pm \sqrt{n^2 + 4a^2(a^2-1)k^2} \Big] + O\Big(\frac{1}{\kappa^3}\Big) , \tag{6.31}$$

so that for any $a = \cosh\rho_0 \geq 1$ there are no unstable modes.

The conclusion is that for a regular large-spin expansion of the energy one needs to have at least one (large) component of spin in S^5 direction. This turns out to be true also in the case of other (folded and circular) spinning string solutions with more complicated σ-dependence.

7. "Inhomogeneous" two-spin solutions $AdS_5 \times S^5$

7.1. *Rotating ansatz in terms of angles*

If we set k_a and m_i or u_r in (6.10) and v_i in (4.17) to zero (i.e. assume that the angles ϕ_a and φ_i do not depend on σ), the $AdS_5 \times S^5$ NR system reduces to the sum of the two $n = 3$ Neumann systems. Then rotating strings carrying 2+3 charges $(S_1, S_2; J_1, J_2, J_3)$ are described by the following ansatz in terms of angles in (2.15) [14]

$$t = \kappa\tau \ , \quad \phi_a = \omega_a\tau \ , \quad \varphi_i = w_i\tau \ , \quad \rho(\sigma) = \rho(\sigma + 2\pi) \ . \tag{7.1}$$

The remaining angles may depend only on σ, i.e. $\theta = \theta(\sigma)$, $\gamma = \gamma(\sigma)$ and $\psi = \psi(\sigma)$ and may be periodic modulo 2π, e.g., $\psi(\sigma + 2\pi) = \psi(\sigma) + 2\pi n$. If $n = 0$ we get *folded* solutions, if $n \neq 0$ we get *circular* solutions [18].

The conserved charges in (2.18), (2.19) then have the following explicit form

$$\begin{aligned}
\mathcal{S}_1 &= \omega_1 \int_0^{2\pi} \tfrac{d\sigma}{2\pi} \sinh^2\!\rho \, \cos^2\!\theta \, , \qquad & \mathcal{J}_1 &= w_1 \int_0^{2\pi} \tfrac{d\sigma}{2\pi} \sin^2\!\gamma \, \cos^2\!\psi \, , \\
\mathcal{S}_2 &= \omega_2 \int_0^{2\pi} \tfrac{d\sigma}{2\pi} \sinh^2\!\rho \, \sin^2\!\theta \, , \qquad & \mathcal{J}_2 &= w_2 \int_0^{2\pi} \tfrac{d\sigma}{2\pi} \sin^2\!\gamma \, \sin^2\!\psi \, , \\
\mathcal{E} &= \kappa \int_0^{2\pi} \tfrac{d\sigma}{2\pi} \cosh^2\!\rho \, , \qquad & \mathcal{J}_3 &= w_3 \int_0^{2\pi} \tfrac{d\sigma}{2\pi} \cos^2\!\gamma \, .
\end{aligned} \tag{7.2}$$

The sigma model equations for the σ-dependent angles (ρ, θ)

$$\begin{gathered}
\rho'' - \sinh\rho \, \cosh\rho \, (\kappa^2 + \theta'^2 - \omega_1^2 \cos^2\theta - \omega_2^2 \sin^2\theta) = 0 \, , \\
(\sinh^2\!\rho \, \theta')' - (\omega_1^2 - \omega_2^2) \sinh^2\!\rho \, \sin\theta \, \cos\theta = 0 \, ,
\end{gathered} \tag{7.3}$$

and (γ, ψ)

$$\begin{gathered}
\gamma'' - \sin\gamma \, \cos\gamma \, (w_3^2 + \psi'^2 - w_1^2 \cos^2\!\psi - w_2^2 \sin^2\!\psi) = 0 \, , \\
(\sin^2\!\gamma \, \psi')' - (w_1^2 - w_2^2) \sin^2\!\gamma \, \sin\psi \, \cos\psi = 0 \, ,
\end{gathered} \tag{7.4}$$

are decoupled from each other. As explained above and in [18], the resulting system of equations is completely integrable, being equivalent to a combination of the two Neumann dynamical systems. As a result, there are 2+2 "hidden" integrals of motion, reducing the general problem to the solution of two independent systems of two coupled first-order equations, with parameters related through the one nontrivial conformal gauge constraint

$$\rho'^2 - \kappa^2 \cosh^2\rho + \sinh^2\rho\ (\theta'^2 + \omega_1^2 \cos^2\theta + \omega_2^2 \sin^2\theta)$$
$$+ \gamma'^2 + w_3^2 \cos^2\gamma + \sin^2\gamma\ (\psi'^2 + w_1^2 \cos^2\psi + w_2^2 \sin^2\psi) = 0\ . \qquad (7.5)$$

Note that the two metrics in (2.16), (2.17) are related by the obvious analytic continuation and change of the overall sign, which is equivalent for the present rotational ansatz (7.1) to

$$\rho \leftrightarrow i\gamma\ , \qquad \theta \leftrightarrow \psi\ , \qquad \kappa \leftrightarrow -w_3\ , \qquad \omega_1 \leftrightarrow -w_1\ , \qquad \omega_2 \leftrightarrow -w_2\ . \ (7.6)$$

This transformation maps the system (7.3) into the system (7.4) and also preserves the constraint (7.5). Thus it formally maps solutions into solutions [18, 19]. Under (7.6) the conserved charges (7.2) transform as follows

$$S_1 \leftrightarrow J_1\ , \qquad S_2 \leftrightarrow J_2\ , \qquad E \leftrightarrow -J_3\ . \qquad (7.7)$$

This corresponds to interchanging different $SO(2)$ generators of the symmetry group $SO(2,4) \times SO(6)$. One can find also other similar transformations that map solutions into solutions by combining (7.6) with discrete $SO(2,4) \times SO(6)$ isometries such as interchanging the angular coordinates (see below).

7.2. *Folded two-spin solutions:* (S, J) *and* (J_1, J_2)

Let us now review the two non-trivial two-spin folded string solutions which are, in fact, related by the above analytic continuation.

The first is the "(S, J)" solution [11]

$$\kappa, \omega_1, w_3 \neq 0\ , \qquad \rho = \rho(\sigma)\ , \qquad \theta = 0\ , \qquad \gamma = 0\ , \qquad \psi = 0\ , \qquad (7.8)$$

where the string is stretched in the radial direction ρ of AdS_5 and rotates (ω_1) in AdS_5 about its center of mass. The latter in turn moves (w_3) along a large circle of S^5. In the limit of a point-like string ($\mathcal{S} = 0$) this becomes a massless geodesic in S^5 as in [4, 8]. In the case $w_3 = 0$ this becomes the folded string rotating in AdS_5 [8, 36]. The gauge constraint (7.5) and the integrals of motion (7.2) here become

$$\rho'^2 - \kappa^2 \cosh^2\rho + \omega_1^2 \sinh^2\rho = -w_3^2\ , \qquad \mathcal{J} \equiv \mathcal{J}_3 = w_3\ ,$$
$$\mathcal{S} \equiv \mathcal{S}_1 = \omega_1 \int_0^{2\pi} \frac{d\sigma}{2\pi}\ \sinh^2\rho\ \ , \qquad \mathcal{E} = \kappa \int_0^{2\pi} \frac{d\sigma}{2\pi}\ \cosh^2\rho\ . \qquad (7.9)$$

For the second "(J_1, J_2)" solution one has [14,17]

$$\kappa, w_1, w_2 \neq 0 \ , \qquad \rho = 0 \ , \qquad \theta = 0 \ , \qquad \gamma = \frac{\pi}{2} \ , \qquad \psi = \psi(\sigma) \ . \tag{7.10}$$

Here the string is located at the center of AdS_5 while it is stretched (ψ) along a great circle of S^5 and rotates (w_2) about its center of mass which moves (w_1) along an orthogonal great circle of S^5. The gauge constraint (7.5) and the integrals of motion (7.2) here are

$$\begin{gathered} \psi'^2 + w_1^2 \cos^2\psi + w_2^2 \sin^2\psi = \kappa^2 \ , \qquad \mathcal{E} = \kappa \ , \\ \mathcal{J}_1 = w_1 \int_0^{2\pi} \frac{d\sigma}{2\pi} \ \cos^2\psi \ , \qquad \mathcal{J}_2 = w_2 \int_0^{2\pi} \frac{d\sigma}{2\pi} \ \sin^2\psi \ . \end{gathered} \tag{7.11}$$

In view of (7.6), (7.7) we conclude that these two solutions are related by the following analytic continuation:

$$\begin{gathered} \rho \to i\psi \ , \qquad \kappa \to -w_1 \ , \qquad \omega_1 \to -w_2 \ , \qquad w_3 \to -\kappa \ , \\ E \to -J_1 \ , \qquad S \to J_2 \ , \qquad J \to -E \ . \end{gathered} \tag{7.12}$$

Here we are assuming that $\rho(\sigma + 2\pi) = \rho(\sigma)$ *and* $\psi(\sigma + 2\pi) = \psi(\sigma)$, i.e. ψ does not have a winding number. This choice corresponds to a folded string solution (a two-spin generalization of the solution of [8,36]).

The first-order equations in (7.9) and (7.11) are first integrals of the sinh-Gordon and sine-Gordon equations for ρ and ψ, respectively, which are the only non-trivial equations of the Neumann systems that one has to solve in the present two cases: here the related hyperelliptic curve (see section 4.3) reduces to an elliptic one. Indeed, their solutions can be readily expressed in terms of elliptic functions (see below).

One can also directly relate [19] the systems of equations expressing the periodicity condition and the respective energies and spins. In the first case [11] we get, introducing a modular parameter $\mathrm{q} < 0$ related to the maximal value of the radial AdS_5 coordinate ρ_0

$$\mathrm{q} \equiv -\sinh^2\rho_0 = \frac{\kappa^2 - w_3^2}{\kappa^2 - \omega_1^2} < 0 \ , \tag{7.13}$$

$$\sqrt{\kappa^2 - w_3^2} = \frac{2\sqrt{-\mathrm{q}}}{\pi} \, \mathrm{K(q)} \ , \qquad \mathcal{E} = \kappa + \frac{\kappa}{\omega_1} \mathcal{S} = \frac{2\kappa\sqrt{-\mathrm{q}}}{\pi\sqrt{\kappa^2 - w_3^2}} \, \mathrm{E(q)} \ ,$$

where K(q) and E(q) are the standard complete elliptic integrals of the first and the second kind.* Solving for ω_1 and κ in terms of $\mathcal{J}$ and q we

* They are defined by $\mathrm{K(q)} = \int_0^{\frac{\pi}{2}} \frac{d\alpha}{\sqrt{1-\mathrm{q}\sin^2\alpha}}$ and $\mathrm{E(q)} = \int_0^{\frac{\pi}{2}} d\alpha \sqrt{1 - \mathrm{q}\sin^2\alpha}$ and are related to

get the system of two equations for the energy as a function of the spins $\mathcal{E} = \mathcal{E}(\mathcal{S}, \mathcal{J})$ [19]

$$\left(\frac{\mathcal{J}}{\mathrm{K(q)}}\right)^2 - \left(\frac{\mathcal{E}}{\mathrm{E(q)}}\right)^2 = \frac{4}{\pi^2}\,\mathrm{q}\ , \tag{7.14}$$

$$\left(\frac{\mathcal{S}}{\mathrm{K(q)} - \mathrm{E(q)}}\right)^2 - \left(\frac{\mathcal{J}}{\mathrm{K(q)}}\right)^2 = \frac{4}{\pi^2}\,(1 - \mathrm{q})\ , \tag{7.15}$$

where the parameter q is negative for a physical folded solution. The second of these two parametric equations determines q in terms of $\mathcal{S}$ and $\mathcal{J}$, while the first one then gives the energy as a function of the spins.

Similarly, for the (J_1, J_2) solution (7.10) one finds [17] (we assume $\omega_2^2 > \omega_1^2$; here ψ_0 is the maximal value of ψ)

$$\mathrm{q} \equiv \sin^2\psi_0 = \frac{\kappa^2 - w_1^2}{w_2^2 - w_1^2} > 0\ , \qquad 1 = \frac{\mathcal{J}_1}{w_1} + \frac{\mathcal{J}_2}{w_2}, \qquad \mathcal{E} = \kappa\ ,$$
$$\mathcal{J}_1 = \frac{2w_1}{\pi\sqrt{w_2^2 - w_1^2}}\,\mathrm{E(q)}\ , \qquad \sqrt{w_2^2 - w_1^2} = \frac{2}{\pi}\,\mathrm{K(q)}\ . \tag{7.16}$$

The solution of the equation in (7.11) for ψ can be written as follows

$$\cos\psi(\sigma) = r_1(\sigma) = \mathrm{dn}(A\sigma, \mathrm{q})\ , \qquad \sin\psi(\sigma) = r_2(\sigma) = \sqrt{q}\,\mathrm{sn}(A\sigma, \mathrm{q})\ , \tag{7.17}$$

where $r_3(\sigma) = 0$ ($\gamma = \frac{\pi}{2}$, cf. (2.15)), $A \equiv \frac{2}{\pi}\,\mathrm{K(q)}$ and dn and sn are the standard elliptic functions.† Here we end up with a system of two equations determining $\mathcal{E} = \mathcal{E}(\mathcal{J}_1, \mathcal{J}_2)$

$$\left(\frac{\mathcal{E}}{\mathrm{K(q)}}\right)^2 - \left(\frac{\mathcal{J}_1}{\mathrm{E(q)}}\right)^2 = \frac{4}{\pi^2}\,\mathrm{q}\ , \tag{7.18}$$

$$\left(\frac{\mathcal{J}_2}{\mathrm{K(q)} - \mathrm{E(q)}}\right)^2 - \left(\frac{\mathcal{J}_1}{\mathrm{E(q)}}\right)^2 = \frac{4}{\pi^2}\ , \tag{7.19}$$

where $\mathrm{q} > 0$. A manifestation of the analytic continuation relation (7.12) between the two solutions is the equivalence [19] of the systems (7.14), (7.15)

hypergeometric functions ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1, \mathrm{q}) = \frac{2}{\pi}\,\mathrm{K(q)}$, ${}_2F_1(-\frac{1}{2}, \frac{1}{2}; 1; \mathrm{q}) = \frac{2}{\pi}\,\mathrm{E(q)}$. Let us note also that the elliptic integral of the third kind is defined by $\Pi(m^2, \mathrm{q}) = \int_0^{\pi/2} \frac{d\alpha}{(1-m^2\sin^2\alpha)\sqrt{1-\mathrm{q}\sin^2\alpha}}$.

† The Jacobi elliptic function $\mathrm{sn}(u, \mathrm{q})$ is defined by $u = \int_0^{\mathrm{sn}(u,\mathrm{q})} \frac{dy}{\sqrt{(1-y^2)(1-\mathrm{q}y^2)}}$. Equivalently, if $\mathrm{sn}\ u = \sin\phi$ then $u = \int_0^{\phi} \frac{d\alpha}{\sqrt{1-\mathrm{q}\sin^2\alpha}}$. One has also $\mathrm{dn}^2(u, \mathrm{q}) +\ \mathrm{q}\ \mathrm{sn}^2(u, \mathrm{q}) = 1$ and $\mathrm{cn}^2(u, \mathrm{q}) + \mathrm{sn}^2(u, \mathrm{q}) = 1$. These three functions (given by ratios of theta-functions) are meromorphic. Also, $\mathrm{sn}(-u, \mathrm{q}) = -\mathrm{sn}(u, q)$ and $\mathrm{sn}(u + 2I, q) = -\mathrm{sn}(u, q)$, where the half-period is $I = \mathrm{K(q)}$.

and (7.18), (7.19) under the substitution

$$\mathcal{E} \mapsto -\mathcal{J}_1 \ , \qquad \mathcal{S} \mapsto \mathcal{J}_2 \ , \qquad \mathcal{J} \mapsto -\mathcal{E} \ , \tag{7.20}$$

and the analytic continuation from $\mathrm{q} > 0$ to $\mathrm{q} < 0$ in the elliptic integrals.

7.3. *Energy as a function of spins*

Depending on the region of parameter space (or values of the integrals of motion) one finds different functional form of dependence of the energy on the two spins. A direct comparison with gauge theory is possible in the case when the two spins are large compared to $\sqrt{\lambda}$, i.e. $\mathcal{S} \gg 1, \ \mathcal{J} \gg 1$ in the (S,J) case and $\mathcal{J}_1 \gg 1, \ \mathcal{J}_2 \gg 1$ in the (J_1,J_2) case. We can then expand the energies, e.g., in powers of the total S^5 spin $\mathcal{J}$. This amounts to an expansion in (inverse) powers of $\mathcal{J} \equiv \mathcal{J}_3 = \frac{J_3}{\sqrt{\lambda}}$ in the (S,J) case and of $\mathcal{J} \equiv \mathcal{J}_1 + \mathcal{J}_2 = \frac{1}{\sqrt{\lambda}}(J_1+J_2)$ in the (J_1,J_2) case, respectively,

$$E = S+J+\frac{\lambda}{J}\,\tilde{\epsilon}_1\Big(\frac{S}{J}\Big)+\frac{\lambda^2}{J^3}\,\tilde{\epsilon}_2\Big(\frac{S}{J}\Big)+\dots \ , \qquad J \equiv J_3, \ \ J,S \gg \sqrt{\lambda} \ , \tag{7.21}$$

$$E = J+\frac{\lambda}{J}\,\epsilon_1\Big(\frac{J_2}{J}\Big)+\frac{\lambda^2}{J^3}\,\epsilon_2\Big(\frac{J_2}{J}\Big)+\dots \ , \qquad J \equiv J_1+J_2, \ \ J_1,J_2 \gg \sqrt{\lambda} \ . \tag{7.22}$$

The coefficient functions $\tilde{\epsilon}_n$ and ϵ_n in (7.21) and (7.22) (the analogs of c_n in (1.2)) can be related, given that the two solutions are related by the analytic continuation (7.20). By expanding in large spins one finds a simple relation between the leading order ("one-loop") corrections for the energies of the two solutions [19]:

$$\tilde{\epsilon}_1(y) = -\epsilon_1(-y) \ . \tag{7.23}$$

The same relation is obtained also on the gauge theory side [19].

Equation (7.23) follows also directly from (7.14) and (7.16) or the systems (7.14), (7.15) and (7.18), (7.19). In the (J_1,J_2) case, expanding the parameter q for large $\mathcal{J}$ as (with $\mathcal{J}$ being $\mathcal{J}_1+\mathcal{J}_2$)

$$\mathrm{q} = \mathrm{q}_0 + \frac{\mathrm{q}_1}{\mathcal{J}^2} + \frac{\mathrm{q}_2}{\mathcal{J}^4} + \dots \ , \tag{7.24}$$

one finds that q_0 is given by the solution of the transcendental equation

$$\frac{\mathrm{E}(\mathrm{q}_0)}{\mathrm{K}(\mathrm{q}_0)} = 1 - \frac{J_2}{J} \ , \qquad \mathrm{q}_0 = \mathrm{q}_0\Big(\frac{J_2}{J}\Big) \ . \tag{7.25}$$

The rest of the expansion coefficients in q and the energy (7.22) are then determined simply by linear algebra. In particular, one finds [19]

$$\epsilon_1 = \frac{2}{\pi^2}\,\mathrm{K}(\mathrm{q}_0)\big[\mathrm{E}(\mathrm{q}_0) - (1-\mathrm{q}_0)\mathrm{K}(\mathrm{q}_0)\big] \ . \tag{7.26}$$

In the (S,J) case, using the same expansion (7.24) for the corresponding parameter q in (7.13) where now $\mathcal{J}=\mathcal{J}_3$ we find that q_0 satisfies

$$\frac{\mathrm{E}(\mathrm{q}_0)}{\mathrm{K}(\mathrm{q}_0)} = 1 + \frac{S}{J} \ , \qquad\qquad \mathrm{q}_0 = \mathrm{q}_0\Big(\frac{S}{J}\Big) \ , \tag{7.27}$$

and also

$$\tilde\epsilon_1 = -\frac{2}{\pi^2}\,\mathrm{K}(\mathrm{q}_0)\big[\mathrm{E}(\mathrm{q}_0) - (1-\mathrm{q}_0)\mathrm{K}(\mathrm{q}_0)\big] \ . \tag{7.28}$$

Comparing (7.25), (7.26) to (7.27), (7.28) and observing that to the leading order (7.12) implies $J_2 \to S,\ J \to -J$, we indeed observe the relation (7.23), or $(\tilde\epsilon_1)_{\mathrm{q}_0} = -(\epsilon_1)_{-\mathrm{q}_0}$.

Let us now comment on the dependence of the energy on the spins in other regions of the parameter space. Let us start with the (S,J) solution. In the limit of short strings with $\mathcal{J} \ll 1,\ \mathcal{S} \ll 1$ one finds [11]

$$E = \sqrt{J^2 + 2\sqrt{\lambda}\ S} + \dots \ . \tag{7.29}$$

This limit probes a small-curvature region of AdS_5 where $\rho \approx 0$, and where the energy spectrum should thus be approximately the same as in flat space. Indeed, (7.29) is the standard relativistic expression for the energy of a string in flat space moving with momentum $\mathcal{J}$ and rotating in a 2-plane with spin $\mathcal{S}$. If the boost energy is smaller than the rotation one, i.e. if $\mathcal{J}^2 \ll \mathcal{S}$, then we get the flat-space Regge trajectory relation $E \approx \sqrt{2\sqrt{\lambda}\ S} + \frac{J^2}{2\sqrt{2\sqrt{\lambda}\ S}}$. Such an expression (with a non-analytic dependence on λ) cannot be directly compared to SYM theory without computing all quantum string sigma model $\frac{1}{\sqrt{\lambda}}$ corrections and resumming them to get a regular $\lambda \to 0$ limit.

For short strings with $\mathcal{J} \gg 1$ (and thus with $\mathcal{J} \gg \mathcal{S}$)

$$E = J + S + \frac{\lambda S}{2J^2} + \dots \ . \tag{7.30}$$

This corresponds to the BMN limit with S playing the role of the string excitation number [11].[‡] One may also consider a "near BMN" limit $\frac{S}{J} \ll 1$

[‡] The BMN case corresponds to expanding near a point-like string moving along a great circle of S^5. In the limit $J \to \infty$, $\frac{\lambda}{J^2}$=fixed one may drop all but quadratic fluctuation terms in the string action (which becomes then equivalent to the plane-wave [5] action [6] in the light-cone gauge).

of this two-spin solution [11, 19]

$$E = J + S\sqrt{1 + \frac{\lambda}{J^2}} - \frac{\lambda\, S^2}{2J^3} \cdot \frac{1 + \frac{\lambda}{2J^2}}{1 + \frac{\lambda}{J^2}} + \dots \ , \qquad\qquad S \ll J \ . \qquad (7.31)$$

This represents the near BMN limit for a total of S excitations of the oscillation modes with $n = \pm 1$. Thus solving non-linear classical sigma model equations gives the same semiclassical spectrum as expanding the sigma model action near a point-like geodesic and then quantizing the small-fluctuation Lagrangian. We see that there is an overlap between the leading order (large $\sqrt{\lambda}$) quantum spectrum obtained by expanding near S^5-boosted point-like string state with *no* rotation in AdS_5 and a classical spectrum obtained by expanding near a highly boosted *and* rotating string solution. This supports the suggestion [8,11,12] that parts of the semiclassical $AdS_5 \times S^5$ string spectrum can be captured by expanding near different classical string solutions.

Other asymptotic expressions are found when S is large. In this case the string can become very long and approach the boundary of AdS_5, i.e. $\rho_0 \to \infty$. For $\mathcal{J} \ll \ln \mathcal{S},\ \mathcal{S} \gg 1$ one finds $\mathcal{E} \approx \mathcal{S} + \frac{1}{\pi} \ln \mathcal{S} \ + \ \frac{\pi \mathcal{J}^2}{2 \ln \mathcal{S}}$, i.e. [11]

$$E \approx S + \frac{\sqrt{\lambda}}{\pi} \ln \frac{S}{\sqrt{\lambda}} \ + \ \frac{\pi J^2}{2\sqrt{\lambda} \ln \frac{S}{\sqrt{\lambda}}} \ . \qquad (7.32)$$

In the limit of $J = 0$ this reduces to the remarkable $\ln S$ behavior found in [8] for the single-spin AdS_5 rotating string solution in AdS_5. Having only one large AdS_5 spin thus does not lead to an analytic dependence of the energy on λ *and*, not surprisingly, is not enough to suppress quantum string sigma model corrections. Indeed, the one-loop string correction shifts the coefficient of the $\ln S$ term by a constant [11], and, in general, the classical $\sqrt{\lambda}$ coefficient should be replaced by an *"interpolating"* function[§]

$$E = S + f(\lambda) \ln S + \dots \ , \qquad f(\lambda)_{\lambda \gg 1} = \frac{\sqrt{\lambda}}{\pi} + a_1 + \frac{a_2}{\sqrt{\lambda}} + \dots \ . \qquad (7.33)$$

The AdS/CFT correspondence implies that after a resummation $f(\lambda)$ should admit a regular weak-coupling expansion $f(\lambda)_{\lambda \ll 1} = q_1 \lambda + q_2 \lambda^2 + \dots$, with (7.33) reproducing the anomalous dimension of the corresponding gauge theory operators such as $\mathrm{tr}(\bar{\Phi} D^S \Phi) + \dots$ (see [13]).

The energies of fluctuations above the BPS ground state $E = J$ are then determined by the string fluctuation masses given by $\mathrm{m}^2 = \frac{1}{\mathcal{J}^2} = \frac{\lambda}{J^2}$.

[§] The one-loop coefficient computed in [11] is $a_1 \approx -\frac{3}{2\pi} \ln 2$. We use this opportunity to correct factor of 1/2 misprints in equations (6.6) and (6.9) in [11].

In the intermediate case where $\ln \frac{\mathcal{S}}{\mathcal{J}} \ll \mathcal{J} \ll \mathcal{S}$ we get [11]

$$E = S + J + \frac{\lambda}{2\pi^2 J} \ln^2 \frac{S}{J} + \dots \ . \qquad (7.34)$$

In contrast to the large $\mathcal{J}$ limit of the short string (small $\mathcal{S}$) case (7.31) here the third correction term is not related to the BMN-type spectrum: there the boost is large and string oscillations are small, while in the long-string case the spin $\mathcal{S}$ is always larger than the boost parameter $\mathcal{J}$. Equation (7.34) appears to be analytic in λ and, assuming that string loop corrections to the coefficient of the $\ln^2 \frac{S}{J}$ term are suppressed in the limit $S \gg 1, \ J \gg 1$, one could hope to relate the $\frac{\lambda}{2\pi^2 J} \ln^2 \frac{S}{J}$ term to the one-loop anomalous dimension of the gauge theory operators with large spin and large R-charge. Indeed, (7.34) may be viewed as a special case of (7.21), where $\tilde{\epsilon}_1 \approx \frac{1}{2\pi^2} \ln^2 \frac{S}{J}$. This asymptotic behavior is indeed observed on the gauge theory side as a special case of the general relation between the string theory and the gauge theory results for the function $\tilde{\epsilon}_1(\frac{S}{J})$ established in [19]. One concludes, in particular, that the coefficient of the $\ln S$ term in the anomalous dimensions of the corresponding $\mathcal{N} = 4$ SYM operators with large spin *and* large R-charge J is indeed suppressed also at weak 't Hooft coupling.

A similar analysis can be repeated for the (J_1, J_2) solution. The energy of a short string rotating in S^5 with $\mathcal{J}_1 \gg 1, \ \mathcal{J}_1 \gg \mathcal{J}_2$ is given by a BMN type expression (cf. (7.30))

$$E = J + \frac{\lambda J_2}{2 J^2} + \dots , \qquad J_2 \ll J_1 \ , \quad J = J_1 + J_2 \, . \qquad (7.35)$$

The full expression in the near BMN limit is (cf. (7.31))

$$E = J_1 + J_2 \sqrt{1 + \frac{\lambda}{J_1^2}} - \frac{\lambda J_2^2}{2 J_1^3} \cdot \frac{1 + \frac{3\lambda}{2 J_1^2}}{1 + \frac{\lambda}{J_1^2}} + \dots , \qquad J_2 \ll J_1 \ . \qquad (7.36)$$

To compare to the BMN case we may set $J_1 = J$ and then J_2 represents the number of excitations.

Making the string longer corresponds to increasing the spin J_2. For example, at $J_1 = J_2$ we get [17]

$$E = J + c_1 \frac{\lambda}{J} + \dots , \quad c_1 \equiv \epsilon_1\Big(\frac{1}{2}\Big) = 0.356\dots , \qquad J_2 = J_1 = \frac{1}{2} J \ . \ (7.37)$$

When $J_2 \to J = J_1 + J_2$, i.e. J_1 becomes small, the string extends over half a great S^5 circle and [19]

$$E = J + \frac{2\lambda}{\pi^2 J (1 - J_2/J)} + \dots = J + \frac{2\lambda}{\pi^2 J_1} + \dots , \qquad J_2 \approx J \ . \qquad (7.38)$$

The point where $J_2 = J$ can be viewed as a transition point: one half of the string can be unfolded to give a circular string which is discussed below. Alternatively, the case of $J_1 = 0$ can be studied by starting with a single-spin folded rotating S^5 solution with its center of mass at rest at a pole of S^5 [8]. In this case for $\mathcal{J} = \mathcal{J}_2 \gg 1$ one finds [8]

$$E = J + \frac{2}{\pi}\sqrt{\lambda} + ... , \qquad \frac{\sqrt{\lambda}}{J} \ll 1 . \tag{7.39}$$

As in the single-spin AdS_5 case (cf.(7.32)) here the expansion of the energy is not analytic in λ, and one expects that quantum sigma model corrections should promote the subleading $\sqrt{\lambda}$ term into a nontrivial function $h(\lambda) = \frac{2}{\pi}\sqrt{\lambda} + k_1 + \frac{k_2}{\sqrt{\lambda}} + \cdots$.

7.4. *Circular two-spin solution*

In addition to the "homogeneous" circular two-spin solutions discussed in section 6 there are also different circular two-spin solutions of the Neumann system (4.17) with $v_i = 0$ that generalize the "round circle" $J_1 = J_2$ solution of [14] to the case of $J_1 \neq J_2$ [18]. The circular string solution is given by the same ansatz (7.10) as for the folded string but now $\psi(\sigma)$ is assumed to be periodic modulo 2π

$$\psi(\sigma + 2\pi) = \psi(\sigma) + 2\pi k . \tag{7.40}$$

In what follows we shall set the winding number k to be 1. In general, in spherical coordinates (γ, ψ) the equations of motion (7.4) describing this type of string are $\gamma = \frac{\pi}{2}$ and $\psi'' + \frac{1}{2}w_{21}^2 \sin 2\psi = 0$, $w_{21}^2 = w_2^2 - w_1^2$. Integrating once, we get $\psi'^2 = w_{21}^2(\mathrm{q}^{-1} - \sin^2\psi)$, where q is an integration constant. If $\mathrm{q} > 1$, then $\mathrm{q}^{-1} = \sin^2\psi_0$ and this solution describes a *folded* string extending from $-\psi_0$ to ψ_0. If instead $\mathrm{q} < 1$, then there is no turning point where $\psi' = 0$, and the solution describes a circular string extending all the way around the equator $\gamma = \frac{\pi}{2}$ with ψ from 0 to 2π: instead of folding back onto itself, the string wraps completely around a great circle of S^5. In the limit $\mathrm{q} \to 0$, this solution approaches the circular string with $J_1 = J_2$. Thus the parameter q provides an interpolation between the circular and the folded string configurations. Note that after a rescaling $\psi \to \frac{1}{2}\psi$ the equation for ψ describes the plane motion of a pendulum in a gravitational field. Clearly, the rotation of the pendulum requires more energy than the oscillatory motion and this explains why the energy of the circular string is bigger than that of the folded one.

The radial coordinates in (4.2) in the circular case are given by (cf. (7.17))

$$\cos\psi(\sigma) = r_1(\sigma) = \mathrm{sn}(A\sigma, \mathrm{q}) \ , \qquad \sin\psi(\sigma) = r_2(\sigma) = \mathrm{cn}(A\sigma, \mathrm{q}) \ , \quad (7.41)$$

where again $r_3 = 0$ and $A \equiv \frac{2}{\pi}\mathrm{K}(\mathrm{q})$. The set of equations for the energy and spins of this solution is (cf. (7.16)) [18]

$$\mathcal{J}_2 = \frac{w_2}{\mathrm{q}}\left(1 - \frac{\mathrm{E}(\mathrm{q})}{\mathrm{K}(\mathrm{q})}\right) \ , \qquad \mathcal{J}_1 = \frac{w_1}{\mathrm{q}}\left(\mathrm{q} - 1 + \frac{\mathrm{E}(\mathrm{q})}{\mathrm{K}(\mathrm{q})}\right), \quad (7.42)$$

$$\mathcal{E}^2 = w_1^2 + \frac{1}{\mathrm{q}}\,(w_2^2 - w_1^2) \ , \qquad \mathrm{K}(\mathrm{q}) = \frac{\pi}{2}\sqrt{\frac{1}{\mathrm{q}}\,(w_2^2 - w_1^2)} \ . \quad (7.43)$$

Solving for w_1, w_2 we get a system of two equations for $\mathcal{E} = \mathcal{E}(\mathcal{J}_1, \mathcal{J}_2)$ similar to the one in (7.18), (7.19)

$$\left(\frac{\mathcal{E}}{\mathrm{K}(\mathrm{q})}\right)^2 - \left(\frac{\mathrm{q}\mathcal{J}_1}{(1-\mathrm{q})\mathrm{K}(\mathrm{q}) - \mathrm{E}(\mathrm{q})}\right)^2 = \frac{4}{\pi^2} \ , \quad (7.44)$$

$$\left(\frac{\mathrm{q}\mathcal{J}_2}{\mathrm{K}(\mathrm{q}) - \mathrm{E}(\mathrm{q})}\right)^2 - \left(\frac{\mathrm{q}\mathcal{J}_1}{(1-\mathrm{q})\mathrm{K}(\mathrm{q}) - \mathrm{E}(\mathrm{q})}\right)^2 = \frac{4}{\pi^2}\,\mathrm{q} \ . \quad (7.45)$$

Note that the ansatz for the circular solution is symmetric under $\mathcal{J}_1 \leftrightarrow \mathcal{J}_2$,[¶] and this symmetry may be seen by applying a modular transformation to the elliptic integrals [18,19]: $\mathrm{K}(\mathrm{q}) = \sqrt{1-\mathrm{q}'}\ \mathrm{K}(\mathrm{q}')$, $\mathrm{E}(\mathrm{q}) = \frac{\mathrm{E}(q')}{\sqrt{1-\mathrm{q}'}}$, $1-\mathrm{q} = \frac{1}{1-\mathrm{q}'}$.

In the limit when both spins are large we can expand q and the energy in powers of $\frac{1}{\mathcal{J}^2}$, i.e. $\mathrm{q} = \mathrm{q}_0 + \frac{\mathrm{q}_1}{\mathcal{J}^2} + ...$, $\mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2$, and (cf. (7.24), (7.25), (7.26))

$$E = J + \frac{\lambda}{J}\ \hat{\epsilon}_1\Big(\frac{J_2}{J}\Big) + ... \ , \quad (7.46)$$

$$\hat{\epsilon}_1 = \frac{2}{\pi^2}\mathrm{K}(\mathrm{q}_0)\mathrm{E}(\mathrm{q}_0) \ , \qquad \frac{J_2}{J} = \frac{1}{\mathrm{q}_0}\left[1 - \frac{\mathrm{E}(\mathrm{q}_0)}{\mathrm{K}(\mathrm{q}_0)}\right] \ . \quad (7.47)$$

The same relations for $\hat{\epsilon}_1(\frac{J_2}{J})$ in (7.47) were reproduced for the corresponding one-loop anomalous dimension on the gauge theory side [16, 18, 19].

One may also construct other similar solutions, for example by combining a folded string solution in AdS_5 with a folded or circular (J_1, J_2) S^5 solution. In this case the energy will be given by a system of three parametric equations involving $\mathcal{E}, \mathcal{S}, \mathcal{J}_1, \mathcal{J}_2$. Such (S, J_1, J_2) solutions may be related by an analytic continuation to special (J_1, J_2, J_3) solutions.

To conclude, as we have seen on the examples discussed above, to have regular (1.2) dependence of the string energy on $\frac{\lambda}{J^2} \ll 1$ we need at least one

[¶] The direct limit $J_2 = 0$ is not, however, well-defined for the circular (J_1, J_2) case.

large "center-of-mass" momentum in S^5. In such cases quantum sigma model corrections are expected to be suppressed in the limit $J \gg 1$, and thus the classical string energy should represent the exact gauge theory anomalous dimension computed in the limit $J \gg 1$, to all orders in perturbative expansion in λ. This was explicitly verified (for the leading "one-loop" $\frac{\lambda}{J}$ term) for several types of such "regular" spinning string solutions [16, 18, 19, 21].

Having large spins in AdS_5 only or only one large spin (with center of mass being at rest) in S^5 appears not to be enough for the energy to have an expansion in even powers of $\frac{\sqrt{\lambda}}{J}$ (note that the circular (S, J) solution of the NR system discussed in section 6.2 represents an exception from this rule, cf.(6.29)). In these latter cases quantum sigma model corrections are not expected to be suppressed in the large spin limit and thus the classical $\sqrt{\lambda}$ -coefficients of the leading terms in the expansion of the energy should become promoted by the string quantum $\frac{1}{(\sqrt{\lambda}\ )^n}$ corrections to non-trivial "interpolating" functions of λ. The latter should be resummed before one may try to compare to perturbative gauge theory results. Comparing string theory to gauge theory at a *quantitative* level in such cases remains a challenge.

8. Open questions and generalizations

The above discussion of particular string solutions with "regular" expansion of the energy E in powers of $\frac{\lambda}{J^2}$ raises several questions.

Since the direct comparison with gauge theory at present can be done only in the $J \to \infty$ limit and in an expansion in $\frac{\lambda}{J^2} \ll 1$ it would be interesting to classify all possible solutions with such a property (1.2) of the energy. One may also try to derive the general expression for the leading order coefficient in $E = J + \frac{\lambda}{J} c_1 + ...$, i.e. for c_1 as a functional on a space of such solutions. Interesting work in this direction [42] utilizes the observation [41] that the induced world-sheet metric of rotating strings with large J becomes degenerate, and that one can then develop a perturbative expansion near such a world sheet. Deriving equations for the functional c_1 (with the expressions in (6.27), (7.26), (7.28) and (7.47) as special solutions) may help to establish the correspondence with spin chain energy eigenstates in a more universal way than the presently known procedure based on association of a particular Bethe root distribution with a particular string solution [16,19,21].

It remains also to prove that for all solutions with "regular" expansion of the energy in $\frac{\lambda}{J^2} \ll 1$ quantum superstring sigma model corrections are indeed suppressed by extra $\frac{1}{J^n}$ factors as in (1.5). The underlying supersymmetry of the $AdS_5 \times S^5$ string theory is certainly important for that conclusion, and a possible role of asymptotic supersymmetry at finite J and $\lambda \to 0$ observed in [41] for simple S^5 rotating solutions remains to be clarified. The string/gauge theory matching for a pulsating solution in [21] (when the $\lambda \to 0$ limit does not give a BPS state) seems to indicate that suppression of quantum sigma model corrections may occur even under more general conditions.

More generally, as discussed in section 1, the full expression for the classical energy of a "regular" solution $E = \sqrt{\lambda}\, \mathcal{E}(\frac{J}{\sqrt{\lambda}}, ...)$ should be representing the exact dimensions of the corresponding gauge theory operators computed in the large J limit. Here we assume that as for the energy on the string side, the anomalous dimension on the gauge theory side should admit a regular double expansion in $\frac{\lambda}{J^2} < 1$ and $\frac{1}{J} \to 0$, i.e. should have the form (1.7), (1.8). This remains to be proved in general for multi R-charge/spin SYM operators. The full expression for the classical energy should be a solution of some differential equations or an equivalent system of algebraic or transcendental equations involving moduli parameters of the string solutions (cf. (6.22)-(6.24) or (7.14), (7.15) and (7.44), (7.45), see also [34]). Thus, as in the examples discussed above, the full expression for the energy E should be effectively determined by its leading order term c_1. This suggests that it may be possible to derive, in the large J limit, expressions for the corresponding anomalous dimensions in SYM theory which are exact in λ. By analogy with the way the simple square root expression $\sqrt{1 + \frac{\lambda}{J^2} n^2}$ of the near-BPS BMN case was reproduced in [10], one may expect that in the $J \to \infty$ limit there may then be a relation between the values of anomalous dimensions (or, in fact, the expressions for the dilatation operator restricted to a particular subsector of states) at different orders of the expansion in λ.

Let us add that while the full expression for the classical string energy comes from the conformal gauge constraint (and looks like a "relativistic" expression $E^2 = J^2 + 2\sqrt{\lambda}\ c_1 + ...$) the one-loop anomalous dimensions on the SYM side are obtained by solving the quantum spin chain Hamiltonian eigenvalue problem ($\Delta - J = a_1 \frac{\lambda}{J} + ...$), which looks like a first term in a "non-relativistic" expansion. It would be important to understand how the perturbative series on the SYM side can be summed up, i.e. how the one-loop expression for anomalous dimension can be promoted to the full "relativistic" expression without order-by-order analysis of modification of the dilatation operator (interpreted as a generalized "non-local" spin chain Hamiltonian, cf. [44, 45]).

Another interesting problem is to compare subleading terms in the $1/J$

expansion, as was done in the BMN case in [39, 40].* This will involve computing one-loop correction to the classical string energy (1.5) and comparing it with the subleading correction to the "thermodynamic" limit of the one-loop Bethe energies. Note that the one-loop (order λ) SYM result for the anomalous dimension for any value of J, i.e. $q_1(J)$ in (1.7), should represent the sum of all string sigma model loop corrections to the leading $\frac{\lambda}{J}$ term (c_1 coefficient) in (1.2). The subleading $1/J$ terms should be governed by the same integrable structures on the two sides of the duality. For any value of J, one certainly expects that, in view of the conformal invariance of the $AdS_5 \times S^5$ string theory (absence of mass generation), the classical integrability of the $AdS_5 \times S^5$ sigma model [56] should have a direct extension to the quantum level. On the $\mathcal{N} = 4$ SYM side, there are strong indications that the one- and two-loop integrability of the dilatation operator extends to all loop orders [44, 45, 68].

It is important to understand if the precise check of the string theory / gauge theory correspondence in the large spin sector of states may be extended to other semiclassical string states with large oscillation numbers. An indication that this is indeed the case comes from recent work [21]. As was noticed earlier in [38], the circular string oscillating in S^5 (but not in AdS_5) has energy that admits a regular expansion in $\frac{\lambda}{\mathrm{N}^2}$, where N is the oscillation level number. The leading term in this expansion was matched in [16, 21] onto a particular eigenvalue of the corresponding [22] $SO(6)$ spin chain Hamiltonian.

This raises the question of generalization of the rotation ansatz (4.2), (6.1) of the previous sections to include the possibility of string oscillations, i.e. of changing of string shape in time. It is not a priori clear which should be the most general rotation/oscillation ansatz for the σ and τ dependence of the $AdS_5 \times S^5$ coordinates consistent with the full 2-d classical sigma model equations of motion, but for each consistent ansatz one should expect that the 2-d sigma model should again reduce to an integrable 1-d system, whose solutions (and thus their energies) could be found in a relatively explicit way.†

An example is provided by a "2d-dual" version of the rotation ansatz

* Since the classical energy of multi-spin (say (J_1, J_2)) solutions reproduces the near-BMN spectrum in the limit $J_1 \gg J_2$ (see (7.36)), computing the one-loop sigma model correction to the energy would effectively determine the $1/J$ (two-loop, in the BMN case) correction to the BMN spectrum and thus could be compared to the result of [39, 40].

† An alternative to this direct procedure of finding classical solutions may be the semiclassical quantization method used in [21, 35, 38].

(4.2) with τ and σ interchanged (but keeping the AdS_5 time as $t = \kappa\tau$), i.e. [34]

$$\mathrm{X}_i = z_i(\tau)\ e^{im_i\sigma} = r_i(\tau)e^{i\alpha_i(\tau)+im_i\sigma}\ , \qquad\qquad \sum_{i=1}^{3} r_i^2(\tau) = 1\ . \qquad (8.1)$$

In this case the radial directions depend on τ instead of σ and the "frequencies" m_i must take integer values in order to satisfy the closed string periodicity condition. This ansatz describes "oscillating" or "pulsating" S^5 string configurations, special cases of which (with motion in both AdS_5 and S^5) were discussed previously in [8,21,35,38,70]. Since the sigma model Lagrangian (2.6) is formally invariant under $\sigma \leftrightarrow \tau$, the resulting 1-d effective Lagrangian will have essentially the same form as (4.11), (4.15)

$$L = \frac{1}{2}\sum_{i=1}^{3}(\dot{z}_i\dot{z}_i^* - m_i^2 z_i z_i^*) + \frac{1}{2}\Lambda\Big(\sum_{i=1}^{3} z_i z_i^* - 1\Big)\,. \qquad (8.2)$$

Solving for $\dot{\alpha}_i$ as in (4.16) we get $r_i^2\dot{\alpha}_i = \mathcal{J}_i$=const, where the counterparts of the integration constants v_i are now the angular momenta in (4.7). Then we end up with the following analogue of (4.17)

$$L = \frac{1}{2}\sum_{i=1}^{3}\Big(\dot{r}_i^2 - m_i^2 r_i^2 - \frac{\mathcal{J}_i^2}{r_i^2}\Big) + \frac{1}{2}\Lambda\Big(\sum_{i=1}^{3} r_i^2 - 1\Big)\,. \qquad (8.3)$$

Thus pulsating solutions (carrying also 3 spins $\mathcal{J}_i$) are again described by a special Neumann-Rosochatius integrable system [34]. Since the corresponding conformal gauge constraints are also $\tau \leftrightarrow \sigma$ symmetric, they take the form similar to (2.7), (2.8) or (4.19), (4.20): $\kappa^2 = \sum_{i=1}^{3}(\dot{r}_i^2 + m_i^2 r_i^2 + \frac{\mathcal{J}_i^2}{r_i^2})$ and $\sum_{i=1}^{3} m_i\mathcal{J}_i = 0$. One may then look for periodic solutions of the above NR system (8.3) subject to the above constraint, i.e. having finite 1-d energy. The resulting class of pulsating string solutions deserves a detailed study. In the simplest ("elliptic") case reducing to a sine-Gordon type system we may follow [35,38,71] and introduce, as for any periodic solitonic solution, an oscillation "level number" N. In the case of the S^5 pulsating solution in [21,38] the expansion of the energy at large level $\mathrm{N} \gg 1$ appears to be regular in $\frac{\lambda}{\mathrm{N}^2}$ and, moreover, the leading $\frac{\lambda}{\mathrm{N}}$ term in E can be matched onto a particular anomalous dimension on the SYM side [16,21].

One would certainly like to go beyond comparison of particular string states to particular SYM operators and to establish a more general relation between the string sigma model and the dilatation operator on the SYM side, implied by the emergence of similar integrable structures on the two sides [20, 21]. The spin chain Hamiltonian may be associated (in the thermodynamic limit) to an effective coset sigma model with the same global symmetries. Then one may hope to relate this sigma model to (the $J \to \infty$ limit of) the string sigma model by a kind of non-local duality transformation.‡

Finally, one would like also to extend the successes of checking the gauge/string duality in the non-supersymmetric semiclassical sectors of states from $\mathcal{N} = 4$ SYM theory to less supersymmetric gauge theories. As was already mentioned in the introduction, evidence of integrable structures in the high-energy (near-conformal) limit of QCD appeared in [24–28], and so the spin chain relation of the the one-loop dilatation operator of $\mathcal{N} = 4$ SYM theory [22, 23, 33] should have generalizations to other $\mathcal{N} = 1, 2$ supersymmetric theories (and not only in twist 2 sector [72]). On the string side, while finding similar classical rotating string solutions in other less supersymmetric conformal $AdS_5 \times M^5$ (and non-conformal, see, e.g., [36, 73]) backgrounds in type IIB theory or its orbifolds is, in principle, straightforward, it is not clear if the string sigma model corrections to the leading terms in the classical energy are again suppressed in the $J \to \infty$ limit. For example, in the single S^5 spin point-like string (BMN) case in the type 0 string theory setting [74] there is a non-trivial one-loop string correction to the energy of the twisted-sector states (which is non-analytic in $\frac{\lambda}{J^2}$ but going to zero when $\frac{\lambda}{J^2} \to 0$) [75]. Similar corrections are expected also for extended spinning string solutions, complicating direct comparison to perturbative gauge theory results.

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‡ The spatial direction of the spin chain may be interpreted as a "momentum" direction from the point of view of the string sigma model. Note also that there is a known relation between a discrete version of the Neumann model and spin chains [32], but discretizing σ in the string sigma model seems to imply a departure from the planar limit on the dual SYM side.

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