LOW-DIMENSIONAL SISTERS OF SEIBERG–WITTEN EFFECTIVE THEORY

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We consider the theories obtained by dimensional reduction to $D = 1, 2, 3$ of $4D$ supersymmetric Yang–Mills theories and calculate there the effective low-energy lagrangians describing moduli space dynamics — the low-dimensional analogs of the Seiberg–Witten effective lagrangian. The effective theories thus obtained are rather beautiful and interesting from a mathematical viewpoint. In addition, their study allows one to understand better some essential features of $4D$ supersymmetric theories, in particular the non-renormalization theorems.

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1. Introduction

Ian's scientific style had two attractive features: *(i)* his works used, more often than not, rather non-trivial modern mathematical constructions; *(ii)* they were always based on a solid and clear physical idea. This text also represents an exercise (a review of exercises) in "physical mathematics", involving an interplay between purely mathematical geometric constructions and the simple physical notion of effective lagrangians.

Effective lagrangians/hamiltonians arise naturally in theories involving two energy scales. Integrating out the "fast" variables (the degrees of freedom with large characteristic excitation energy), one obtains the effective lagrangian involving only "slow" variables which describes the low–energy dynamics. The classic example is the Born–Oppenheimer effective hamiltonian describing the dynamics of nuclei in a molecule, obtained after integrating out the electronic degrees of freedom. The Euler–Heisenberg effective lagrangian describing nonlinear soft photon interactions, the effective chiral lagrangian for QCD, the Wilsonean renormalized effective lagrangian (where modes with high frequency up to Λ_{UV} are integrated out) all belong to this class.

Another lagrangian in this class is the famous Seiberg–Witten effective lagrangian [1]. Let us remind ourselves of its salient features. Consider the pure $4D$ $\mathcal{N} = 2$ supersymmetric Yang–Mills theory. The lagrangian written in terms of $\mathcal{N} = 1$ superfields is [a]

$$\mathcal{L} \;=\; \frac{1}{g^2}\mathrm{Tr}\left\{\int d^2\theta\, W^\alpha W_\alpha + 2\int d^2\theta d^2\bar\theta\; \Phi e^{-V}\bar\Phi e^V\right\} \tag{1}$$

In the bosonic sector, it includes the gauge field A_μ and a complex scalar ϕ belonging to the adjoint representation of the gauge group,

$$g^2\mathcal{L} \;=\; -\frac{1}{2}\mathrm{Tr}\{F_{\mu\nu}^2\} + 2\mathrm{Tr}\{\mathcal{D}_\mu\bar\phi\mathcal{D}_\mu\phi\} - \mathrm{Tr}\{[\bar\phi,\phi]^2\} \;+\text{fermions} \tag{2}$$

The lagrangian is most economically expressed as (see e.g. [3])

$$\mathcal{L} \;=\; \frac{1}{g^2}\mathrm{Tr}\int d^2\theta d^2\tilde\theta\,\mathcal{W}^2\;, \tag{3}$$

[a] Our convention is close to that of Ref. [2], $\theta^2 = \theta^\alpha\theta_\alpha\,\bar\theta^2 = \theta_{\dot\alpha}\theta^{\dot\alpha}$, $\int d^2\theta\,\theta^2 = \int d^2\bar\theta\,\bar\theta^2 = 1$. In the following we will also use $(\sigma^\mu)_{\alpha\dot\beta} = \{1,\boldsymbol{\tau}\}_{\alpha\dot\beta}$, $(\bar\sigma^\mu)^{\dot\beta\alpha} = \{1,-\boldsymbol{\tau}\}^{\dot\beta\alpha}$. Our Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ differs in sign from Wess' and Bagger's conventions and we include an extra factor of 2 in the definition of V.

where $\mathcal{W}(x_L, \theta_\alpha, \tilde{\theta}_\alpha)$ is an $\mathcal{N}=2$ chiral superfield

$$\mathcal{W} \;=\; \Phi + i\sqrt{2}\tilde{\theta}^\alpha W_\alpha - \frac{\tilde{\theta}^2}{4}\bar{D}^2\left(e^{-V}\bar{\Phi}e^V\right) \;. \tag{4}$$

The superfields $V, W_\alpha, \Phi, \bar{\Phi}$ live in ordinary superspace, $(x, \theta, \bar{\theta})$. Besides the chirality conditions, $\bar{D}^a_{\dot{\alpha}}\mathcal{W} = 0$, the superfield (4) satisfies the constraints

$$D^{a\alpha}D^b_\alpha \mathcal{W} \;=\; \bar{D}^a_{\dot{\alpha}}\bar{D}^{b\dot{\alpha}}\bar{\mathcal{W}} \;, \tag{5}$$

where $a, b =$ (*no tilde, tilde*) are the global $SU(2)$ indices. The superfield $\mathcal{W}$ can be naturally expressed in the framework of the harmonic superspace approach (see the monograph [4] and also [5]), but do not themselves depend on harmonics in the chosen basis.

This theory has (infinitely) many different classical vacua. The supersymmetric vacuum has zero energy. At the classical level, it has zero potential energy. Note that the potential commutator term in (2) vanishes when $[\bar{\phi}, \phi] = 0$, implying that ϕ belongs to the Cartan subalgebra of the corresponding Lie algebra. Factorizing over gauge transformations, this gives r physical complex parameters (where r is the rank of the group) characterizing the classical vacuum moduli space. When quantum corrections are taken into account, one could in principle expect the appearance of a nontrivial effective potential on the moduli space, such that the energy would generically be shifted from zero. For supersymmetric theories, quantum corrections *vanish* at any order of perturbation theory; for the $\mathcal{N}=2$ theory, non-perturbative corrections to the effective potential also vanish. However, corrections to the kinetic part of the lagrangian need not vanish and they do not. The relevant slow variables are r complex parameters ϕ^A, mentioned above, and their $\mathcal{N}=2$ superpartners involving fermions and also r Abelian gauge fields. In the simplest $SU(2)$ case, they can be combined into one $\mathcal{N}=2$ superfield $\mathcal{W} = \phi + \ldots$.[b] The effective Seiberg–Witten lagrangian has the form

$$\mathcal{L} = \int d^4\theta F(\mathcal{W}) + \text{ c.c. } . \tag{6}$$

When expressed in components, this gives a non-trivial metric on moduli space. $F(\mathcal{W})$ is a non-trivial, elliptic function taking account of the instanton contributions, etc. Its asymptotic behavior at large $\mathcal{W}$ is simple, however: $F(\mathcal{W}) = \frac{\mathcal{W}^2}{4\pi^2}\ln\mathcal{W}$. This takes into account only the perturbative corrections, which appear only at the one-loop level.

[b] No spinor or matrix indices here!

This paper is devoted to evaluation of the effective lagrangians in the theories obtained by the dimensional reduction of (2) and also by the dimensional reduction of $\mathcal{N} = 1$ SYM theories.

Let us start by discussing the latter. In four dimensions, pure SYM theories do not possess a vacuum moduli space. The number of quantum vacua is finite, given by the dual Coxeter number (or, equivalently, the adjoint Casimir operator c_V) of the gauge group [6]. However, a moduli space does appear after dimensional reduction. Consider first the theory reduced to $(0+1)$ dimensions. In such a theory, new gauge invariants made of the spatial components of the gauge potential appear. The simplest such invariant is $\mathrm{Tr}\{A_i^2\}$. Indeed, the gauge transformation of A_i^a is reduced now to multiplication by a group matrix O_{ab} and does not involve the derivative term. The tree potential term $\propto \mathrm{Tr}\,[A_i, A_j]^2$ vanishes when $[A_i, A_j] = 0$, i.e. when A_i belongs to the Cartan subalgebra. For $SU(2)$, this means that A_i can be gauge rotated to the form $c_i t^3$. The three variables c_i characterize the vacuum moduli space. For an arbitrary gauge group, the moduli space is characterized by $3r$ parameters.

Consider now the reduction to $(1+1)$ dimensions. Only two components of A_i do not involve the derivative term in their gauge transformation law and we have $2r$ physical moduli space parameters. When reducing to $D = 3$, only one component of the vector potential for each unit of rank is left, but there are also r Abelian gauge fields which are dual in three dimensions to scalars, $\epsilon_{ijk}F_{jk} \leftrightarrow \partial_i \Psi$. Thus, in three dimensions we have $r + r = 2r$ parameters in the vacuum moduli space.

For $\mathcal{N} = 2$ theories, the counting is basically the same, only we have to add $2r$ parameters associated with the scalar fields. In other words, the corresponding effective lagrangians involve $5r$ bosonic degrees of freedom in the $1D$ case and $4r$ degrees of freedom in the $2D$ and $3D$ cases.

The paper is organized as follows. In the next section, we describe the supersymmetric quantum-mechanical models representing effective lagrangians for the theories obtained after reduction to $(0+1)$ dimensions. They represent non-standard (so called *symplectic*) supersymmetric sigma models. They are characterized by a mismatch between the number of bosonic and fermionic degrees of freedom: for example, in the symplectic σ models of the first kind (obtained from $\mathcal{N} = 1$ theories), we have 3 bosonic and 2 fermionic degrees of freedom for each unit of rank, while for symplectic σ models of the second kind (obtained from $\mathcal{N} = 2$ theories), we have $5r$ bosonic and $4r$ fermionic degrees of freedom. We hasten to mention that the number of bosonic and fermionic *quantum states* is still equal, as dictated by supersymmetry. We will explain later why the existence of such an unusual $\mathcal{N} = 2$ sigma model[c] (it is not Kähler !) does not contradict the no-go theorem proven in [7].

In Sect. 3, we discuss 2-dimensional effective theories. The theories obtained from $\mathcal{N} = 1$ $4D$ SYM represent conventional Kähler sigma models. For extended SYM, the effective theories are more interesting since they enjoy $\mathcal{N} = 4$ supersymmetry, but are not hyper-Kähler, belonging to the class of so-called twisted sigma models [8].

Sect. 4 is devoted to $3D$ effective theories. They are hyper–Kähler sigma models. In the simplest $SU(2)$ case, the corresponding target space is the Atiyah–Hitchin manifold (the $(0+1)$ version of this sigma model describes also the dynamic of two BPS monopoles). In the $SU(N)$ case, the target space represents a generalized Atiyah–Hitchin manifold associated with the dynamics of N BPS monopoles.[d] For an arbitrary gauge group, the corresponding hyper–Kähler manifolds (not studied by mathematicians previously) are obtained after certain factorizations (hyper–Kähler reductions) of generalized AH manifolds.

In Sect. 5 we discuss the relationship between effective lagrangians in different dimensions and discuss in detail the *non-renormalization theorems* for $D = 1, 2, 3$ and their relationship to the conventional non-renormalization theorems in four dimensions.

2. $D = 1$: Symplectic Sigma Models

2.1. $\mathcal{N} = 1$

Consider the simplest example, namely massless $\mathcal{N} = 1$ $4D$ SQED, with the lagrangian

$$\mathcal{L} = \frac{1}{2e^2}\int d^2\theta\, W^2 + \int d^4\theta\, \left[\bar{S}\, e^V S + \bar{T}\, e^{-V} T\right] , \tag{7}$$

(S, T are chiral multiplets carrying opposite electric charges) reduced to $(0+1)$ dimensions. The effective lagrangian (determined in [9]) depends on the gauge potentials $A_i(t)$ and their superpartners: the photino fields $\psi_\alpha(t)$, $\alpha = 1, 2$. The charged scalar and spinor fields represent fast variables that

[c] Our counting of $\mathcal{N}$ always refers to a number of minimal supercharge representations in a given dimension. Thus, for $D = 1$, $\mathcal{N}$ counts the number of *complex* supercharges, for $D = 4$ it counts the number of Weyl spinors, etc.

[d] To avoid confusion, we note that they are the standard monopoles of the $O(3)$ Georgi–Glashow model characterized by spatial position and a single $U(1)$ phase.

should be integrated over. Now A_i, the auxiliary field D and the spinor fields ψ_α can be combined in a single $\mathcal{N}=2$ $1D$ superfield [10] (see also [11]) [e]

$$\Gamma_k = A_k + \bar\theta\sigma_k\psi + \bar\psi\sigma_k\theta + \epsilon_{kjp}\dot A_j\bar\theta\sigma_p\theta + D\bar\theta\sigma_k\theta$$
$$+ i(\bar\theta\sigma_k\dot\psi - \dot{\bar\psi}\sigma_k\theta)\bar\theta\theta + \frac{\ddot A_k}{4}\theta^2\bar\theta^2 \; . \tag{8}$$

The field (8) satisfies the constraints

$$D_{(\alpha}\Gamma_{\beta\gamma)} \;=\; 0\,, \quad \bar D_{(\alpha}\Gamma_{\beta\gamma)} = 0\;, \tag{9}$$

where $\Gamma_{\alpha\beta} = \Gamma_{\beta\alpha} \;=\; i(\sigma_k)_\alpha{}^\gamma\epsilon_{\beta\gamma}\Gamma_k$ and

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\bar\theta_\alpha\frac{\partial}{\partial t}\,, \quad \bar D_\alpha = \frac{\partial}{\partial\bar\theta^\alpha} - i\theta_\alpha\frac{\partial}{\partial t} \tag{10}$$

are the covariant derivatives. Actually, Γ_k are nothing but the spatial components of the former $4D$ superconnections

$$\Gamma_\mu = \frac{1}{4}(\bar\sigma_\mu)^{\dot\beta\alpha}\bar D_{\dot\beta}D_\alpha V = \; A_\mu + \dots \;, \tag{11}$$

the covariant background derivatives having the form $\nabla_\mu = \partial_\mu - i\Gamma_\mu$ [12]. In one-dimensional theory, Γ_k is gauge invariant,

$$\delta\Gamma_{\alpha\beta} \;\sim\; (D_\alpha\bar D_\beta + D_\beta\bar D_\alpha)(\Lambda - \bar\Lambda) = 0$$

as follows from the (anti-)chirality of $\Lambda(\bar\Lambda)$ and the $1D$ relationship $\{D_\alpha, \bar D_\beta\} \;=\; 2i\epsilon_{\alpha\beta}\partial_t$.

The effective supersymmetric and gauge–invariant action is presented in the form

$$S \;=\; \int dt \int d^2\theta d^2\bar\theta \; F(\Gamma_k). \tag{12}$$

By construction, it enjoys $\mathcal{N} = 2$ supersymmetry. The lagrangian is expressed in components as follows

$$\mathcal{L} = \frac{h}{2}\dot A_k\dot A_k + \frac{ih}{2}\left(\bar\psi\dot\psi - \dot{\bar\psi}\psi\right) + \frac{\partial_k h}{2}\epsilon_{kjp}\dot A_j\bar\psi\sigma_p\psi$$
$$+\frac{hD^2}{2} - \frac{D\partial_k h}{2}\bar\psi\sigma_k\psi - \frac{\partial^2 h}{8}\bar\psi^2\psi^2\;, \tag{13}$$

[e] We follow the notations of Ref. [5],

$$\bar\theta^\alpha = (\theta_\alpha)^\dagger, \quad \bar\theta\theta = \bar\theta^\alpha\theta_\alpha, \;\; \bar\theta\sigma_k\theta = \bar\theta^\alpha(\sigma_k)_\alpha{}^\beta\theta_\beta$$

and the indices are raised and lowered with the help of the invariant tensors $\epsilon^{\alpha\beta} = -\epsilon_{\alpha\beta}$.

where

$$h(\mathbf{A}) = -\frac{1}{2}\,\partial^2 F(\mathbf{A}) \;. \tag{14}$$

This is a supersymmetric sigma model with conformally-flat $3D$ target space, $ds^2 = hd\mathbf{A}^2$. However, it is not the conventional supersymmetric sigma model associated with the de Rahm complex. The latter has only one pair of complex supercharges $(Q, Q^\dagger) \equiv (d, d^\dagger)$. When the target space represents a Kähler manifold, one can define an extra pair of supercharges (three such extra pairs for hyper–Kähler manifolds), but in our case the target space is 3–dimensional and definitely not Kähler.

One also notices that the number of bosonic and fermionic degrees of freedom are not matched in the usual way: in a conventional sigma model one has a complex fermion for each boson while the lagrangian (13) involves three bosonic dynamic variables and only two fermionic ones. In field theory, where each field is associated with an asymptotic quantum state, such a mismatch would not be allowed by supersymmetry. But in supersymmetric quantum mechanics, there are no problems with a mismatch of this kind: for each nonzero eigenvalue of the hamiltonian we still have two bosonic and two fermionic degenerate states $|n\rangle = \Phi_n(\mathbf{A}, \psi_\alpha)$.[f]

A reader might be somewhat confused at this point. The widely-known theorem [7] seems to assert that $\mathcal{N} = 2$ sigma models can *only be defined* on Kähler manifolds (and $\mathcal{N} = 4$ models only on hyper–Kähler manifolds). However, this theorem relies on two assumptions : *(i)* the theory considered should be a real field theory with at least 2 spacetime dimensions and *(ii)* the kinetic term should have the standard form $\propto g_{ab}\partial_\mu\phi^a\partial_\mu\phi^b$. For quantum mechanics, the first condition is not satisfied and there are no restrictions.

In a standard sigma model, fermions are vectors in the tangent space. In our case, they belong to the spinor representation of $SO(3) \equiv Sp(2)$. We will call this model a symplectic sigma model of the first kind (see below for the second kind).

In our case, the function $F(\Gamma_k)$ has a particular form. At the tree level, $F(\mathbf{\Gamma}) = \mathbf{\Gamma}^2/(6e^2)$ and $h = 1/e^2$. This gives the lagrangian of dimensionally-reduced photodynamics. Let us evaluate the one-loop correction to the metric. To this end, we need to calculate the loops of charged superfields S, T in a gauge background. It is convenient to do this in components. We choose

[f] As is well known, vacuum states with zero energy need not be paired. In Ref. [13], we considered SQM models with non-standard "weak" supersymmetric algebra. For such models, the exact pairing is absent also for the first excited state. But the algebra of all models considered in this paper is standard.

the background $A_i = C_i + E_i t,\ \psi_\alpha = 0$ and calculate the charged scalar and fermion loops. The corresponding contributions to the effective action have the form

$$\Delta S_{\rm eff} = -i \ln \frac{\det^{\frac{1}{2}}(-D^2 I + \frac{i}{2}\sigma_{\mu\nu}F_{\mu\nu})}{\det^{\frac{1}{2}}(-D^2 I)} \ , \tag{15}$$

where I is the 4×4 unity matrix and the identity

$$({\rm i}\not{D})^2 = -D^2 I + \frac{\rm i}{2}\sigma_{\mu\nu}F_{\mu\nu} \ , \tag{16}$$

$\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]$, has been used. We observe that a non-zero correction is due solely to magnetic interactions $\propto \sigma_{\mu\nu}\, F_{\mu\nu}$. If the latter were absent, the fermion and scalar contributions would exactly cancel. This feature is common to all supersymmetric gauge theories, both non-Abelian and Abelian (see [14] for more details). This fact is related to another known fact, namely that when the supersymmetric β function is calculated in an *instanton* background, only the contribution of the zero modes survives [15].

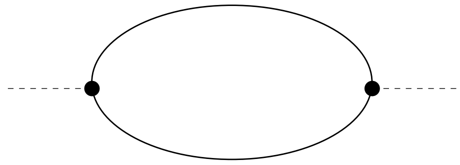

Figure 1. One-loop renormalization of the kinetic term in SQED. The internal lines are Green's functions of the operator $(-D^2)$ with constant $A_i = C_i$. The vertices involve the magnetic interaction $\propto \sigma_{0i}\, E_i$.

To lowest order in $F_{\mu\nu}$ (or E_i), the contribution (15) can be represented by the graph in Fig.1. The constant background $\mathbf{C}$ gives a "mass" to the charged fields and the Euclidean propagator has the form $1/(\omega^2 + \mathbf{C}^2)$. The calculation gives

$$\begin{aligned} \Delta S_{\rm eff} &= -i \cdot i_{\rm Wick} \cdot i^2 \cdot \frac{1}{2} \cdot \left(-\frac{1}{2}\right) \\ \cdot E_j E_k \,{\rm Tr}\{\sigma_{0j}\sigma_{0k}\} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{(\omega^2 + \mathbf{C}^2)^2} &= \frac{\mathbf{E}^2}{4|\mathbf{C}|^3} \ , \end{aligned} \tag{17}$$

where the factor $1/2$ is the power of the determinant and the factor $-1/2$

comes from the expansion

$$\ln \det \|I+\alpha\| = \ln \left[1 - \frac{1}{2}\mathrm{Tr}\,\alpha^2 + \frac{1}{3}\mathrm{Tr}\,\alpha^3 + \ldots\right] \approx -\frac{1}{2}\mathrm{Tr}\,\alpha^2$$

(Tr $\alpha = 0$ in our case). This immediately gives

$$e^2 h(\mathbf{C}) \;=\; 1 + \frac{e^2}{2|\mathbf{C}|^3} + \ldots \tag{18}$$

Let us discuss now non–Abelian theories. In the simplest case of the group $SU(2)$, the moduli space involves the variables $c_k = A_k^3$ and their superpartners, which are combined in the superfield Γ_k^3. The effective action again has the form (12), but the function $F(\mathbf{\Gamma}^3)$ is now different. Like in the Abelian case, it can be determined by calculating the loops of gauge and fermion fields in an Abelian background $A^{\rm cl}(t) = (C_i + E_i t)t^3$ (where C stands not only for "constant", but also for "Cartan").

The graphs are conveniently calculated in the background gauge. We represent $A_\mu = A_\mu^{\rm cl} + \mathcal{A}_\mu$, where $\mathcal{A}_\mu$ is the quantum fluctuation and and add to the Lagrangian the gauge-fixing term

$$-\frac{1}{2g^2}(D_\mu^{\rm cl}\mathcal{A}_\mu)^2 \;, \tag{19}$$

where $D_\mu^{\rm cl} = \partial_\mu - \mathrm{i}\left[A_\mu^{\rm cl},\, \cdot\right]$. The coefficient chosen in Eq. (19) defines the "Feynman background gauge", which is more convenient than others. Adding (19) to the lagrangian and integrating by parts, we obtain for the gauge–field–dependent part of the Lagrangian

$$\mathcal{L}_{\mathcal{A}} = -\frac{1}{2g^2}\,\mathrm{Tr}\,\left(F_{\mu\nu}^2\right) + \frac{1}{g^2}\,\mathrm{Tr}\,\left\{\mathcal{A}_\mu\left(D^2 g_{\mu\nu}\mathcal{A}_\nu - 2i\left[F_{\mu\nu}, \mathcal{A}_\nu\right]\right)\right\} + \ldots\;, \tag{20}$$

where the dots stand for the terms of higher order in $\mathcal{A}_\mu$. The ghost part of the Lagrangian is

$$\mathcal{L}_{ghost} = -2\,\mathrm{Tr}\,\left(\bar{c}D^2 c\right)\; + \text{higher order terms.} \tag{21}$$

Now we can integrate over the quantum fields $\mathcal{A}_\mu$, c and over the fermions using the relation (16). We obtain

$$\delta S_{\rm eff} = -i\ln\left(\frac{\det^{\frac{1}{4}}\left(-D^2\, I + \frac{i}{2}\sigma_{\mu\nu}\left[F_{\mu\nu},\,\cdot\,\right]\right)\det^{1/4}\left(-D^2 I\right)}{\det^{\frac{1}{2}}\left(-D^2\, g_{\mu\nu} + 2i\,\left[F_{\mu\nu},\,\cdot\right]\right)}\right)\;. \tag{22}$$

Again, the result would be zero in the absence of magnetic interactions. In this case, besides the fermion loop, the gauge field loop should also be taken into account. The non-zero commutators, $[F_{\mu\nu},\, \mathcal{A}_\nu]$, $[F_{\mu\nu},\, \lambda_\alpha]$, imply that the quantum fields are charged with respect to the background, i.e. that their color indices a acquire the values $1, 2$.

The fermion loop gives the same contribution to $S_{\rm eff}$ as in the Abelian theory: the power of the determinant is now $1/4$ rather than $1/2$ (the theory involves a Weyl, rather than Dirac, fermion), but this is compensated by the extra color factor 2. One can be convinced that the gauge boson loop contribution involves the factor -4 compared to the fermion one (the factor -2, coming from the power of determinant $-1/2$ vs. $1/4$, is explicitly seen in (22) and another factor 2 comes from spin. This gives

$$g^2 h^{SU(2)}(\mathbf{C}) \ = \ 1 - \frac{3g^2}{2\mathbf{C}^2} + \dots \tag{23}$$

One notices at this point that exactly the same graphs determine the one–loop renormalization of the effective charge in the corresponding $4D$ theories. The only difference is that, in four dimensions, we have to substitute

$$\int \frac{d\omega}{2\pi} \frac{1}{(\omega^2 + \mathbf{C}^2)^2} \ \longrightarrow \ \int \frac{d^2 p}{(2\pi)^4} \frac{1}{(p^2 + \mathbf{C}^2)^2} \ \propto \ln \frac{\Lambda}{|\mathbf{C}|} \ .$$

In other words, the coefficients in (18), (23) are rigidly related to the one–loop β function coefficients in the parent $4D$ theories. Indeed, the β function in non–Abelian SYM theory with $SU(2)$ gauge group involves the factor -3 compared to SQED. We will return to the discussion of this point in the last section.

The metrics (18), (23) and the relation (14) allow one to restore the corresponding prepotentials:

$$\begin{aligned} e^2 F^{\rm SQED}(\mathbf{\Gamma}) &= -\frac{\mathbf{\Gamma}^2}{3} + \frac{e^2 \ln|\mathbf{\Gamma}|}{|\mathbf{\Gamma}|} + ... \\ g^2 F^{SU(2)}(\mathbf{\Gamma}) &= -\frac{\mathbf{\Gamma}^2}{3} - \frac{3g^2 \ln|\mathbf{\Gamma}|}{|\mathbf{\Gamma}|} + ... \end{aligned} \tag{24}$$

(we replaced $\mathbf{\Gamma}^3 \to \mathbf{\Gamma}$ in the non–Abelian case). Consider now an arbitrary simple, compact Lie group. The classical potential energy vanishes when $[A_j, A_k] = 0$, which implies that A_j lies in the Cartan subalgebra (and is effectively Abelian). This gives $3r$ bosonic variables in the effective lagrangian. They are supplemented by $2r$ Abelian gluino variables. These variables are organized in r superfields $\mathbf{\Gamma}^{A=1,\dots,r}$ defined as in Eqs. (8), (9). $\mathbf{\Gamma}^A$ represent dimensionally-reduced Abelian superconnections. Thus, the effective lagrangian has the form[g] $\int d^4\theta F(\mathbf{\Gamma}^A)$ and the only question is what the

[g] It is interesting that such a lagrangian describes also the dynamics of r extremal Reissner–

function $F(\mathbf{\Gamma}^A)$ is. Again, we choose an Abelian gauge field background and perform the calculation over quantum fields. The latter must have non-zero commutators with the background. They are classified according to the *roots* of the corresponding Lie algebra. Actually, we have to add the contributions of the loops corresponding to each such (positive) root. The result is (see [17] for more details)

$$g^2 F(\mathbf{\Gamma}^A) \;=\; -\sum_j \left[\frac{2}{3c_V} \left(\mathbf{\Gamma}^{(j)}\right)^2 + \frac{3g^2}{|\mathbf{\Gamma}^{(j)}|} \ln |\mathbf{\Gamma}^{(j)}| \right] , \tag{25}$$

where $\mathbf{\Gamma}^j = \alpha_j(\mathbf{\Gamma}^A)$ and α_j are the roots. For example, for $SU(3)$ we have the sum of three terms with

$$\alpha_1(\mathbf{\Gamma}^A) = \mathbf{\Gamma}^3 , \quad \alpha_2(\mathbf{\Gamma}^A) = \frac{-\mathbf{\Gamma}^3 + \sqrt{3}\mathbf{\Gamma}^8}{2} , \quad \alpha_3(\mathbf{\Gamma}^A) = \frac{\mathbf{\Gamma}^3 + \sqrt{3}\mathbf{\Gamma}^8}{2} . \tag{26}$$

2.2. $\mathcal{N} = 2$

The same program can be carried out for SQM models obtained by dimensional reduction from $\mathcal{N} = 2$ $4D$ theories. Consider first Abelian theory. $\mathcal{N} = 2$ SQED has the same charged matter content as $\mathcal{N} = 1$ theory, but involves an extra neutral chiral multiplet Φ. The lagrangian acquires two new terms

$$\Delta L \;=\; \int d^4\theta\, \bar{\Phi}\Phi + \left[\sqrt{2}e \int d^2\theta\, \Phi S T + \text{c.c.} \right] . \tag{27}$$

The lowest component of Φ gives two extra degrees of freedom in the vacuum moduli space, which thereby becomes 5–dimensional.[h] The vector superfield V and the chiral superfield Φ can be unified in a single $\mathcal{N} = 4$ (in SQM sense) harmonic gauge superfield and the effective lagrangian can be formulated in the terms of the latter [5]. We use here a more conventional approach, using $\mathcal{N} = 2$ superfields. The effective action depends on $\Gamma_J = (\mathbf{\Gamma}, \sqrt{2}\,\mathrm{Re}\{\Phi\}, \sqrt{2}\,\mathrm{Im}\{\Phi\})$ (interpreted as superconnection in the "grandmother" $6D$ theory) and must have the form

$$S = \int dt \int d^2\theta d^2\bar{\theta}\; \mathcal{K}(\mathbf{\Gamma}, \bar{\Phi}, \Phi) . \tag{28}$$

Nordström black holes (representing classical solutions in $\mathcal{N} = 2$ $4D$ supergravity) [16].

[h] The moduli can be represented as spatial components of the gauge potential in $6D$ SQED, from which the $\mathcal{N} = 2$ $4D$ theory is obtained by dimensional reduction.

Now $\mathcal{N} = 2$ symmetry is manifest here. The action (28) is invariant under the *additional* $\mathcal{N} = 2$ supersymmetry transformations

$$\begin{aligned}
\delta\bar{\Phi} &= \frac{2i}{3}\epsilon^\alpha (\sigma_k)_\alpha^{\ \beta} D_\beta \Gamma_k \,, \\
\delta\Phi &= \frac{2i}{3}\bar{\epsilon}_\alpha (\sigma_k)_\beta^{\ \alpha} \bar{D}^\beta \Gamma_k \,, \\
\delta\Gamma_k &= -i\epsilon^\alpha (\sigma_k)_\alpha^{\ \beta} D_\beta \Phi - i\bar{\epsilon}_\alpha (\sigma_k)_\beta^{\ \alpha} \bar{D}^\beta \bar{\Phi} \,,
\end{aligned} \tag{29}$$

provided that

$$\frac{\partial^2 \mathcal{K}}{\partial \Gamma_k^2} + 2\frac{\partial^2 \mathcal{K}}{\partial \bar{\Phi}\partial\Phi} \equiv \frac{\partial^2 \mathcal{K}}{\partial \Gamma_J^2} \ = \ 0 \ , \tag{30}$$

i.e. $\mathcal{K}$ is a 5–dimensional harmonic function [18]. Unifying $\mathbf{A}$ and $\phi, \bar{\phi}$ in a single 5-dimensional vector A_J and two spinors from the multiplets Γ_k and Φ in a single 4–component complex spinor η_α lying in the fundamental (spinor) representation of $SO(5) \equiv Sp(4)$, we can write the following component expression for the lagrangian [19]

$$\begin{aligned}
\mathcal{L} \ = \ h \left[\frac{1}{2}\dot{A}_J^2 + \frac{i}{2}(\bar{\eta}\dot{\eta} - \dot{\bar{\eta}}\eta) \right] + \frac{i}{2}\partial_J h \dot{A}_K \ \bar{\eta}\sigma_{JK}\eta + \\
\frac{1}{24}\left(2\partial_J\partial_K h - \frac{3}{h}\partial_J h \partial_K h \right) (\bar{\eta}\gamma_J\eta \ \bar{\eta}\gamma_K\eta - \eta C\gamma_J\eta \ \bar{\eta}\gamma_K C\bar{\eta}) \ ,
\end{aligned} \tag{31}$$

where γ_K are 5–dim Dirac matrices, $\sigma_{JK} = (1/2)(\gamma_J\gamma_K - \gamma_K\gamma_J)$ and C is the antisymmetric matrix of charge conjugation, $C\gamma_J^T = -\gamma_J C$. The metric h is related to $\mathcal{K}$ by $h = -(1/2)\partial^2\mathcal{K}/\partial\mathbf{A}^2$.

The lagrangian (31) describes a sigma model defined on a conformally-flat 5–dimensional target space. We will call it a symplectic sigma model of the second kind. A generalized symplectic model of the second kind depends in this approach on r sets of $\mathcal{N} = 2$ superfields $\Gamma_J \equiv (\mathbf{\Gamma}^A, \Phi^A, \bar{\Phi}^A)$. The action

$$S = \int dt \int d^2\theta d^2\bar{\theta} \ \mathcal{K}(\Gamma_J^A) \,. \tag{32}$$

enjoys extended $\mathcal{N} = 4$ supersymmetry, provided the following generalized harmonicity conditions are satisfied [5]

$$\frac{\partial^2 \mathcal{K}}{\partial \Gamma_I^A \partial \Gamma_I^B} \ = \ 0 \ , \qquad\qquad \frac{\partial^2 \mathcal{K}}{\partial \Gamma_I^{[A} \partial \Gamma_J^{B]}} \ = \ 0 \,. \tag{33}$$

In the Abelian case, the effective action has the form (31) with the *same* metric h as in the $\mathcal{N} = 1$ $4D$ SQED case discussed above. Indeed, we can choose the background with zero ϕ, in which case the effective action is given by the graph drawn in Fig.1. Now, $O(5)$ invariance dictates that the metric also has the form (18) in a generic background C_J with $\mathbf{C}^2$ being substituted by C_J^2. The prepotential can be chosen as

$$e^2\mathcal{K} \;=\; -\frac{R^2}{3}+\frac{\rho^2}{2} \;+\; \frac{e^2}{R}\ln\left(R+\sqrt{R^2+\rho^2}\right) \;, \tag{34}$$

where $R^2 = \mathbf{\Gamma}^2$ and $\rho^2 = 2\bar\Phi\Phi$. Note that $\mathcal{K}$ need not be (and is not) $O(5)$ invariant.

In non-Abelian $\mathcal{N}=2$ SYM theory with $SU(2)$ gauge group, the calculations are readily done in the same way as before. The only modification is that there are now two Weyl fermions and an additional adjoint scalar. The ghost determinant is canceled by the adjoint scalar determinant and we obtain

$$\delta S_{\rm eff} = -i\ln\left(\frac{\det^{\frac12}\left(-D^2\, I+\frac{i}{2}\sigma_{\mu\nu}\,[F_{\mu\nu},\,\cdot\,]\right)}{\det^{\frac12}\left(-D^2\, g_{\mu\nu}+2i\;[F_{\mu\nu},\,\cdot]\right)}\right)\;. \tag{35}$$

This gives the expression

$$g^2 h^{SU(2)}_{\mathcal{N}=2}(C_J) \;=\; 1-\frac{g^2}{|C_J|^3} \tag{36}$$

for the metric. The respective coefficients in the correction in the Abelian and non-Abelian cases conform with the respective coefficients in the corresponding $4D$ beta functions.

The structure of the expressions (36) and (23) is similar, but there is one essential difference. Eq. (36) contains no ellipsis! The expression for the metric is *exact*: higher-loop corrections vanish. The proof of this *non-renormalization theorem* is simple. Dimensional counting tells us that an n–loop correction to the metric should be proportional to $(A_J A_J)^{-3n/2}$. But this is not harmonic for $n \geq 2$ and is excluded by supersymmetry requirements. We will discuss the relationship of this non-renormalization theorem to the $4D$ non-renormalization theorem (in $\mathcal{N}=2$ theories two and higher loop contributions to the beta function vanish) in Sect. 5.

We want to emphasize that the absence of the corrections to the metric does not mean the absence of the corrections to the effective lagrangian. The latter involves higher derivative corrections, which do not vanish either at the one–loop or at the two- and higher-loop level [20]. Thus, the singularity of the metric at $A_J^2 = 0$ has no great physical meaning: the effective lagrangian involves uncontrollable higher–derivative corrections there anyway.

The effective lagrangian can also be found for an arbitrary gauge group. Again we have to sum over all positive roots. The prepotential is

$$g^2\mathcal{K} \;=\; -\sum_j \left\{ \frac{2}{3c_V}\left[\left(R^{(j)}\right)^2 - \frac{3}{2}\left(\rho^{(j)}\right)^2\right] + \frac{2g^2}{R^{(j)}} \ln\left[R^{(j)} + \sqrt{\left(R^{(j)}\right)^2 + \left(\rho^{(j)}\right)^2}\right]\right\} , \qquad (37)$$

where $\left(R^{(j)}\right)^2 = \left(\mathbf{\Gamma}^{(j)}\right)^2$, $\left(\rho^{(j)}\right)^2 = 2\bar{\Phi}^{(j)}\Phi^{(j)}$ and $\mathbf{\Gamma}^{(j)} = \alpha_j(\mathbf{\Gamma}^A)$, $\Phi^{(j)} = \alpha_j(\Phi^A)$.

3. $D = 2:$ Kähler and Twisted Models

3.1. $\mathcal{N} = 1:$ *Unfolding the ring*

Consider first Abelian theory. As was noted previously, in two dimensions we have two, rather than three, moduli, representing the components of the gauge potential in the reduced dimensions. The bosonic part of the effective lagrangian can be evaluated in the same way as in the $1D$ case by calculating the loop diagram in Fig.1. The only difference is that the loop integral is now two–dimensional. We obtain

$$e^2\mathcal{L}^{\rm bos}_{\rm eff} \;=\; \frac{(\partial_\alpha A_j)^2}{2}\left[1 + \frac{e^2}{2\pi A_j^2} + \ldots\right] , \qquad (38)$$

$\alpha = 1, 2$ and $j = 1, 2$. This describes a sigma model on a 2–dimensional target space. One can, of course, introduce the complex coordinate $\sigma = (A_1 + iA_2)/\sqrt{2}$.

The full effective lagrangian involves, besides A_j, their supersymmetric partners, which are two-component photino fields. We see that, in this case, there is perfect matching between the number of bosonic and fermionic degrees of freedom. Actually, the Alvarez–Gaume–Freedman theorem [7] dictates that the only two–dimensional $\mathcal{N} = 2$ supersymmetric theory with standard sigma–model kinetic term like in (38) is the supersymmetric Kähler sigma model. The Kähler potential $\mathcal{K}$ ($\mathcal{L} = \int d^4\theta\mathcal{K}$) can be recovered from the metric. In the case under consideration, it can be chosen as

$$e^2\mathcal{K}(\bar{\Phi}, \Phi) \;=\; \bar{\Phi}\Phi + \frac{e^2}{4\pi}\ln\Phi \ln\bar{\Phi} \; . \qquad (39)$$

Now, Φ is a chiral superfield that is related to the gauge–invariant super-connections Γ_j in reduced dimensions in the following way. Consider the superfield $\Sigma = (\Gamma_1 + i\Gamma_2)/\sqrt{2}$. From the definition (11) and the $2D$ anti-commutation relations between D_α and $\bar{D}_{\dot\alpha}$, we deduce that Σ satisfies the constraints

$$\bar{\mathcal{D}}_1 \Sigma \; = \; \mathcal{D}_2 \Sigma = 0 \tag{40}$$

and represents a so-called *twisted* chiral multiplet. It differs from the standard one by a pure convention: Σ is obtained from Φ by interchanging θ_2 and $\bar\theta^2$. This means that the change $\Phi \to \Sigma$, in any standard action involving Φ, would not change anything except the sign due to the change of sign of $d^4\theta$. For example, the tree Lagrangian is expressed as

$$e^2 \mathcal{L}^{2d}_{\rm tree} = -\frac{1}{2}\int d^4\theta (\Gamma_1^2 + \Gamma_2^2) = \; -\int d^4\theta \bar\Sigma \Sigma \; \equiv \; \int d^4\theta \bar\Phi \Phi. \tag{41}$$

It is very instructive to derive the effective $2D$ lagrangian *directly*, elucidating its relationship with the SQM effective lagrangian (13) discussed in the previous section. To do this, consider the original theory, not on R^2 and not on R^1, but rather on $R^1 \times S^1$. By adjusting the length L of the circle, one can interpolate between the $1D$ and $2D$ pictures [14].

The Lagrangian (13) was obtained after integrating out the charged fields in $1D$ theory. Thinking in $1D$ terms, we now have an infinite number of charged fields, representing the coefficients in the Fourier series

$$f(z,t) \; = \; \sum_{n=-\infty}^{\infty} f_n(t) e^{inz/L} \; . \tag{42}$$

The relevant variables in the effective Lagrangian are still the zero Fourier modes of the vector potential $\mathbf{A} \equiv (A_{j=1,2}, A_3)$ and its superpartners. The expression (13) is replaced by the infinite sum [i]

$$e_1^2 \mathcal{L} \; = \; \left[1 + \sum_{n=-\infty}^{\infty} \delta h \left(A_j, A_3 + \frac{2\pi n}{L} \right) \right] (\dot A_j^2 + \dot A_3^2) + \text{ other terms.} \tag{43}$$

In the limit $L \to 0$, only one term in the sum survives and we reproduce the previous $1D$ result (with $\delta h = e_1^2/(2|\mathbf{A}|^3)$). But for large $L \gg e_1^{-2/3}$, all terms are essential. In the limit $L \to \infty$, we can replace the sum by an integral, $\sum_n \longrightarrow \frac{L}{2\pi}\int dA_3$. This integral depends on A_j, but not on A_3: the

[i] The notation e_1 indicates that we are dealing with the coupling constant in $1D$ theory, $[e_1] \sim m^{3/2}$.

expression in square brackets in Eq. (43) gives

$$\tilde{h} \;=\; 1 + \frac{e_2^2}{2\pi A_j^2}\;, \tag{44}$$

with $e_2^2 = e_1^2 L$. This agrees, of course, with (38). Actually, in the limit $L \to \infty$, the effective lagrangian cannot depend on A_3. For large L, the range where A_3 changes is very small, $0 \le A_3 \le 2\pi/L$, and the eigenmodes of the hamiltonian $\Psi_n(A_3) \sim \exp\{inA_3\}$ with $n \neq 0$ acquire large energy and decouple; only the mode $n = 0$ survives. To take this limit carefully, we cannot just set $A_3 = 0$ in Eq. (43), however, but should perform the functional integral of e^{iS} (S is obtained from Eq. (13) by substituting $\tilde{h}$ for h) over $\prod_t dA_3(t)$ first. Doing this and integrating out also the auxiliary field D, we arrive at the result

$$e_2^2 \mathcal{L}_{2D} = \frac{1}{2} g_{jk} \dot{A}^j \dot{A}^k + \frac{i\tilde{h}}{2}\left(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi\right) + i\tilde{h}\omega_j^{ab}\dot{A}^j\bar{\psi}\sigma^{ab}\psi + \\ \frac{1}{8\tilde{h}}\left[(\partial_j\tilde{h})^2 - \tilde{h}(\partial^2\tilde{h})\right](\bar{\psi})^2(\psi)^2\;, \tag{45}$$

where we have raised the index of the vector A^j indicating its contravariant nature, $g_{jk} = \tilde{h}\delta_{jk}$, $\sigma^{ab} = \frac{i}{2}\epsilon^{abc}\sigma^c = \frac{i}{2}\epsilon^{ab}\sigma^3$ $(a, b = 1, 2\ )$ is the generator of rotations in the tangent space, and

$$\omega_i^{ab} = \frac{1}{2}\left[\delta_i^a\,\partial^b \log(\tilde{h}) - \delta_i^b\,\partial^a \log(\tilde{h})\right] \tag{46}$$

is the spin connection on a conformally flat manifold with the natural choice of the zweibein, $e_j^a = \sqrt{\tilde{h}}\delta_j^a$.

In deriving (45), we went over from the lagrangian $\mathcal{L}_{1D}$ to the $2D$ lagrangian density $\mathcal{L}_{2D} = \mathcal{L}_{1D}/L$. (Normally, $\mathcal{L}_{1D}$ is a spatial integral of $\mathcal{L}_{2D}$, but we have dealt up to now only with the terms depending on zero spatial Fourier modes, in which case the spatial integral is reduced to multiplication by L.)

The lagrangian (45) coincides with the standard lagrangian of the Kähler supersymmetric sigma model [21] in the QM limit. In particular, the coefficient of the 4–fermion term represents a $2D$ scalar curvature.[j] The full $(1 + 1)$ effective lagrangian could be obtained by taking into account the

[j] Incidentally, although the bifermion term in (13) can be interpreted in terms of a $3D$ spin connection, the 4–fermion term (before or after integrating out D) is *not* expressed in terms of $3D$ curvature.

higher Fourier harmonics $\propto \exp\{inz/L\}$ of $A_j(z,t)$ and $\psi_\alpha(z,t)$ in the background.

The result (39) can be readily generalized for an arbitrary non–Abelian gauge group. The Kähler potential depends on r complex chiral superfields Φ^A and has the same sum–over–the–roots structure as the $1D$ prepotential in Eq. (25),

$$g^2\mathcal{K}(\Phi^A) \;=\; \sum_j \left[\frac{2}{c_V}\bar\Phi^{(j)}\Phi^{(j)} - \frac{3g^2}{4\pi}\ln\bar\Phi^{(j)}\ln\Phi^{(j)}\right] , \tag{47}$$

where $\Phi^{(j)} = \alpha_j(\Phi^A)$.

3.2. $\mathcal{N} = 2$: *Twisted Sigma Model*

We start again by analyzing the Abelian theory. The effective lagrangian now involves two complex bosonic variables

$$\sigma = (A_1 + iA_2)/\sqrt{2}, \qquad \phi = (A_4 + iA_5)/\sqrt{2} \; . \tag{48}$$

The one–loop calculation brings about a non-trivial metric in the target space $(\sigma, \bar\sigma, \phi, \bar\phi)$. This metric can be related to the SQM 5–dimensional metric by integrating the latter over A_3 in the same way as the Kähler metric (38) was obtained from the metric of the SQM model in the $\mathcal{N} = 1$ case:

$$e^2\, ds^2_{1+1}\big|_{\mathcal{N}=2} \;=\; \left(1 + \int_{-\infty}^{\infty}\frac{dA_3}{2\pi}\delta h_{0+1}\right) \;=\; \left[1 + \frac{e^2}{4\pi(\bar\phi\phi + \bar\sigma\sigma)}\right](2d\bar\sigma d\sigma + 2d\bar\phi d\phi) \; . \tag{49}$$

We expect the effective action to have a σ model form. One might worry at this point, because the metric (49) is not hyper-Kähler (the Ricci tensor and the scalar curvature do not vanish), while the hyper-Kähler property of the metric was shown to be necessary in order for the standard (1+1) σ model to enjoy $\mathcal{N} = 4$ supersymmetry [7]. In our case, $\mathcal{N} = 4$ supersymmetry is present but the metric is not hyper-Kählerian, and this seems to present a paradox. The resolution is that the σ model to hand is not standard [19]. Indeed, the bosonic part of the Lagrangian involves, besides the standard kinetic term $h\left(\partial_\alpha\bar\sigma\partial_\alpha\sigma + \partial_\alpha\bar\phi\partial_\alpha\phi\right)$, the "twisted" term $\propto \epsilon_{\alpha\beta}\partial_\alpha\sigma\partial_\beta\phi$ and $\propto \epsilon_{\alpha\beta}\partial_\alpha\bar\sigma\partial_\beta\bar\phi$. To understand where the twisted term comes from, consider a charged-fermion loop in the background

$$\sigma = \sigma_0 + \sigma_\tau\tau + \sigma_z z, \qquad \phi = \phi_0 + \phi_\tau\tau + \phi_z z \tag{50}$$

(τ is the Euclidean time). The contribution to the effective action is $\propto$ $\ln\det\|\mathfrak{D}\|$, where $\mathfrak{D}$ is the 6–dimensional Euclidean Dirac operator, which can be written in the form

$$\mathfrak{D} \;=\; i\frac{\partial}{\partial\tau} + \gamma_3\frac{\partial}{\partial z} - i(\gamma_1 A_1 + \gamma_2 A_2 + \gamma_4 A_4 + \gamma_5 A_5)\;. \tag{51}$$

Now, if A_4 and A_5 were absent, we could write $\mathfrak{D} = \gamma_4(i\tilde{\gamma}_\mu \mathcal{D}_\mu)$, with

$$\mu = 1,2,3,4;\ \mathcal{D}_4 = \frac{\partial}{\partial\tau},\ \mathcal{D}_3 = \frac{\partial}{\partial z},\ \mathcal{D}_{1,2} = -iA_{1,2};\ \tilde{\gamma_4} = \gamma_4, \tilde{\gamma}_{1,2,3} = -i\gamma_4\gamma_{1,2,3}$$

and then use the squaring trick

$$\det\|\mathfrak{D}\| \;=\; \det\|i\tilde{\gamma}_\mu\mathcal{D}_\mu\| \;=\; \det{}^{1/2}\left\|-\mathcal{D}^2 + \frac{i}{2}\tilde{\sigma}_{\mu\nu}F_{\mu\nu}\right\|\;, \tag{52}$$

with $F_{14} = -\partial A_1/\partial\tau$, etc. The effective action would be proportional to

$$\mathrm{Tr}\{\sigma_{\mu\nu}\sigma_{\alpha\beta}\}F_{\mu\nu}F_{\alpha\beta}\int\frac{d^2p}{4\pi^2}\frac{1}{(p^2+2\bar{\sigma}\sigma)^2}\;\propto F^2_{\mu\nu}\;, \tag{53}$$

which gives the renormalization of the kinetic term, while the twisted term does not appear. The squaring trick also works in the case where $A_{4,5}$ are non-zero, but do not depend on τ, z. Then $2\bar{\phi}\phi$ is just added to $-\mathcal{D}^2$ in Eq. (52) and to $2\bar{\sigma}\sigma$ in Eq. (53), leading to Eq. (49). But in the generic case, the fermion determinant cannot be reduced to $\det^{1/2}\| - \mathcal{D}^2 + \frac{i}{2}\sigma_{\mu\nu}F_{\mu\nu}\|$. The basic reason for this impasse is that one cannot adequately "serve" six components of the gradient with only five γ matrices.[k] As a result, the extra twisted term in the determinant appears.

We need not perform an explicit calculation here, as the twisted and all other terms in the Lagrangian are fixed by supersymmetry. The twisted $\mathcal{N} = 4$ supersymmetric σ model was constructed almost 20 years ago [8]. At that time it did not attract much attention. Recently, there has been a revival of interest in the GHR model: it arose in some string–related problems [22,23]. It also arises as the effective $(1+1)$ Lagrangian in the case under study.

It was shown that, for the $\mathcal{N} = 4$ supersymmetric generalization to be possible, the conformal factor in the metric $h(\bar{\sigma},\sigma,\bar{\phi},\phi)$ should satisfy the harmonicity condition

$$\frac{\partial^2 h}{\partial\bar{\sigma}\partial\sigma} + \frac{\partial^2 h}{\partial\bar{\phi}\partial\phi} \;=\; 0\;. \tag{54}$$

[k] By the same reasoning, the squaring trick does not work for Weyl 2–component fermions in 4 dimensions: the three Pauli matrices that are available in that case are not enough to do the job.

Obviously, (49) satisfies the condition everywhere except at the origin. The relationship between (54) and the 5–dimensional harmonicity condition for the metric in the effective SQM model (31) is also obvious. Indeed, integrating a D–dimensional harmonic function over one of the coordinates, like in (49), we always arrive at a $(D-1)$–dimensional harmonic function.

To construct the full action, consider, along with the standard chiral multiplet Φ satisfying the conditions $\mathcal{D}_\alpha \Phi = 0$, a *twisted* chiral multiplet Σ, which satisfies the constraints (40). As we have seen, the action (depending on only $\bar{\Sigma}$ and Σ) can be expressed in terms of standard chiral multiplets. However, one can write non-trivial Lagrangians involving *both* Φ and Σ. The twisted σ model is determined by the expression

$$\mathcal{L} \;=\; \int d^2\theta d^2\bar{\theta}\; \mathcal{K}(\bar{\Phi}, \Phi; \bar{\Sigma}, \Sigma) \;, \tag{55}$$

where the prepotential $\mathcal{K}$ satisfies the harmonicity condition,

$$\frac{\partial^2 \mathcal{K}}{\partial \bar{\Sigma} \partial \Sigma} + \frac{\partial^2 \mathcal{K}}{\partial \bar{\Phi} \partial \Phi} \;=\; 0 \;. \tag{56}$$

The condition (56) is required if we want the theory to be $\mathcal{N}=4$ supersymmetric. This is best seen by expressing the lagrangian in components [19] and observing that the lagrangian is symmetric under interchange of fermionic variables entering the twisted and untwisted multiplets only for harmonic $\mathcal{K}$. The composition of this discrete symmetry and $\mathcal{N}=2$ supersymmetry, which is manifest in (55), brings about two extra supersymmetries, mixing ϕ and σ with the fermion components of "alien" $\mathcal{N}=2$ multiplets.

One of the possible choices for $\mathcal{K}$ (two functions $\mathcal{K}$ and

$$\mathcal{K}' \;=\; \mathcal{K} + f(\bar{\sigma}, \phi) + \bar{f}(\sigma, \bar{\phi}) + g(\sigma, \phi) + \bar{g}(\bar{\sigma}, \bar{\phi})$$

result, up to a total derivative, in one and the same lagrangian.) leading to the metric (49) is [23, 24]

$$e^2 \mathcal{K} \;=\; \bar{\Sigma}\Sigma - \bar{\Phi}\Phi \;+ \frac{e^2}{4\pi} \left[F\left(\frac{\bar{\Sigma}\Sigma}{\bar{\Phi}\Phi} \right) - \ln \Phi \ln \bar{\Phi} \right] \;, \tag{57}$$

where

$$F(\eta) \;=\; \int_1^\eta \frac{\ln(1+\xi)}{\xi}\, d\xi \tag{58}$$

is the Spence function. This gives, besides (49), a twisted term

$$\mathcal{L}^{\rm twisted} \;=\; -\frac{e^2}{4\pi(\bar{\sigma}\sigma + \bar{\phi}\phi)} \left[\frac{\sigma}{\bar{\phi}} \epsilon_{\alpha\beta} (\partial_\alpha \bar{\sigma})(\partial_\beta \bar{\phi}) + \frac{\bar{\sigma}}{\phi} \epsilon_{\alpha\beta} (\partial_\alpha \sigma)(\partial_\beta \phi) \right] \tag{59}$$

in the lagrangian. The twisted term is a 2–form F. Its external derivative dF can be associated with the torsion. The above–mentioned freedom to choose $\mathcal{K}$ corresponds to adding to F the external derivative of the 1-form generated from the functions $f, \bar{f}, g, \bar{g}$. The torsion is invariant under such a change.

Consider now a generic non–Abelian case. For a simple Lie group of rank r, the effective lagrangian is

$$\mathcal{L} \;=\; \int d^2\theta d^2\bar{\theta}\; \mathcal{K}(\bar{\Phi}^A, \Phi^A; \bar{\Sigma}^A, \Sigma^A)\ , \tag{60}$$

where $A = 1, \ldots, r$ and the expression for $\mathcal{K}$ is derived exactly in the same way as in previous cases. We have

$$\begin{aligned} g^2\mathcal{K} \;&=\; \sum_j \left\{ \frac{2}{c_V} \left[\bar{\Sigma}^{(j)}\Sigma^{(j)} - \bar{\Phi}^{(j)}\Phi^{(j)} \right] \right. \\ &\left. -\frac{g^2}{2\pi} \left[F\left(\frac{\bar{\Sigma}^{(j)}\Sigma^{(j)}}{\bar{\Phi}^{(j)}\Phi^{(j)}} \right) - \ln\Phi^{(j)} \ln\bar{\Phi}^{(j)} \right] \right\}\ , \end{aligned} \tag{61}$$

where $\Sigma^{(j)} \;=\; \alpha_j(\Sigma^A)$, etc. The prepotential (61) satisfies a generalized harmonicity condition

$$\frac{\partial^2\mathcal{K}}{\partial\bar{\Sigma}^A\partial\Sigma^B} + \frac{\partial^2\mathcal{K}}{\partial\bar{\Phi}^A\partial\Phi^B} \;=\; 0 \tag{62}$$

for all A, B.

4. $D = 3$: Kähler and Hyper–Kähler Models

4.1. $\mathcal{N} = 1$: *Dual photon*

The effective lagrangian for $3D$, $\mathcal{N} = 2$ (in $3d$ sense) SQED depends on only one gauge-invariant superconnection in the reduced dimension Γ_3. Its component expansion (in Wess–Bagger notation) is

$$\Gamma_3 \;=\; A_3 - \frac{1}{2}\epsilon_{\mu\rho\alpha}F_{\mu\rho}\,\theta\sigma_\alpha\bar{\theta} - D\theta\sigma_3\bar{\theta} + \frac{1}{4}(\partial^2 A_3)\theta^2\bar{\theta}^2 + \text{fermion terms} \tag{63}$$

where $F_{\mu\rho}$ is the $3D$ electromagnetic field ($\mu, \rho = 0, 1, 2$). The bosonic terms in the effective lagrangian are

$$\mathcal{L} \;=\; \int \mathcal{F}(\Gamma_3)d^4\theta \;=\; h(A_3)\left[\frac{1}{2}(\partial_\mu A_3)^2 - \frac{1}{4}F_{\mu\rho}F_{\mu\rho} + \frac{D^2}{2} \right]\ , \tag{64}$$

where $h = -\mathcal{F}''/2$.

It is convenient at this stage to perform a *duality transformation.* To this end, write the functional integral corresponding to the lagrangian (64) in the form

$$\int \prod dF\, d\Psi\, \exp\left\{ i \int d^3x \left(\mathcal{L} + \frac{1}{2}\epsilon_{\mu\rho\alpha} F_{\mu\rho} \partial_\alpha \Psi \right) \right\} . \tag{65}$$

Integrating this over $\prod d\Psi$ brings about the Bianchi constraints $\epsilon_{\mu\rho\alpha}\partial_\alpha F_{\mu\rho} = 0$, which are solved by the standard relation $F_{\mu\rho} = \partial_{[\mu} A_{\rho]}$. But let us instead do the integral in Eq. (65) over $\prod dF$ first. We are left with

$$\prod d\Psi \exp\left\{ i \int d^3x \left[\frac{h}{2}(\partial_\mu A_3)^2 + \frac{1}{2h}(\partial_\mu \Psi)^2 \right] \right\} . \tag{66}$$

The integrand in the exponent is the dual lagrangian (the bosonic part thereof). The scalar field Ψ is the dual photon.

Let us now introduce the field

$$B \;=\; -\frac{\mathcal{F}'(A_3)}{2}, \tag{67}$$

so that $\partial_\mu B = h\, \partial_\mu A_3$. Introducing a complex variable $\phi = (B + i\Psi)/\sqrt{2}$, we can write $\mathcal{L}_{\rm dual}$ in the Kähler form $\int d^4\theta\, \mathcal{K}(\bar\Phi, \Phi)$. The relation between the Kähler potential $\mathcal{K}$ and the function $\mathcal{F}$ can be inferred from Eq. (67).

For the effective lagrangian of $3D$ SQED, the particular form of the metric and prepotentials $\mathcal{F}$, $\mathcal{K}$ can be found via the "unfolding the ring" procedure. We have to integrate the one–loop correction to the $2D$ metric in (44) over one of the components A_j, in the same way as we earlier integrated the correction to the $1D$ metric to derive (44). We obtain

$$e^2 h_{3D} \;=\; 1 + \frac{e^2}{4\pi |A_3|} + \ldots \tag{68}$$

(with 3–dimensional charge e). The metric is singular at $A_3 = 0$. This point separates two completely independent sectors in the moduli space (and in the theory !) with positive and negative A_3. We will assume for definiteness that A_3 is positive. The prepotential entering (64) can be recovered from the metric as

$$-e^2 \mathcal{F}(\Gamma_3) \;=\; \Gamma_3^2 + \frac{e^2}{2\pi}\Gamma_3 \ln \Gamma_3 + \ldots \;. \tag{69}$$

The Kähler potential depends only on $\Delta = (\bar\Phi + \Phi)/\sqrt{2}$ and is given at the one–loop level by a similar formula, namely

$$\mathcal{K} \;=\; e^2 \left[\Delta^2 + \frac{1}{2\pi} \Delta \ln \Delta \right] . \tag{70}$$

The generalization to the non–Abelian case is straightforward. The generalized Kähler potential is

$$\mathcal{K}(\Delta^A) \;=\; g^2 \sum_j \left\{ \frac{2}{c_V} (\Delta^{(j)})^2 - \frac{3}{2\pi} \Delta^{(j)} \ln \Delta^{(j)} \right\} + \dots \;, \tag{71}$$

where $\Delta^A = (\bar{\Phi}^A + \Phi^A)/\sqrt{2}$ and $\Delta^{(j)} = \alpha_j(\Delta^A)$.

This is not yet the end of the story, however. Considerations of supersymmetry alone do not exclude the presence of a *superpotential* $\sim$ Re $\int d^2\theta \, F(\Delta^A)$ on top of the Kähler potential in the effective lagrangian. Indeed, such a superpotential is generated in non-Abelian $3D$ theories by a *non-perturbative* mechanism [25]. The mechanism is roughly the same as the known instanton mechanism for generating a superpotential in $4D$ $\mathcal{N} = 1$ SYM theory with matter [26]. In three dimensions, instantons are t' Hooft–Polyakov monopoles. They have two fermion "legs" (zero modes) which lead to generation of gluino condensate. The superpotential can be recovered from the condensate. In the simplest $SU(2)$ case, it has the form

$$F(\Delta) \;\sim\; g^4 \exp\left\{ -2\sqrt{2}\pi\Phi \right\}. \tag{72}$$

The superpotential (72) lifts the degeneracy of the valley. Actually, the scalar potential $U \sim \exp\left\{-2\sqrt{2}\pi(\phi + \bar{\phi})\right\}$ corresponding to the superpotential (72) (the exponent $2\sqrt{2}\pi(\phi + \bar{\phi}) = 4\pi A_3/g^2$ is nothing but the $3D$ instanton action) does *not* have a minimum at a finite value of ϕ (as it does not in the massless $\mathcal{N} = 1$ supersymmetric QCD — this is a typical "runaway vacuum" phenomenon). In 4-dimensional SQCD, this can be cured by giving a mass to the matter fields: the supersymmetric vacuum would then occur at a finite value of ϕ. But in the framework given here, the form of the lagrangian of the descendants is dictated by the original $4D$ theory and we have to conclude that the $3D$ $\mathcal{N} = 1$ sister simply does not exist as a consistent theory.

In principle, this could also happen in the $2D$ $\mathcal{N} = 1$ theory, where the appearance of the superpotential is also not excluded by the symmetry considerations. Moreover, non-perturbative instanton solutions also exist in the $2D$ case [27, 28]. They appear in any $2D$ gauge theory involving only adjoint fields, due to the non-triviality of $\pi_1(\text{gauge group})$, by the same token as ordinary BPST instantons in four dimensions appear, due to non-triviality of $\pi_3(G)$. Now, $\pi_1[SU(N)] = 0$, but if only adjoint fields are present, the gauge group is globally $SU(N)/Z_N$ and involves $N - 1$ topologically distinct non-contractible loops and, correspondingly, $N - 1$ different types of instantons.

However, these instantons do not generate a superpotential in this case for two reasons.

- The minima of the classical action are realized on delocalized constant gauge field strength configurations (like in the Schwinger model). The non–Abelian $2D$ instantons do not know about the scalar fields (unlike the monopoles) and cannot lift the degeneracy on the moduli space.
- As was shown in [28], the instantons involve $N-1$ *pairs* of fermion zero modes for each fermion flavor. In our case, there are two flavors and an instanton involves altogether $4(N-1)$ instanton legs, which is too many to generate the fermion condensate and superpotential.

4.2. $\mathcal{N}=2$: *Taub–NUT, Atiyah–Hitchin and their Relatives*

The effective lagrangian here involves $4r$ moduli: three components of the vector-potential in reduced dimensions and a dual photon for each unit of the rank. One way to derive the effective lagrangian is to first determine the effective lagrangian for the theory defined on $R^2 \times S^1$ (it represents a twisted sigma model involving an infinite sequence of the Fourier modes associated with the circle) and unfold the circle as was explained in detail in Sect. 3. When the length L of the circle becomes large, we can replace the sum over the modes by the integral. In the Abelian case, we obtain [29]

$$e_3^2\mathcal{L} \;=\; \left(\frac{1}{2}+\frac{e^2}{8\pi|\mathbf{A}|}\right)\left[(\partial_\mu \mathbf{A})^2+(\partial_\mu \tau)^2\right] \\ -\frac{ie^2}{4\pi}\boldsymbol{\omega}(\mathbf{A})\epsilon_{\mu\nu}\partial_\mu\tau\partial_\nu\mathbf{A}\;, \tag{73}$$

where $g_3^2 = g_2^2 L$ is the 3–dimensional gauge coupling constant, $\mu = 1, 2$ (we have not yet added excited Fourier modes of the slow variables), and $\boldsymbol{\omega}(\mathbf{A}) \equiv \omega(\mathbf{A}) = cos\theta\, d\phi$ is a 1-form which coincides with the Abelian connection describing a Dirac monopole in the space of $\mathbf{A}$. The variables $\mathbf{A}$ live on R^3, whereas the variable τ lives on the dual circle, $0 \le \tau \le 2\pi/L$. The second term in Eq. (73) comes from the twisted term (59).

When L is very large, the size of the dual circle is very small which would normally imply that the excitations related to nonzero Fourier modes of τ become heavy and decouple. This is exactly what happened when we reconstructed the $2D$ effective Lagrangian from the $1D$ one in Sect. 3 according to this method. But in the case under consideration, it would not be correct just to cross out the terms involving $\partial_\mu\tau$: The presence of the twisted term $\propto \epsilon_{\mu\nu}$ prevents us from doing it.

To understand this, consider a trivial toy model,

$$\mathcal{L} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + Bx\dot{y} \implies H = \frac{1}{2}\left[p_x^2 + (p_y - Bx)^2\right] , \tag{74}$$

where $x \in R^1$, while y is restricted to lie on a small circle, $0 \leq y \leq \alpha$. The lagrangian (74) describes a particle living on a cylinder and moving in a constant magnetic field . Now, if the magnetic field B were absent, the higher Fourier modes of the variable y would be heavy and the low–energy spectrum would be continuous, corresponding to free motion along x direction. When $B \neq 0$, for *each* Fourier mode of the variable y, we obtain the *same* oscillatory spectrum. Only the position of the center of the orbit, and not the energy, depends on $p_y^{(n)} = 2\pi n/\alpha$.

Thus, we cannot suppress the variable y in the Lagrangian (74). Likewise, we cannot suppress the variable τ in Eq. (73). What we can do, however, is to trade it with another variable using the duality trick. Performing the same transformations as in the $\mathcal{N} = 1$ case, we arrive at the lagrangian of a sigma model living on a target space with the metric

$$ds^2 \;=\; \left(1 + \frac{e^2}{4\pi|\mathbf{A}|}\right) d\mathbf{A}^2 + \frac{\left(d\Psi - \frac{e^2}{4\pi}\omega\right)^2}{\left(1 + \frac{e^2}{4\pi|\mathbf{A}|}\right)} . \tag{75}$$

The dual variable Ψ describes the dual photon.

We want to emphasize that the effective lagrangian thus obtained represents a *conventional* sigma model — the twisted term disappears after the duality transformation. A conventional $\mathcal{N} = 4$ sigma model must be hyper–Kähler. Indeed, the metric (75) describes a well-known hyper–Kähler Taub–NUT manifold [30].

Consider now SYM theory and let us start with the case of $SU(2)$. The result is immediately written down by substituting $-2g^2$ for e^2 in all above formulae. The metric thus obtained (Taub–NUT with negative mass term) is also hyper–Kähler. However, in contrast to the regular Taub–NUT metric, it is singular at $|\mathbf{A}| = 2\pi/g^2$ and represents an *orbifold*. Note that the singularity occurs at small values of $|\mathbf{A}|$, where the Born–Oppenheimer approximation is not valid. As was explained earlier, we cannot neglect higher–derivative terms in the effective lagrangian in this region. Moreover, the very notion of the effective lagrangian makes no sense any more. Still, the presence of a singularity in the double–derivative part of the lagrangian is somewhat irritating.

Remarkably, the singularity actually disappears when one takes into account *non-perturbative* effects associated with instantons (coinciding with

’t Hooft–Polyakov monopoles). Instantons bring about corrections to the metric of the form $\sim \exp\{-4\pi n|\mathbf{A}|/g^2\}$ (n is the topological charge). They are irrelevant asymptotically, but are very important for small values of $|\mathbf{A}|$: their re-summation gives a *smooth* hyper–Kähler Atiyah–Hitchin metric.[1] There exists an explicit expression for the AH metric. It involves elliptic functions and is not so simple. An interested reader may look it up in [33] where it is shown that its asymptotics coincide with Taub–NUT and that the corrections to these asymptotics are exponential.

Another remarkable fact is that the same AH metric describes the low–energy dynamics of two BPS monopoles [34]. In this case, the vector $\mathbf{A}$ acquires the meaning of the monopole separation $\mathbf{r}$ and Ψ of their relative phase. The (smoothened) singularity occurs when the distance between the monopoles is of the same order as the size of the monopole cores. The classical trajectories of the monopoles represent geodesics on the AH manifold.

In the same way as before, we can write the effective lagrangian for an arbitrary gauge group,

$$g^2\mathcal{L} \;=\; \frac{1}{2}(\partial_\mu \mathbf{A}^A)(\partial_\mu \mathbf{A}^B)Q_{AB} + \frac{1}{2}J^A_\mu Q^{-1}_{AB} J^B_\mu \ , \tag{76}$$

where

$$\begin{aligned} Q_{AB} &= \delta_{AB} - \frac{g^2}{2\pi}\sum_j \frac{\alpha_j^A \alpha_j^B}{|\mathbf{A}^{(j)}|} \ , \\ J^A_\mu &= \partial_\mu \Psi^A + \frac{g^2}{2\pi}\sum_j \boldsymbol{\omega}(\mathbf{A}^{(j)})\partial_\mu \mathbf{A}^{(j)} \alpha_j^A \ , \end{aligned} \tag{77}$$

$\mathbf{A}^{(j)} = \alpha_j(\mathbf{A}^A) \equiv \alpha_j^A \mathbf{A}^A$.

These are asymptotic expressions for the metric. They involve singularities and their structure is complicated. However, re-summation of instanton corrections should patch up these singularities. The result of such a re-summation gives a smooth hyper–Kähler manifold. Let me give arguments in favor of this conclusion.

(i) Consider first the unitary groups [35]. The Cartan subalgebra of $SU(N)$ consists of traceless $N \times N$ diagonal matrices. As far as the effective lagrangian (76) is concerned, we have four such matrices: diag $(\mathbf{A}_1, \ldots, \mathbf{A}_N)$ and diag $(\Psi_1, \ldots, \Psi_N)$, $\sum_m \mathbf{A}_m = \sum_m \Psi_m = 0$. There are $N(N-1)/2$

[1] This was the conjecture of Ref. [31], confirmed by direct evaluation of the one-instanton contribution to the metric in Ref. [32]. (Multi-instanton contributions need not be calculated. The Hyper-Kähler nature of the metric fixes them once the one–instanton contribution is known.)

positive roots, $\alpha_{ml}(\mathbf{A}) = \mathbf{A}_m - \mathbf{A}_l, m < l = 1, \ldots, N$. Substituting it in Eq. (76), we obtain the metric

$$ds^2 = A_{ml} d\mathbf{A}_m d\mathbf{A}_l + A^{-1}_{ml} \Lambda_m \Lambda_l \ , \tag{78}$$

where A is the following $N \times N$ matrix:

$$\begin{aligned} A_{mm} &= 1 - \frac{g^2}{4\pi} \sum_{l \neq m} \frac{1}{|\mathbf{A}_m - \mathbf{A}_l|} && \text{(no summation over } m), \\ A_{ml} &= \frac{g^2}{4\pi |\mathbf{A}_m - \mathbf{A}_l|}, && (m \neq l) \ , \end{aligned} \tag{79}$$

and

$$\Lambda_m = d\Psi_m + \frac{g^2}{4\pi} \sum_{l \neq m} \omega(\mathbf{A}_m - \mathbf{A}_l) \ .$$

This metric happens to describe the dynamics of N well–separated BPS monopoles [34]. $\mathbf{A}_m \equiv \mathbf{r}_m$ and Ψ_i are interpreted as the positions and phases of individual monopoles. The condition $\sum_m \mathbf{r}_m = 0$ (and similarly for phases) means that the trivial center of mass motion is separated out. The classical dynamics is described by the following equations of motion [m]

$$\begin{aligned} \ddot{\mathbf{r}}_l - \frac{g^2}{4\pi} \sum_{m \neq l}^{N} \frac{\ddot{\mathbf{r}}_{lm}}{\mathbf{r}_{ml}} + \frac{g^2}{8\pi} \sum_{l \neq m = 1}^{N} \frac{2\,[\dot{\mathbf{r}}_{ml} \times \mathbf{r}_{ml}] \cdot \dot{\mathbf{r}}_{ml} - \mathbf{r}_{ml}(\dot{\mathbf{r}}^2_{ml})}{r^3_{ml}} & \\ - \frac{g}{4\pi} \sum_{m \neq l} (q_{ml}) \dot{\mathbf{r}}_{ml} \times \frac{\mathbf{r}_{ml}}{r^3_{ml}} + \frac{1}{8\pi} \sum_{m \neq l} \frac{q^2_{ml} \mathbf{r}_{ml}}{r^3_{ml}} &= 0 \ , \\ q_l = g A^{-1}_{lm} \left[\dot{\Psi}_m + \frac{g^2}{4\pi} \sum_{n \neq m} \boldsymbol{\omega}(\mathbf{r}_{nm}) \dot{\mathbf{r}}_{nm} \right] &= \text{const} \ , \end{aligned} \tag{80}$$

where $\mathbf{r}_{ml} = \mathbf{r}_m - \mathbf{r}_l$, $q_{ml} = q_m - q_l$. The equations of motion for the effective lagrangian (76) have a similar form, with time derivatives being replaced by ∂_μ (and $\mathbf{r}$ by $\mathbf{A}$).

The metric (78) is singular for certain small values of the distances between the monopoles $|\mathbf{r}_{ml}|$. These singularities can be patched, however, and with all probability *are* patched by the instanton corrections. A conjecture

[m] Here g is interpreted as the monopole magnetic charge. The equations (80) are classical as far as the variables $\mathbf{r}_l$ are concerned, but the quantization of the dynamic variables Ψ_l has already been carried out. The spectral parameters q_l are quantized to (integer)/g and are interpreted as the electric charges of the corresponding dyons.

of existence and uniqueness can now be formulated: there is only one smooth hyper-Kähler manifold of dimension $4(N-1)$ (a *generalized* Atiyah–Hitchin manifold) with the asymptotics (78). (I bet there is, though, as far as I know, this has not yet been proven mathematically in an absolutely rigorous way.) An explicit expression for the generalized AH metric is not known.

(ii) $Sp(2r)$. There are r long positive roots $\alpha_m(\mathbf{r}) = \mathbf{r}_m$ and $r(r-1)$ short positive roots $\alpha_{ml}(\mathbf{r}) = (\mathbf{r}_m \pm \mathbf{r}_l)/2$ ($m < l = 1, \ldots, r$; $\mathbf{r}_m$ are mutually orthogonal and linearly independent). The metric reads

$$ds^2 = \sum_m (d\mathbf{r}_m)^2 - \frac{g^2}{4\pi} \sum_\pm \sum_{m<l} \frac{(d\mathbf{r}_l \pm d\mathbf{r}_m)^2}{|\mathbf{r}_l \pm \mathbf{r}_m|} - \frac{g^2}{2\pi} \sum_m \frac{(d\mathbf{r}_m)^2}{r_m} + \text{ phase part}$$
$$\equiv Q_{ml} d\mathbf{r}_m d\mathbf{r}_l + \text{ phase part} . \tag{81}$$

The full metric is restored from Eqs. (76, 77).

An important observation is that the corresponding effective Lagrangian (the QM version thereof) is obtained from the effective Lagrangian describing the dynamics of $2r+1$ BPS monopoles numbered by the integers $j = -r, \ldots, r$ by imposing the constraints

$$\begin{aligned} \mathbf{r}_{-r} + \mathbf{r}_r \ &= \cdots = \mathbf{r}_{-1} + \mathbf{r}_1 \ = \ 2\mathbf{r}_0 \ = \ 0 \ , \\ \Psi_{-r} + \Psi_r \ &= \cdots = \Psi_{-1} + \Psi_1 \ = \ 2\Psi_0 \ = \ 0 \ . \end{aligned} \tag{82}$$

We *are* allowed to impose these constraints because they are compatible with the equations of motion (80). The corresponding metric is hyper-Kähler. It has to be, due to $\mathcal{N} = 4$ supersymmetry, absence of the twisted term and the theorem [7]. One can also prove it more directly, reproducing the result (81) by the hyper-Kähler reduction procedure worked out in [36].

(iii) $SO(2r+1)$. The system of roots is the same as for $Sp(2r)$, only the long and short roots are interchanged: there are now $r(r-1)$ long roots $(\mathbf{r}_m \pm \mathbf{r}_l)/\sqrt{2}$ and r short roots $\mathbf{r}_m/\sqrt{2}$. The metric reads

$$ds^2 \ = \ \sum_m (d\mathbf{r}_m)^2 - \frac{g^2}{2\pi\sqrt{2}} \left[\sum_\pm \sum_{m<l} \frac{(d\mathbf{r}_l \pm d\mathbf{r}_m)^2}{|\mathbf{r}_l \pm \mathbf{r}_m|} + \sum_m \frac{(d\mathbf{r}_m)^2}{r_m} \right] + \text{ phase part} . \tag{83}$$

This metric is obtained from the Gibbons-Manton type metric for $2r$ BPS monopoles numbered by the integers $j = -r, \ldots, r,\ j \neq 0$ by imposing the constraints

$$\begin{aligned} \mathbf{r}_{-r} + \mathbf{r}_r \ &= \cdots = \mathbf{r}_{-1} + \mathbf{r}_1 \ = \ 0 \ , \\ \Psi_{-r} + \Psi_r \ &= \cdots = \Psi_{-1} + \Psi_1 \ = \ 0 \end{aligned} \tag{84}$$

and rescaling ds^2 and g^2. The constraints (84) are compatible with the equations of motion.

Note that we obtained the effective Lagrangian for $Sp(2r)$ out of that for $SU(2r+1)$ and not $SU(2r)$, as one could naively expect in view of the embedding $Sp(2r) \subset SU(2r)$. Likewise, the moduli space for $SO(2r+1)$ is obtained out of $SU(2r)$ and not $SU(2r+1)$. This is due to the fact that magnetic charges are coupled to co-roots rather than roots.

The effective lagrangian for $SO(2r)$ can also be readily written. It can also be interpreted in monopole terms and the corresponding manifold is related to a generalized AH manifold for the system of $2r$ monopoles by hyper–Kähler reduction accompanied by a certain mass *deformation* (suppressing interactions between certain monopoles) [29].

The constraints (82), (84) have the form of "mirrors" in the monopole configuration space. In the $Sp(2r)$ case, the mirror passes through one of the monopoles while in the $SO(2r+1)$ case it does not. Such "mirrors" appeared earlier in string–related problems and were christened *orientifolds* [37]. For me, they just represent graphical pictures describing the embedding of the symplectic and orthogonal groups into unitary ones.

(iv) G_2. This is the simplest exceptional group. There are three long ($\mathbf{r}_1 - \mathbf{r}_2, \mathbf{r}_1 - \mathbf{r}_3, \mathbf{r}_2 - \mathbf{r}_3$) and three short ($\mathbf{r}_{1,2,3}$) positive roots (the constraint $\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 = 0$ being imposed). The metric reads

$$ds^2 \;=\; \sum_{m=1}^{3} d\mathbf{r}_m^2 - \frac{g^2}{2\pi}\left(\sum_{m>l=1}^{3} \frac{(d\mathbf{r}_m - d\mathbf{r}_l)^2}{|\mathbf{r}_m - \mathbf{r}_l|} + 3\sum_{m=1}^{3} \frac{d\mathbf{r}_m^2}{|\mathbf{r}_m|}\right) + \ldots\,. \quad (85)$$

It can be obtained out of the metric for $Sp(6)$

$$ds^2 \;=\; \sum_{m=1}^{3} d\mathbf{r}_m^2 - \frac{g^2}{4\pi}\left(\sum_{\pm}\sum_{m>l=1}^{3} \frac{(d\mathbf{r}_m \pm d\mathbf{r}_l)^2}{|\mathbf{r}_m \pm \mathbf{r}_l|} + 2\sum_{m=1}^{3} \frac{d\mathbf{r}_m^2}{|\mathbf{r}_m|}\right) + \ldots \quad (86)$$

by rescaling and imposing the [compatible with $Sp(6)$ equations of motion] constraints

$$\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 \;=\; 0\,, \qquad \Psi_1 + \Psi_2 + \Psi_3 \;=\; 0\,. \quad (87)$$

This "3–fold mirror" is (or could be called) an orientifold of new type. Again, we obtained $\mathcal{L}_{\text{eff}}$ for G_2 out of $\mathcal{L}_{\text{eff}}$ for $Sp(6)$, though G_2 is embedded not into $Sp(6)$, but into the dual algebra $SO(7)$.

The effective Lagrangian for F_4 can be related to the moduli space of 26 monopoles. (26 is the lowest dimension of a unitary group in which F_4 can be embedded. This follows from the fact that the representation **26** of

F_4 has the lowest dimension.) E_6 can be embedded into $SU(27)$ and hence the corresponding effective Lagrangian is related to the moduli space of 27 monopoles. Now, the shortest representation in E_7 has the dimension 56 and we need at least 56 monopoles in this case. Finally, $E_8 \subset SU(248)$ and for this we need 248 monopoles. The moduli space of 248 monopoles can also be used as a universal starting point to describe the dynamics of F_4, E_6 and E_7, if one follows the chain of embeddings $F_4 \subset E_6 \subset E_7 \subset E_8 \subset SU(248)$.

The explicit formulae we have written refer to the asymptotic region where non-perturbative effects are suppressed. The corresponding metrics involve singularities at small $|\mathbf{r}^{(j)}|$. As for the $SU(N)$ case, a reasonable conjecture is that these singularities are sewn up by instantons for any simple Lie group, giving a unique smooth hyper-Kähler metric with the asymptotics

$$ds^2 \ = \ d\mathbf{r}^A Q_{AB} d\mathbf{r}^B + \ldots \ . \tag{88}$$

It is natural to conjecture that this metric is obtained from the multi-monopole Atiyah-Hitchin metrics by the same hyper-Kähler reduction procedure as above. To the best of my knowledge, hyper–Kähler manifolds thus obtained have not been studied before by mathematicians.

5. Non–renormalization Theorems

There are several proofs of the well–known fact that two– and higher–order corrections to the β function in the $4D$ $\mathcal{N}=2$ supersymmetric Yang–Mills theory vanish. We will discuss here two such proofs: *(i)* diagrammatic (historically, this was the first) and *(ii)* the one following from holomorphy.

Supergraphs. The diagrammatic proof is based on the technique of supergraphs. The simplest non-renormalization theorem states that all loop corrections to the *superpotential* (the term $\int d^2\theta F(\Phi_i)$ in the lagrangian, Φ_i are chiral superfields) vanish. We refer the reader to the textbooks [2,12,38] for its proof, recalling in more detail how it is done for gauge couplings (following Refs. [39]), the subject of our interest here.

Consider for simplicity SQED.[n] We explained above [see Eq. (15)] how the one–loop correction to the effective action in any dimension can be evaluated. It does not vanish here. The effective charge in $\mathcal{N}=1$ theory is given by

$$\frac{1}{e^2_{\text{phys}}} \ = \ \frac{1}{e_0^2} + \frac{1}{4\pi^2}\ln\frac{\Lambda_{UV}}{m_0} + \ldots \ , \tag{89}$$

[n] The generalization to the non–Abelian case is relatively straightforward, but it involves some subtleties associated with infra-red singularities of the theory [39, 40] which we do not discuss here.

where m_0 is the *bare* charged field's mass.

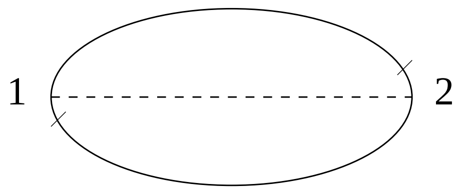

Figure 2. Two-loop contribution to the effective action. Solid lines are chiral field superpropagators $\langle\Phi\bar{\Phi}\rangle$ evaluated in the classical gauge background and the dashed line is the propagator of the (quantum part of the) vector superfield V. The bar on the solid line marks the $\bar{\Phi}$ end.

The relevant two–loop graph is drawn in Fig. 2 (there are actually two such supergraphs giving the same contribution with the superfields S or T in the loop). According to the supergraph Feynman rules [12, 38], each vertex involves the integral $\int d^8z = \int d^4x\, d^2\theta d^2\bar{\theta}$ and the whole contribution of the graph in Fig. 2 is $\int d^4x_1\, d^2\theta_1 d^2\bar{\theta}_1 K$, where

$$\mathcal{K} = \frac{i}{2}\int d^8z_2 \langle\Phi_1\bar{\Phi}_2\rangle\langle\Phi_2\bar{\Phi}_1\rangle\langle v_1 v_2\rangle \ .$$

Here Φ stands for charged chiral superfields S, T, v is quantum vector superfield and $\langle\Phi_1\bar{\Phi}_2\rangle$, $\langle v_1 v_2\rangle$ are quantum superpropagators evaluated in the external background $V_{\rm cl}$. Now, $\langle v_1 v_2\rangle$ does not depend on the external field or its gauge. The charged field propagators are gauge–dependent:

$$\begin{aligned} \langle S_1\bar{S}_2\rangle &\to e^{i\Lambda_1}\langle S_1\bar{S}_2\rangle e^{-i\bar{\Lambda}_2} \ , \\ \langle T_1\bar{T}_2\rangle &\to e^{-i\Lambda_1}\langle T_1\bar{T}_2\rangle e^{i\bar{\Lambda}_2} \ . \end{aligned} \tag{90}$$

The point is, however, that the *integrand* K is gauge–independent and should thereby be *locally* [o] expressed via the gauge–invariant superfield W_α. But W_α is a chiral superfield and the integral over $d^2\bar{\theta}$ of any function of W vanishes. Therefore $\int d^2\theta d^2\bar{\theta}\,\mathcal{K} = 0$, Q.E.D. The same reasoning applies also to an arbitrary multi-loop graph.[p]

[o] Locality follows from the presence of an infra-red cut-off (non-zero mass) in the theory.

[p] To be precise, the integrand could depend on *both* W and $\bar{W}$, in which case the integral $\int d^2\theta d^2\bar{\theta}\,\mathcal{K}$ need not vanish. One can be convinced, however, that such contribution is a supersymmetric generalization of higher derivative $\sim F^4$ terms in the Euler–Heisenberg effective lagrangian. *Such* corrections to the effective lagrangian are indeed present, but this does not affect the renormalization of the gauge coupling.

We hasten to comment that this does *not* mean that multi-loop contributions to β function in $\mathcal{N}=1$ supersymmetric QED vanish. Higher loops appear when expressing m_0, entering Eq. (89), into $m_{\rm phys}$. As was already mentioned above, the physical mass *is* renormalized in spite of the fact that the mass term in the Lagrangian is not. Indeed, the physical mass is defined as the pole of the fermion propagator $\propto 1/(Z\not{p}-m_0)$, where Z describes the renormalization of the *kinetic* term

$$\propto \int d^2\theta d^2\bar\theta \left(\bar S e^V S + \bar T e^{-V} T\right) \ .$$

We have $m_0 = Z m_{\rm phys}$ which leads to an exact relation expressing the charge renormalization via the matter Z factor,

$$\frac{1}{e^2_{\rm phys}} \ = \ \frac{1}{e_0^2} + \frac{1}{4\pi^2}\ln\frac{\Lambda}{m_{\rm phys}} - \frac{1}{4\pi^2}\ln Z \ . \tag{91}$$

In particular, using knowledge of Z at the one–loop level

$$Z \ = \ 1 - \frac{e_0^2}{4\pi^2}\ln\frac{\Lambda}{m_{\rm phys}} \ , \tag{92}$$

we obtain the two–loop renormalization of the charge

$$\frac{1}{e^2_{\rm phys}} \ = \ \frac{1}{e_0^2} + \frac{1}{4\pi^2}\ln\frac{\Lambda}{m_{\rm phys}} + \frac{e_0^2}{16\pi^4}\ln\frac{\Lambda}{m_{\rm phys}} + \dots \ . \tag{93}$$

Now, $\mathcal{N}=2$ supersymmetric electrodynamics involves an extra neutral chiral superfield Υ. The Lagrangian involves its kinetic term and the extra superpotential term $\propto \int d^2\theta\, \Upsilon ST$. The latter is not renormalized: this is the standard F term non-renormalization theorem. The point is that this superpotential term is related by extended supersymmetry to the charged-field kinetic term. Hence, non-renormalization of the superpotential *implies* in $\mathcal{N}=2$ theory non-renormalization of the kinetic term, which implies the absence of the mass renormalization. In other words, in $\mathcal{N}=2$ theory, $m_{\rm phys} = m_0$ and hence only the first term in the β function survives. All $\mathcal{N}=2$ theories with vanishing 1–loop contribution to the β function are finite. The $\mathcal{N}=4$ SYM theory belongs to this class.

The proof just given uses the formalism of $\mathcal{N}=1$ supergraphs. $\mathcal{N}=2$ symmetry is used indirectly via the requirement of equality of renormalization factors for the standard kinetic and superpotential terms. One can also define and calculate supergraphs in $\mathcal{N}=2$ (harmonic) superspace. In this case, the absence of the corrections is manifest [4].

Holomorphy. An alternative elegant proof comes from the analysis of the Seiberg–Witten effective lagrangian. As was mentioned in the beginning, $\mathcal{N} = 2$ supersymmetry dictates the form (6) for the effective lagrangian, where $F(\mathcal{W})$ is an *holomorphic* function of the $\mathcal{N} = 2$ superfield (4) (the Abelian version thereof). Consider this function for large $\mathcal{W}$. Going around the large circle (multiplying $\mathcal{W}$ by $e^{2i\pi}$), we should obtain the *same* theory.[q] Knowing that, asymptotically,

$$\mathcal{L}_{\text{eff}} \sim \text{Re} \int d^4\theta \, \frac{1}{2g^2(\mathcal{W})} \, \mathcal{W}^2 \tag{94}$$

(This is written for non–Abelian $SU(2)$ theory. The same expression with $e^2(\mathcal{W})$ substituted for $g^2(\mathcal{W})$ holds in $\mathcal{N} = 2$ SQED), only two possibilities are allowed: *(i)* $g^2(\mathcal{W})$ is a constant (this possibility is realized for $\mathcal{N} = 4$ non–Abelian gauge theories) or *(ii)* . $g^{-2}(\mathcal{W})$ involves a term $\sim \ln \mathcal{W}$, which corresponds to one–loop renormalization. When multiplying $\mathcal{W}$ by $e^{2i\pi}$, the logarithm is shifted by an imaginary constant. This gives a change $\sim \text{Im} \int d^4\theta \, \mathcal{W}^2$ in the lagrangian, which is a total derivative (θ term). In Abelian theory, the θ term is never relevant. In non–Abelian theory it *might* have been relevant, but it is not in this case: one can be convinced that multiplying $\mathcal{W}^2$ by $e^{2i\pi}$ amounts to the shift $\theta \to \theta + 4\pi$.

The higher–order coefficients β_2, β_3, etc should vanish. Were, e.g., β_2 nonzero, the coefficient $g^{-2}(\mathcal{W})$ in $\mathcal{L}_{\text{eff}}$ would involve the contribution $\sim \ln|1 + c\ln(\Lambda_{\text{UV}}/\mathcal{W})|$, which is not holomorphic and not allowed. On the other hand, nothing prevents the function $f(\mathcal{W})$ from having contributions $\sim \mathcal{W}^{-n}$, which vanish asymptotically. Indeed, $f(\mathcal{W})$ *does* involve such contributions brought about by instantons [1].

We have seen that low-dimensional sisters of $4D$ $\mathcal{N} = 2$ have similar properties: the perturbative corrections to the effective lagrangians vanish beyond one loop. In sect. 2.2 we explained why: extended supersymmetry dictates a special form for the prepotentials. In $1D$, resp. $2D$ theories, the prepotentials in (28), resp. (55) living in $\boldsymbol{R}^5$, resp. $\boldsymbol{R}^4$ must be harmonic functions of their arguments . For $3D$ theories, extended supersymmetry requires the metric to be hyper–Kähler. In the asymptotics, the metric (75) involves a harmonic function $1 + e^2/(4\pi|\mathbf{A}|)$ living in $\boldsymbol{R}^3$. A more detailed analysis shows that the harmonicity follows from the hyper–Kähler nature of the metric (i.e. extended supersymmetry) and from its $U(1)$ isometry corresponding to shifting the phase Ψ (this isometry shows up in the asymp-

[q] Actually, it is sufficient to multiply $\mathcal{W}$ by $e^{i\pi}$. It is $\mathcal{W}^2$ rather than $\mathcal{W}$ which has direct physical meaning, the lowest component of $\mathcal{W}^2$ coinciding with the true moduli $u = \text{Tr}\,\phi^2$.

totics). Actually, in the cases when such an isometry is present, the Kähler potential of a hyper–Kähler metric can be obtained from a certain 3-dim harmonic prepotential by a Legendre (physically – by duality) transformation [24, 29]. [r]

Now, in $4D$ theories the moduli space is $\boldsymbol{R}^2 \equiv \boldsymbol{C}^1$ and harmonicity there is the same as analyticity! In other words, the proof of the $4D$ non-renormalization theorem based on holomorphy has direct low–dimensional counterparts. Nonrenormalizability is a family property of all sisters.

What about the diagrammatic proof ?

The $4D$ diagrammatic proof quoted above involved two parts: *(i)* the $\mathcal{N} = 1$ non-renormalization theorem and *(ii)* the $\mathcal{N} = 2$ relationship between the kinetic and superpotential terms. This relationship holds also in low dimensions, but there is no $\mathcal{N} = 1$ non-renormalization theorem anymore. Indeed, the theorem was based on the the fact that the $4D$ effective lagrangian had the form $\int d^2\theta\, W^2$, while the two- and higher-loop supergraphs suggested the form $\int d^2\theta d^2\bar\theta\, X(W, \bar W)$, which could be only reconciled if $X = 0$. But in low dimensions, the effective lagrangian does not have a chiral form, but instead represents an integral $\int d^2\theta d^2\bar\theta$ of a local density depending not on W, but rather on superconnections Γ_k in reduced dimensions [see e.g. Eq. (12)]. This can well be reconciled with what follows from the diagram in Fig. 2.

Indeed, direct component calculations of the two–loop corrections to the effective action in the $D = 1$, $\mathcal{N} = 1$ Abelian theory showed that they do not vanish [42]. One obtains instead of Eq. (18)

$$e^2 h(\mathbf{C}) \;=\; 1 + \frac{e^2}{2|\mathbf{C}|^3} - \frac{3e^4}{4|\mathbf{C}|^6} + \ldots \tag{95}$$

for the metric. In addition, the two–loop contribution is not related to any Z-factor, unlike in four dimensions: the latter just cannot be defined in quantum mechanics.

In Ref. [42], the result (95) was obtained after rather cumbersome calculations where the contribution of several graphs was added. Using the $\mathcal{N} = 1$ supergraph technique, only one graph in Fig. 2 should be evaluated and the calculation is rather simple [40]. In $\mathcal{N} = 2$ theory, a similar graph with the exchange of Υ field should be added. It has exactly the same structure and

[r] The full Atiyah–Hitchin metric, which involves, besides one loop, also non-perturbative instanton corrections does not have this isometry and cannot be expressed via a 3-dim harmonic function. It can be expressed, however, via certain more complicated generalized harmonic functions [41]. Their physical meaning is yet to be revealed.

gives exactly the same contribution, but with the opposite sign. The cancellation is manifest. Unfortunately, this simple cancellation pattern does not hold at the 3–loop level and higher. Again, one obtains zero after adding up several different supergraphs. In other words, it is hardly possible to prove non-renormalization of $\mathcal{N} = 2$ theories using the formalism of $\mathcal{N} = 1$ supergraphs.

On the other hand, it is very reasonable to suppose that the diagrammatic proof of non–renormalizability based on the technique of harmonic supergraphs [4] can be extended to low dimensions. This question is currently under study.

6. Conclusions

To make distinction with the Introduction where main results concerning the nature and character of different sisters were outlined in words and the main body of the paper where the relevant formulae were written, we give here the same information in the form of a table.

Table 1. Pure SYM: the family of effective theories.

	$\mathcal{N} = 1$	$\mathcal{N} = 2$
$D = 1$	Symplectic σ model of the first kind	Symplectic σ model of the second kind
$D = 2$	Kähler σ model	Twisted σ model (GHR)
$D = 3$	Kähler σ model with superpotential. Runaway vacuum	Hyper–Kähler σ model
$D = 4$	No moduli space. Discrete vacua	SW effective theory

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