BLACK HOLES AND THERMODYNAMIC STATES OF MATRIX MODELS

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Some recent work on the thermodynamic behavior of the matrix model of M-theory on a pp-wave background is reviewed. We examine a weak coupling limit where computations can be done explicitly. In the large N limit, we find a phase transition between two distinct phases which resembles a "confinement-deconfinement" transition in gauge theory and which we speculate must be related to a geometric transition in M-theory. We review arguments that the phase transition is also related to the Hagedorn transition of little string theory in a certain limit of the 5-brane geometry.

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In Memory of Ian Kogan

1. Prologue

Ian Kogan was a great and devoted friend and I will miss him dearly. Part of his journey from the Soviet Union to Oxford passed through Vancouver where he spent a few years. I have great memories of that time.

Among Ian's very broad range of scientific interests was a continuing fascination with critical behavior of string theory at high temperature. One universal characteristic of strings is that their density of states increases exponentially at large energies,

$$\rho(E) = E^{\alpha} e^{E/T_H} \,. \tag{1}$$

The constant T_H is called the Hagedorn temperature. A consequence of this large density of states is that, depending on the exponent, a, string theory has either an upper limiting temperature or a phase transition.

Perhaps my favorite of all of Ian's scientific works is an old result [1], (found independently in [2]) about an interpretation of the Hagedorn temperature in string theory. In that work he noted an analogy between the Hagedorn behavior of strings and the Kosterlitz–Thouless phase transition for the unbinding of vortices in the world-sheet sigma model. He interpreted the latter in string theory as a disintegration of the worldsheet by condensation of vortices. The phase transition occurred when a wrapped mode of the string becomes tachyonic. To this day, this remains a common way in which Hagedorn temperatures are identified in string theory. Characteristic of Ian's work, this very original and fascinating idea seemed well ahead of its time, In all likelihood, its full import has yet to be realized.

Ian's interest in the high temperature behavior of strings continued throughout his career. Some of his recent work explored the use of the AdS/CFT correspondence to understand the phase structure of string theory at high temperatures [3–5].

In this Paper, which I dedicate to Ian, I will discuss some of my own recent work on similar topics. But before I begin a discussion of my own work, let me pause to recall Ian's work on the Hagedorn transition.

1.1. *Hagedorn transition of string theory.*

As an illustration of this beautiful idea, let us consider the closed Bosonic string theory at a finite temperature T. As usual, such a theory is described by quantization on a Euclidean target space, where the time direction is identified with period $\beta = 1/T$,

$$X^0(z,\bar{z}) \sim X^0(z,\bar{z}) + \beta\,. \tag{2}$$

The action of the string theory is

$$S = \frac{1}{4\pi\alpha'}\int d\sigma d\tau\ \partial_a X^\mu\,\partial_a X^\mu\,. \tag{3}$$

If the variable X^0 is periodic, we can think of this as a sigma model with target space S^1, a continuum version of the two dimensional X-Y-model. It is known that this model has two phases. One is a conformal invariant phase which normally describes string theory. The other is a gapped phase which does not have long ranged correlations. They are separated by the Berezinski–Kosterlitz–Thouless phase transition. This phase transition occurs if the parameter $\beta/\sqrt{\alpha'}$ is adjusted in such a way that the simplest periodic interaction that one could add to (3)

$$\int d\sigma d\tau \mu^2 \cos\frac{2\pi X^0}{\beta} \tag{4}$$

becomes a relevant operator. This radius is not universal, but in the weak coupling limit, it has the value $\beta_c = \pi\sqrt{\alpha'}$. We will discuss later how this is related to Hagedorn.

The effect of compactification of is to discretize the momentum of the compactified coordinate,

$$p^0 = 2\pi n/\beta\,.$$

Another result of compacitification is to introduce wrapping modes,

$$X^0(\tau,\sigma+2\pi) = X^0(\tau,\sigma) + w\beta\,,$$

where w is an integer corresponding to the number of times that the string worldsheet wraps around the time circle.

The equation of motion for X^0 is solved by decomposing it into a holomorphic and anti-holomorphic part,

$$X^0(z,\bar{z}) = X^0_L(z) + X^0_R(\bar{z})\,. \tag{5}$$

The left- and right-moving bosons are operators defined by the mode expansions:

$$X_L^0(z) = x_L^0 - i\alpha' p_L^0 \ln z + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\alpha_n}{n} z^{-n}\,, \tag{6}$$

$$X_R^0(\bar{z}) = x_R^0 - i\alpha' p_R^0 \ln \bar{z} + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\tilde{\alpha}_n}{n} \bar{z}^{-n}\,. \tag{7}$$

Here, we use the notation $z = e^{\tau + i\sigma}, \bar{z} = e^{\tau - i\sigma}$. The non-vanishing commutators are

$$[x_L, p_L] = i/2\,, \qquad [x_R, p_R] = i/2\,, \tag{8}$$

$$[\alpha_m, \alpha_n] = m\delta_{m+n}\,, \qquad [\tilde{\alpha}_m, \tilde{\alpha}_n] = m\delta_{m+n}\,. \tag{9}$$

The equal-time commutation relation in Euclidean space-time is

$$\left[X^0(\tau,\sigma), \partial_\tau X^0(\tau,\sigma')\right] = 2\pi\delta(\sigma - \sigma')\,. \tag{10}$$

A representation of this algebra is constructed beginning with the vacuum, which is annihilated by all positively moded oscillators with fixed momentum and wrapping number,

$$\alpha_n \left|0, n, m\right\rangle = 0\,, \quad \tilde{\alpha}_n \left|0, n, w\right\rangle = 0\,, \quad n > 0\,, \tag{11}$$

$$(p_L + p_R)\left|0, n, w\right\rangle = \frac{2\pi}{\beta} n \left|0, n, w\right\rangle\,, \tag{12}$$

$$(p_L - p_R)\left|0, n, w\right\rangle = \frac{\beta}{2\pi\alpha'} w \left|0, n, w\right\rangle \tag{13}$$

and creating excited states using the negatively moded oscillators. These states obey the mass-shell and level-matching conditions

$$m^2\alpha' = \frac{4\pi^2\alpha' n^2}{\beta^2} + \frac{\beta^2}{4\pi^2\alpha'} w^2 + 2(N + \tilde{N} - 2)\,, \tag{14}$$

$$0 = nw + N - \tilde{N}\,. \tag{15}$$

The Hagedorn phase transition occurs when the first winding mode of the string becomes tachyonic. Its lowest mass state has $n = 0$ and $w = \pm 1$. It has no excited oscillators, $N = 0 = \tilde{N}$. The mass shell condition for this state is

$$m^2 = \frac{\beta^2}{4\pi^2{\alpha'}^2} - \frac{4}{\alpha'}\,.$$

As we raise the temperature from zero, or lower β from infinity, these two states become massless when $\beta = \beta_H$ where

$$\beta_H = 4\pi\sqrt{\alpha'}\,. \tag{16}$$

Ian's observation also related this appearance of massless states, and a wrapping mode becoming tachyonic with the Berezinsky–Kosterlitz–Thouless phase transition in the S^1 sigma-model. This occurs when the vertex operator

$$V(z,\bar z) =: e^{2\pi i\left(X^0_L(z)+X^0_R(\bar z)\right)/\beta} : \tag{17}$$

becomes a relevant operator. A relevant operator is one of conformal dimension $(1,1)$. Simple computation reveals that $V(z,\bar z)$ has dimension $(\pi^2\alpha'/\beta^2, \pi^2\alpha'/\beta^2)$ which is type $(1,1)$ when $\beta = \beta_c = \pi\sqrt{\alpha'}$. How is this related to Hagedorn?

We can see from the spectrum, that it exhibits T-duality. We can interchange the wrapping number w and the momentum integer n if, at the same time, we replace β with $4\pi^2\alpha'/\beta$. It is interesting that β_c where Kosterlitz–Thouless occurs is T-dual to β_H. It is the place where the first quantized momentum states, with $n = \pm 1$, whose mass increases as we decrease β (or decrease the temperature), become massless. This means that, if we insist on lowering β from infinity, rather than increasing it from zero as in the usual BKT analysis, we encounter a BKT transition at the same temperature as the Hagedorn transition where the T-dual vertex operator

$$\hat V(z,\bar z) =: e^{\beta i\left(X^0_L(z)-X^0_R(\bar z)\right)/2\pi\alpha'} : \tag{18}$$

becomes relevant. These are just the combination of vortices that Kogan described as condensing in his original paper – the vortices of the right and left-moving fields if his vortex configuration

$$X^0 = \pm\frac{i}{2}\ln\left(\frac{z-z_0}{\bar z-\bar z_0}\right) \tag{19}$$

had opposite vorticities.

The Hagedorn behavior is universal for non-interacting string theories. One might wonder what happens when string interactions are turned on. For sure, simple string theories are theories of quantum gravity, and finite temperatures states in gravitational systems are well known to generally be unstable to collapse to black holes. In the remainder of this essay, I will review some modern attempts to interpret the Hagedorn transition as a symptom of this collapse. Generally black hole collapse will happen at a much lower temperature than Hagedorn. We think of it as a first order phase transition, where the finite temperature, hot graviton state competes in free energy with the Hawking radiating black hole state, but there is still an energy barrier between, making the phase transition first order. The Hagedorn transition would occur at a higher temperature, where, if we superheated the hot graviton gas, it would eventually become unstable to fluctuations. This instability is described by just the extra tachyons which appear in the spectrum.

2. Motivation

The basic degrees of freedom of string theory and M-theory are thought to be known and encoded in the BFSS [6] matrix model. The model is supersymmetric matrix quantum mechanics with action

$$S = \int dt \mathrm{Tr} \left[\frac{1}{2R}(DX^i)^2 + \frac{R}{4\ell_{pl}^6}[X^i, X^j]^2 + \bar{\psi} D\psi + \frac{R}{\ell_{pl}^3}\bar{\psi}\Gamma^i[X^i, \psi] \right], \quad (20)$$

where $i, j = 1, ..., 9$ and all degrees of freedom are $N \times N$ Hermitian matrices. This is a gauge invariant theory with covariant time derivative $D = \frac{d}{dt} - i[A, ...$. The gauge theory coupling constant is given in terms of the null compactification radius R and the eleven dimensional Planck length ℓ_{pl} (or the ten dimensional IIA string coupling g_s and string length $\ell_s = \sqrt{\alpha'}$) by

$$g_{YM}^2 = \left(\frac{R}{\ell_{pl}^2} \right)^3 = g_s \ell_s^3 \,, \quad \ell_{pl} = g_s^{1/3} \ell_s \,, \quad R = g_s \ell_s \,. \quad (21)$$

This model has three uses. It is conjectured to describe a discrete light-cone quantization of M-theory [6] where R is the compactification radius of the light-cone, there are N units of light-cone momentum and ℓ_p is the Planck length of 11-dimensional supergravity. Secondly, and historically a little earlier [7], with parameters suitably re-identified, it describes the low energy dynamics of a collection of N D0-branes of type IIA superstring theory. Finally, it is a matrix regularization of the light-cone action for the 11-dimensional supermembrane [8].

One motivation for understanding the behavior of matrix models such as the BFSS model at finite temperature comes from the conjecture that their finite temperature states are related to black hole states of type IIA supergravity [9,10]. This idea was studied in a series of papers by Kabat and Lowe [11]. They begin with the Beckenstein–Hawking entropy of a black D0-brane solution of IIA supergravity – the area of its even horizon in Planck units. They convert the entropy to the free energy, which they then write in terms of gauge theory parameters to obtain

$$F/T = -4.115N^2 \left(\frac{T^3}{g_{YM}^2 N} \right)^{3/5} . \tag{22}$$

A derivation of (22) from the matrix model would be an important result, a first principles computation of non-extremal black hole entropy using string theory. However, one would expect to find (22) in a low temperature and therefore strong coupling limit of the matrix model, making it inaccessible to perturbation theory. A variational technique was applied and claimed approximate agreement with the formula over some range of temperature [11].

The formula (22) is remarkable in three respects. First, it has the correct dependence on N and the 't Hooft coupling $g_{YM}^2 N$ to be the leading order of the 't Hooft limit of the gauge theory. If it could be derived in a perturbative expansion, it would obtain contributions only from planar Feynman diagrams. This means that the 't Hooft limit should be part of the limiting process that would extract the classical physics of black holes from the matrix model.

Secondly, the scaling with N in (22) is as if this 0-dimensional gauge theory were in a de-confined phase. This is particularly true if we assume that we are taking the 't Hooft limit. The use of the word "deconfined" in a theory where there is no spatial extent over which particles can be separated must be justified carefully. The gauge theory has a Gauss law constraint so that quantum states of the Hamiltonian must be singlets under the gauge symmetry. The gauge field in (20) enforces this constraint. The number of singlets at a given energy do not scale like N^2, rather they are of order one. As an example of this, consider a matrix harmonic oscillator with Hamiltonian

$$H = \sum_{i=1}^{d} \omega \mathrm{Tr}(\alpha_i^\dagger \alpha_i) \tag{23}$$

and matrix-valued creation and annihilation operators with algebra

$$\left[\alpha_{ab}^i, \alpha_{cd}^{j\dagger} \right] = \delta^{ij} \delta_{ad} \delta_{bc} \, . \tag{24}$$

States are created by $\alpha_{ab}^{i\dagger}$ operating on a vacuum $|0>$. To get a state with energy $E = n\omega$ we must act with n creation operators.

The analog of gauge invariance is to require a physical state condition of invariance under the unitary transformation

$$\alpha^\dagger \rightarrow u\alpha^\dagger u^\dagger ,$$

where

$$u \in U(N)/U(1) .$$

We assume that the vacuum state is invariant under this gauge transform. Then, physical states are created by operating with invariant combinations of creation operators. In the limit $N \rightarrow \infty$ all such combinations are traces

$$\left[\mathrm{Tr}\left(\alpha_i^\dagger\right)\right]^{n_1}\left[\mathrm{Tr}\left(\alpha_{j_1}^\dagger\alpha_{j_2}^\dagger\right)\right]^{n_2}\left[\mathrm{Tr}\left(\alpha_{k_1}^\dagger\alpha_{k_2}^\dagger\alpha_{k_3}^\dagger\right)\right]^{n_3}\ldots|0\rangle , \tag{25}$$

where the energy is

$$E = \omega(n_1 + 2n_2 + 3n_3 + \ldots) .$$

The number of these traces with a fixed energy, E, does not scale like N^2 as $N \rightarrow \infty$, instead it approaches a constant as N is taken large. Thus for normal thermodynamic states, one would not expect the free energy to be of order N^2.

However, the number of independent traces does increase rapidly with E. It has been shown [12] that, in the large N limit, the oscillator has a Hagedorn-like density of states at high energy,

$$\rho(E) \sim \frac{1}{E}\, e^{E/T_H} ,$$

where the Hagedorn temperature is

$$T_H = \omega/\ln d .$$

A similar result has been found for weakly coupled Yang–Mills theory [13–15].

At temperatures higher than T_H, the thermodynamic canonical ensemble does not exist. It could be made to exist by keeping N large but finite. That would cut off the exponential growth in the asymptotic density of states at some large energy. Then we could consider a temperature that is greater than T_H. Both the energy and entropy would be dominated by states at and above the cutoff scale. Then, the divergence of the free energy $\sim N^2$ occurs as we take the limit $N \rightarrow \infty$ at constant temperature (noting that the Hagedorn temperature does not depend on N).

As we shall show in the following, a behavior like this can indeed by found in the matrix model. In more conventional terms, it occurs as a large

N Gross–Witten type of phase transition [16,17] which is familiar in unitary matrix models. It is this behavior that we call "deconfinement". At this deconfinement transition, $\lim_{N\to\infty} F/N^2$ jumps from being zero to of order one.

In an adjoint gauge theory such as (20), there is an order parameter for confinement, the Polyakov loop [18, 19]. It is the trace of the holonomy of the gauge field around the finite temperature Euclidean time circle,

$$P = \frac{1}{N}\mathrm{Tr}\left(e^{i\oint A}\right). \tag{26}$$

This operator gets a nonzero expectation value when a gauge theory is deconfined. An interesting question is whether it has a nonzero expectation value in the BFSS matrix model. In such a low dimensional theory, it can only have a nonzero expectation value when N is infinite. Indications from weak coupling computations [20] are that it has.

The third remarkable fact about (22) is, though this formula is thought to apply to the black hole only for a range of temperatures [21], the expression at low temperature is reminiscent of critical scaling with a critical temperature $T = 0$ and a simple, rational critical exponent.

In spite of the simplicity of these interesting features, there is no analytic derivation of the formula (22) from the matrix model. One of the difficulties in finding a derivation is the intractability of the model itself. These difficulties are well-known from previous attempts to analyze its thermodynamics [20].

Before we continue with the matrix model, we comment that, if we analyze the thermodynamics of M-theory in the rest frame, we would form the partition function with a Boltzman distribution using the energy $E = \frac{1}{\sqrt{2}}(P^+ + P^-)$,

$$Z = \sum_N e^{-N/\sqrt{R}T} e^{-F/\sqrt{2}T}, \tag{27}$$

where the matrix model free energy is defined by

$$e^{-F/T} = \mathrm{Tr}\, e^{-H/T}. \tag{28}$$

In (27) we have traced over the eigenstates of the light-cone momentum. This gives the sum over matrix model partition functions for each N with the exponential factor $e^{-N/RT}$. The convergence of the sum is clearly dependent on the nature of the large N limit. If there is a sector of the matrix model where this limit is like (22), because of the negative sign, the sum over N diverges. It is tempting to associate this with the non-existence of thermodynamics of a theory of quantum gravity on asymptotically flat space – because of the Jeans instability the space is unstable to collapse to black holes. If there were a phase where the free energy did not become negative and with magnitude growing faster than N, it would be a stable phase.

There are several known behaviors of matrix models in the large N limit. For example, there is the Dijkgraaf–Verlinde–Verlinde [22] limit of matrix string theory. That is a strong coupling limit which kills the off-diagonal degrees of freedom of the matrices. The remaining, diagonal degrees of freedom are N in number and it can be shown explicitly [23–25] that the free energy is negative and is proportional to N. Of course, this is just the correct behavior for a string theory, there will be a Hagedorn temperature where the large N terms in the sum in (27) go from being exponentially suppressed to growing exponentially. There are other versions of matrix string theory [26–29] based on two dimensional Yang-Mills theory where one would expect a similar behavior.

Another limit where we have a quantitative estimate of the large N behavior is the 't Hooft limit where N is taken to infinity at the same time as g_{YM} is taken to zero. Technically this would be done by re-defining $\lambda = g_{YM}^2 N$ and holding λ fixed as when we sum over N in (27). Then, the phase transition that we discuss here is somewhat more violent than the Hagedorn behavior in string theory. The matrix part of the free energy at large N changes from a negative constant to a negative constant times N^2. The linear in N exponent of the momentum part doesn't compete with the N^2-growth of the matrix model contribution.

3. PP-Wave matrix model

Recently a variant of the matrix model which is conjectured to describe a discrete light cone quantization of M-theory on a pp-wave background has been formulated [30]. It is a 1-parameter deformation of (20),

$$S = \int dt \mathrm{Tr} \left[\frac{1}{2R}(DX^i)^2 + \frac{R}{4\ell_{pl}^6}[X^i, X^j]^2 + \bar{\psi} D\psi + \frac{R}{\ell_{pl}^3} \bar{\psi}\Gamma^i [X^i, \psi] \right.$$
$$\left. - \frac{\mu^2}{18R}(X^a)^2 - \frac{\mu^2}{72R}(X^{i'})^2 - \frac{\mu}{4}\bar{\psi}\psi - i\mu\epsilon^{abc} X^a X^b X^c \right], \quad (29)$$

where the indices $a, b, .. = 1, 2, 3$ and $i' = 4, ..., 9$. This matrix model reduces to the BFSS model if we put the parameter μ to zero and can be considered a one-parameter deformation of it. The main difference between the two is that the action in (20) has (super)symmetries identical to those of the residual invariance of 11-dimensional Minkowski space in light-cone quantization whereas (29) has symmetries appropriate to a pp-wave spacetime.

The matrix model in (29) has the great advantage that, unlike (20), it can be analyzed in perturbation theory [31]. In (20), the classical potential $\sim -\mathrm{Tr}\left([X^i,X^j]^2\right)$ has flat directions, any set of matrices which are mutually commuting have zero energy. The behavior of the degrees of freedom in flat directions must be understood at the outset of an honest quantum mechanical treatment of the theory. In (29), these flat directions are removed by the mass terms. Perturbation theory is accurate in the limit where the mass gap μ is large.

The pp-wave space which is a maximally supersymmetric solution of 11-dimensional supergravity is

$$ds^2 = dx^i dx^i - 2dx^+ dx^- - \left(\frac{\mu^2}{9}(x^a)^2 + \frac{\mu^2}{36}(x^{i'})^2\right) dx^+ dx^+$$

with an additional constant background 4-form flux

$$F_{+123} = \mu\,.$$

This space is known to support a spherical membrane solution,

$$x^+ = p^+\tau\,, \quad x^- = \text{const.}\,, \quad \sqrt{(x^a)^2} = \frac{\ell_{pl}^3}{6}p^+\mu$$

and a spherical transverse 5-brane

$$x^+ = p^+\tau\,, \quad x^- = \text{const.}\,, \quad (x^{i'})^2 = \ell_{pl}^3\sqrt{\frac{\mu p^+}{6}}\,.$$

These objects are conjectured to make their appearance in the solutions of the matrix model. The membrane is found immediately in semiclassical quantization. The classical potential is (hereafter we set $\ell_{pl} = 1$ and measure all dimensional quantities in Planck units)

$$V = \frac{R}{2}\mathrm{Tr}\left[\left(\frac{\mu}{3R}X^a + i\epsilon^{abc}X^bX^c\right)^2 + \frac{1}{2}\left(i\left[X^{i'},X^{j'}\right]\right)^2 + \left(i\left[X^{i'},X^a\right]\right)^2 + \left(\frac{\mu}{6R}\right)^2(X^{i'})^2\right]. \tag{30}$$

It is minimized by

$$X^{i'}_{\rm cl} = 0\,, \quad X^a_{\rm cl} = \frac{\mu}{3R}J^a\,, \tag{31}$$

where J^a is an N-dimensional representation of the SU(2) algebra, $[J^a,J^b] = i\epsilon^{abc}J^c$. In addition, the classical solution for gauge field must obey the equation

$$[A_{\rm cl}, J^a_{\rm cl}] = 0\,.$$

If J^a is an irreducible representation of SU(2), by Schur's Lemma, $A_{\rm cl} = 0$. The gauge symmetry is realized by the Higgs mechanism. When the representation is reducible, there are gauge fields which commute with the condensate. This part of $A_{\rm cl}$ remains undetermined and must still be integrated over, even to obtain the leading order in the semi-classical approximation to the partition function.

The configurations in (31) are fuzzy spheres, which are matrix regularizations of the membranes. The 5-branes on the other hand do not seem to appear in the perturbative states of the matrix model. It has been conjectured [32] that the 5-branes indeed appear as the large N limit is taken in a certain way.

First of all, J^a need not be an irreducible representation, but can contain a number of irreducible components,

$$J^a = \begin{pmatrix} s^a_1 & 0 & 0 & 0 \\ 0 & s^a_2 & 0 & 0 \\ 0 & 0 & s^a_3 & 0 \\ 0 & 0 & 0 & s^a_4 \end{pmatrix}\,.$$

To get a (multi-)membrane state, the large N limit is taken by holding the number of representations fixed and sending the dimension of each of the representations to infinity.

To get a five-brane state on the other hand, we hold the dimensions of the representations fixed and repeat them an infinite number of times to get the large N limit.

An important difference between these limits is in the realization of the gauge symmetry. In the classical sectors, the gauge symmetry is partially realized by the Higgs mechanism, with the residual symmetry being that which interchanges representations of the same size. In a membrane state the residual gauge group thus has finite rank, whereas in a 5-brane state its rank always goes to infinity. The single 5-brane state is $X^a = 0$ whereas the state with k coincident 5-branes has the k-dimensional representation repeated $N/k \to \infty$ times. A state with k non-coincident 5-branes has largest representation k-dimensional and a number of smaller representations all repeated an infinite number of times.

The effective coupling constant which governs a semi-classical expansion about one of the classical ground states is

$$\lambda = \left(\frac{3R}{\mu}\right)^3 n\,,$$

where n is the rank of the residual gauge group. The 5-brane limit is where we are required to take the weakest coupling limit. It is the conjecture of Ref. [32] that the membranes and all other degrees of freedom decouple in this limit and it isolates the internal dynamics of the 5-brane.

4. Perturbative expansion

Now, let us consider a perturbative expansion of the pp-wave matrix model at finite temperature. The partition function is the path integral

$$Z = \int [dA][dX^i][d\psi]e^{-\int_0^\beta d\tau L[A,X^i,\psi]}\,,$$

where L is the lagrangian with Euclidean time and $\beta = 1/T$ is the inverse temperature. The bosonic and fermionic variables have periodic and antiperiodic boundary conditions, respectively

$$A(\tau+\beta) = A(\tau)\,,\quad X^i(\tau+\beta) = X^i(\tau)\,,\quad \psi(\tau+\beta) = -\psi(\tau)\,.$$

Since the fermions are antiperiodic, these boundary conditions break supersymmetry. Of course this is expected at finite temperature.

We begin by fixing the gauge. It is most convenient to make the gauge field static and diagonal,

$$\frac{d}{d\tau}A_{ab} = 0\,,\quad A_{ab} = A_a\delta_{ab}\,.$$

The remaining degrees of freedom of the gauge field are just the time-independent eigenvalues A_a.

The Faddeev–Popov determinant for this gauge fixing is

$$\det{}'\left(-\frac{d}{d\tau}\Big(-\frac{d}{d\tau} + i(A_a - A_b)\Big)\right)\,, \tag{32}$$

where the boundary conditions are periodic with period β. The prime means that the zero mode of time derivative operating on periodic functions is omitted from the determinant. The Faddeev–Popov determinant for diagonalizing the gauge field is the familiar vandermonde determinant,

$$\prod_{a\neq b}|A_a - A_b|\,.$$

Together the two determinants are

$$\prod_{a\neq b}\det{}'\left(-\frac{d}{d\tau}\right)\det\left(-\frac{d}{d\tau}+i(A_a-A_b)\right)\,,$$

where the prime on the determinant indicates that the static mode is omitted.[a]

If we expand about the classical vacuum corresponding to the single 5-brane, $X^a_{\rm cl}=0=X^i_{\rm cl}$, we find the partition function in the 1-loop approximation is

$$Z=\int dA_a\prod_{a\neq b}\frac{\det{}'\left(-d/d\tau\right)\det\left(-D_{ab}\right)\det{}^8\left(-D_{ab}+\frac{\mu}{4}\right)}{\det{}^{3/2}\left(-D^2_{ab}+\frac{\mu^2}{9}\right)\det{}^3\left(-D^2_{ab}+\frac{\mu^2}{36}\right)}, \tag{33}$$

where $D_{ab}=\frac{d}{d\tau}-i(A_a-A_b)$. Using the formula

$$\det\left(-\frac{d}{d\tau}+\omega\right)=2\sinh\frac{\beta\omega}{2}$$

with periodic boundary conditions and

$$\det\left(-\frac{d}{d\tau}+\omega\right)=2\cosh\frac{\beta\omega}{2}$$

with antiperiodic boundary conditions, we can write

$$Z=\frac{1}{N!}\int_{-1/2}^{1/2}d\left(\frac{\beta A_a}{2\pi}\right)\prod_{a\neq b}\frac{\left[1-e^{i\beta(A_a-A_b)}\right]\left[1+e^{-\beta\mu/4+i\beta(A_a-A_b)}\right]^8}{\left[1-e^{-\beta\mu/3+i\beta(A_a-A_b)}\right]^3\left[1-e^{-\beta\mu/6+i\beta(A_a-A_b)}\right]^6}\,. \tag{34}$$

The factor of $1/N!$ is the volume of the residual discrete gauge group which permutes the eigenvalues. When N is finite, it might be possible to do this integral using the method of residues.

However, to apply to the 5-brane, we require the integral when $N\to\infty$. There are N integration variables A_a and the action, which is the logarithm of the integrand is generically of order N^2 which is large in the large N limit. For this reason, the integral can be done by saddle point integration. This amounts to finding the configuration of the variables A_a which minimize the

[a] Using zeta-function regularization, $\det{}'\left(-\frac{d}{d\tau}\right)=\beta$.

effective action

$$S_{\rm eff}=\sum_{a\neq b}\Big(-\ln\left[1-e^{i\beta(A_a-A_b)}\right]-8\ln\left[1+e^{-\beta\mu/4+i\beta(A_a-A_b)}\right]$$
$$+3\ln\left[1-e^{-\beta\mu/3+i\beta(A_a-A_b)}\right]+6\ln\left[1-e^{-\beta\mu/6+i\beta(A_a-A_b)}\right]\Big)\,. \quad (35)$$

To study the behavior, it is illuminating to Taylor expand the logarithms in the phases (this requires some assumptions of convergence for the first log)

$$S_{\rm eff}=\sum_{n=1}^{\infty}\frac{1-8(-)^{n+1}e^{-n\beta\mu/4}-3e^{-n\beta\mu/3}-6e^{-n\beta\mu/6}}{n}\left|{\rm Tr}e^{in\beta A}\right|^2. \quad (36)$$

Each term contains the modulus squared of a multiply would Polyakov loop (26). When a coefficient becomes negative, the loop condenses. In fact, as we raise the temperature from zero (and lower β from infinity), the first mode to condense is $n=1$. This occurs when

$$T_C=\frac{\mu}{12}\ln 3\approx .0758533\,\mu\,.$$

The condensate breaks a symmetry under changing the phase of the loop operator.

5. A closer look at the phase transition

At temperatures greater than T_C the eigenvalues A_a distribute themselves so that they are clustered near a particular point on the unit circle. To examine the possibility, we consider the equation of motion for the eigenvalues,

$$\omega(e^{0+}z)+\omega(e^{0-}z)+8\omega(-r^3z)+8\omega(-r^{-3}z)$$
$$=3\omega(r^4z)+3\omega(r^{-4}z)+6\omega(r^2z)+6\omega(r^{-2}z)\,, \quad (37)$$

where $r=e^{\beta\mu/12}$ and the resolvent is defined as

$$\omega(z)=\frac{1}{N}\sum_{a=1}^{N}\frac{z+e^{i\beta A_a}}{z-e^{i\beta A_a}}\,, \quad (38)$$

so $\omega(z)$ is holomorphic for z away from the unit circle and has asymptotic behavior, $\omega(\infty)=1$ and $\omega(0)=-1$. In the large N limit the poles in $\omega(z)$, which occur at the location of the elements $e^{i\beta A_a}$, coalesce to form a cut singularity on a part or perhaps all of the unit circle. $\omega(z)$ remains holomorphic elsewhere in the complex plane.

We must remember that equation (37) is valid only when z is one of the gauge field elements $e^{i\beta A_a}$. In that case, the sum in (38), which turns into an integral in the large N limit, must be defined as a principal value. In (37) this is gotten by averaging over approaching the unit circle from the inside and from the outside.

It is easy to find one exact solution of (37). If we consider the case where $e^{i\beta A_a}$ are uniformly distributed over the unit circle, so that the sum in (38) is symmetric under $z \to e^{i\theta} z$ we can average over the symmetry orbit to get

$$\omega_0(z) = \begin{cases} 1 & |z| > 1\,, \\ -1 & |z| < 1\,. \end{cases} \tag{39}$$

The result is certainly holomorphic everywhere away from the unit circle and is discontinuous on the entire unit circle.

The resolvent (39) is always a solution of (37) for any value of r. This is the symmetric, confining solution of the matrix model, where the Polyakov loop operator, whose expectation value is a particular moment of $\omega(z)$ for large z, vanishes. We would expect that this confining solution is only stable if the temperature is low enough. At some critical temperature it becomes an unstable solution and there should be other solutions which have lower free energy.

The confining phase which we discuss above is stable when $r > 3$ or $r < 1/3$. When $r = 3$ or $r = 1/3$, we can find a 1-parameter family of solutions,

$$\omega_1(z) = \begin{cases} -1 - az & |z| < 1\,, \\ \ \ 1 + a/z & |z| > 1\,. \end{cases} \tag{40}$$

This is an acceptable solution when $|a| < 2$.[b] If we plug it into Eq. (37) and assume that $r > 1$, we obtain

$$(3r^{-4} + 8r^{-3} + 6r^{-2} - 1)(z - 1/z) = 0 \tag{41}$$

which is solved when $r = 3$. If we assume $r < 1$ we find an equation which is solved by $r = 1/3$.

To examine the phase transition further, we expand about $r = \infty$. We expect the transition to occur at $r = 3$ which is not really large, but we will see that corrections are of order $1/r^4$, at the 1-percent level.

[b]Here, $a/2$ is the expectation value of the Polyakov loop operator which must be less than one.

The asymptotic expansions of the resolvent is

$$\omega(z) = 1 + 2\sum_{n=1}^{\infty} \frac{1}{z^n}\eta_n \,, \tag{42}$$

$$\omega(z) = -1 - 2\sum_{n=1}^{\infty} z^n \eta_{-n} \,, \tag{43}$$

where

$$\eta_n \equiv \frac{1}{N}\sum_{a=1}^{N} e^{in\beta A_a}$$

are the expectation values of the Polyakov loop operator for n windings. If we assume that $r > 1$, an asymptotic expansion of the equation (37) is

$$\omega(e^{0+}z) + \omega(e^{0-}z) = 2\sum_{n=1}^{\infty} \left(\frac{6}{r^{2n}} + \frac{3}{r^{4n}} + \frac{8}{r^{3n}}\right)\left(\eta_n z^{-n} - \eta_{-n} z^n\right). \tag{44}$$

Remember that this equation is valid only when z is inside the cut discontinuity of $\omega(z)$ which is assumed to occur on a segment of the unit circle.

In the large r limit, the right-hand-side of this equation can be approximated by the leading terms. It is then similar to the equations for the eigenvalue distributions in adjoint unitary matrix models which have been solved in the literature [17, 34, 35].

It is easy to find a solution of (44) if we truncate the right-hand-side by retaining only the $n = 1$ term. Consider the semi-circle distribution of Gross and Witten [16] [c]

$$\omega_{sc}(z) = \frac{1}{1+t}\left(\frac{1}{z} - z\right) - \frac{1}{(1+t)}\left(1 + \frac{1}{z}\right)\sqrt{1 + 2tz + z^2}\,. \tag{45}$$

This function has a cut singularity on the unit circle between branch points $z_\pm = -t \pm i\sqrt{1-t^2}$ where we take t in the range $-1 < t < 1$. When $t \to 1$

[c] The spectral density is defined by $\rho(\theta) = \frac{1}{N}\sum_{a=1}^{N}\delta(\theta - \beta A_a)$. It is normalized so that $\int_{-\pi}^{\pi} d\theta\rho(\theta) = 1$ In the large N limit, it becomes a continuous function of θ with support on some or all of the interval $[-\pi, \pi]$. An example is the semicircle distribution, which is given by

$$\rho_{sc}(\theta) = \begin{cases} \frac{1}{2\pi(1+t)}\cos\frac{\theta}{2}\sqrt{2(1+t) - 4\sin^2\frac{\theta}{2}}\,, & 0 \le \sin\frac{\theta}{2} \le \sqrt{\frac{1+t}{2}}\,, \\ 0\,, & \sqrt{\frac{1+t}{2}} < \sin\frac{\theta}{2} \le 1\,. \end{cases}$$

To get (45) we integrate

$$\omega_{sc}(z) = \int_{-\pi}^{\pi} d\theta\, \frac{z + e^{i\theta}}{z - e^{i\theta}}\, \rho_{sc}(\theta)\,.$$

the endpoints of the cut touch each other and the cut covers the whole unit circle. This is the where the Gross-Witten phase transition occurs in their unitary matrix model. [16] In their case, it is a third order phase transition. In the present case it is a first order phase transition. The solution that we found above, when $r = 3$, is just a special case of the semicircle (45) when $t = 1$.

Let us explore $\omega_{sc}(z)$ a little more. First we note that it obeys

$$\omega_{sc}(1/z) = -\omega_{sc}(z)$$

(with the appropriate change in the sign of the square root). This means that the $\eta_n = \eta_{-n} = \eta_n^*$ for all n. We can expand for small z,

$$\omega_{sc}(z) = -1 - \frac{3-t}{2}z - \frac{(1-t)^2}{2}z^2 - \frac{(1-t)^2(5t+1)}{8}z^3 + \ldots \tag{46}$$

from which we identify

$$\eta_0 = 1\ ,\quad \eta_1 = \frac{3-t}{4}\ ,\quad \eta_2 = \frac{(1-t)^2}{4}\ ,\quad \eta_3 = \frac{(1-t)^2(5t+1)}{16}\ \ldots\ . \tag{47}$$

We see that, there is a critical point at $t = 1$. At that point, $\eta_0 = 1$, $\eta_{\pm 1} = 1/2$ and $\eta_{|k|>1} = 0$. This is precisely the value of t for which the edges of the cut meet, so that the cut covers the entire unit circle. This is also precisely the exact solution (40) which we found when $r = 3$, here with the special value $a = 1/2$.

In fact, the semicircle distribution gives a good approximation to the solution when r is slightly less than 3. For z in the cut,

$$\omega_{sc}(e^{0+}z) + \omega_{sc}(e^{0-}z) = \frac{2}{1+t}\left(\frac{1}{z} - z\right).$$

If, for the moment, we truncate the right-hand-side of (44) to the term with $n = 1$, we see that the equation is solved by the semi-circle distribution when

$$\frac{4}{(1+t)(3-t)} = \left(8r^{-3} + 3r^{-4} + 6r^{-2}\right). \tag{48}$$

Also, remembering that t falls in the range$-1 < t < 1$, we get the critical value of r, $r_{\rm crit} = 3$ by setting $t = 1$. (48) has a solution only when $r < r_{\rm crit} = 3$ (and, here we have assumed $r > 1$).

When $r = 3$, the $n = 2$ term on the right-hand-side of (44), which we have ignored, contains $\left(8r^{-3} + 3r^{-4} + 6r^{-2}\right) = .086$. Thus, we see that, to an accuracy of about ten percent, the semicircle distribution is an approximate solution of the model for temperatures just above the critical temperature.

5.1. *Systematic improvement of the semicircle*

It is clear what has to be done to improve this approximation. We can begin with an Ansatz for the resolvent which has a single cut singularity placed on the unit circle

$$\omega(z) = \sum_{n=1}^{K} \left(a_n \left(z^{-n} - z^n\right) - b_n \left(z^n - z^{-n-1}\right) \sqrt{1 + 2tz + z^2} \right) . \tag{49}$$

To get the general solution, we should consider all orders by putting $K \to \infty$. An approximate solution is found by truncating at some order K. We will see below that this approximate solution is good near the phase transition. The coefficients in (49) must be arranged so that, in an asymptotic expansion in small z,

(1) all of the poles of order $1/z^K, ..., 1/z$ cancel and $\omega(0) = -1$ so, the asymptotic series then has the form

$$\omega = -1 - 2\eta_1 z - 2\eta_2 z^2 - \dots \; .$$

This gives $K+1$ conditions that the $2K+1$ parameters (a_n, b_n, t) must obey.

(2) From the above expansion, we determine the moments in terms of the parameters

$$\eta_1(a_n, b_n, t) \; , \quad \eta_2(a_n, b_n, t) \; , \quad \dots \; , \quad \eta_K(a_n, b_n, t) \, .$$

(3) Then we use the equation (44), with the right-hand-side truncated to order K to get K conditions

$$a_1 = f(r)\eta_1(a_n, b_n, t) \, , \tag{50}$$
$$a_2 = f(r^2)\eta_2(a_n, b_n, t) \, , \tag{51}$$
$$\dots \, ,$$
$$a_K = f(r^K)\eta_K(a_n, b_n, t) \, , \tag{52}$$

where $f(r) = \left(\frac{8}{r^3} + \frac{3}{r^4} + \frac{6}{r^2}\right)$. This gives K further equations which completely determine the $2K + 1$ parameters of the solution.

For example, if we choose $K = 2$, requiring that the poles cancel and $\omega(0) = -1$ yields the three conditions

$$a_2 - b_2 = 0 \; , \quad a_1 - b_1 - tb_2 = 0 \; , \quad b_1(1 + t) + b_2 \frac{1}{2} (1 - t^2) = 1 \, .$$

We can use these equations to eliminate b_1, a_1, a_2

$$b_1 = \frac{t-1}{2} b_2 + \frac{1}{1+t} \quad , \quad a_1 = \frac{3t-1}{2} b_2 + \frac{1}{1+t} \quad , \quad a_2 = b_2 \ .$$

Then, we can calculate the first and second moments by considering an asymptotic expansion of $\omega(z)$. We get

$$\eta_1 = \frac{(t+1)^3 b_2 - 2(t-3)}{8} ,$$
$$\eta_2 = -\frac{b_2}{16}(3t-5)(1+t)^3 + \frac{1}{4}(1-t)^2$$

and finally, using (50) and (51), we get the equations

$$\frac{3t-1}{2} b_2 + \frac{1}{1+t} = f(r) \left(\frac{(t+1)^3 b_2 - 2(t-3)}{8} \right) ,$$
$$b_2 = f(r^2) \left(-\frac{b_2}{16}(3t-5)(1+t)^3 + \frac{1}{4}(1-t)^2 \right) .$$

Of course, we already know that these equations are solved at the critical point by $b_2 = 0, t = 1, r = 3 \to f(r) = 1$. If we consider a value of r somewhat less than the critical value,

$$f(r) = 1 + \epsilon \, .$$

In this case,

$$f(r^2) = \frac{187}{2187} + \frac{25\epsilon}{162} \ .$$

We get

$$t = 1 - 2\sqrt{\epsilon}$$

and

$$b_2 = \frac{187}{2000} \epsilon \, .$$

This demonstrates that, close to the phase transition, the semicircle distribution gives an accurate description of the de-confined phase and this description can be systematically corrected. It would be interesting to explore this further to determine precise thermodynamic properties of that phase. The next term in the series on the right-hand-side of (44) which we have ignored, since we truncated to order 2, is proportional to $f(r^3)$ with $r \approx 3$, which suggests that in the vicinity of $r = 3$, the error is less than one percent. However, we caution that this is the case only for r close to 3. When $r = 2$, $f(r^3) = .11$.

5.2. *High temperature limit*

Another limit we could consider is the high temperature limit where $r \to 1$. In that case, we expect that the values of $e^{i\beta A_a}$ are concentrated near a point. In fact, the case where they are at a single point

$$\omega(z) = \frac{z+\eta}{z-\eta}$$

is a saddle point for all values of r. However, for $r \neq 1$ it has infinite positive energy, crossing over to infinite negative energy when $r = 1$. To see that it is a solution, we note that, in this case the eigenvalue support is at $z = \eta$ and therefore the variable in (37) is η. Then $\omega(e^{0+}\eta) + \omega(e^{0-}\eta) = 0$. Also $\omega(r^s\eta) + \omega(\frac{1}{r^s}\eta) = 0$ and (37) is solved. It is easy to see from the action that this solution is unstable for all values of r except $r = 1$.

5.3. *Free energy*

This shows the nature of the phase transition. At the critical point, it will turn out that the free energy is continuous, but the expectation of the Polyakov loop is equal to a and is ambiguous. Just below the transition, the theory is approximately described by the semi-circle distribution for which the Polyakov loop is 1/4. So we see that it jumps in value from 0 to 1/4 at the phase transition. It is for this reason that we expect the transition to be of first order. Indeed, by examining the free energy, we see that it is given by

$$\gamma = \frac{1}{N^2}\sum_{a\neq b}\left(-\ln(z_a - z_b) - 8\ln(z^a - r^3 z_b) + 3\ln(z_a - r^4 z_b) + 6\ln(z_a - r^2 z_b)\right). \tag{53}$$

For the symmetric solution, where $\rho_0(\theta) = \frac{1}{2\pi}$,

$$\gamma_0 = 0\,.$$

When r is large, we can expand to get

$$\gamma \approx \frac{1}{N^2}\sum_{a\neq b}\left[-\ln|1 - z_a/z_b| - \left(\frac{6}{r^2} + \frac{8}{r^3} + \frac{3}{r^4}\right)\frac{z_a}{z_b} + \dots\right]. \tag{54}$$

If we keep only the first term in the large r expansion (note that there is another term which competes with $1/r^4$ which we ignore for now), this is approximately an adjoint unitary matrix model of the kind solved in the literature [34]. It is solved by the semicircle distribution or the symmetric distribution. The free energy is

$$\gamma \approx \begin{cases} 0 & r > 3\,, \\ \frac{f(r)}{4}\left(1-\sqrt{1-\frac{1}{f(r)}}\right)-f(r)+\frac{3}{4}+\frac{1}{2}\ln\left[f(r)\left(1+\sqrt{1-\frac{1}{f(r)}}\right)\right] & r < 3\,. \end{cases} \tag{55}$$

The value of the Polyakov loop is

$$\left\langle \frac{1}{N}\,\mathrm{Tr}\, e^{i\oint A} \right\rangle = \begin{cases} 0 & r > 3\,, \\ \frac{1}{2}\left(1+\sqrt{1-\frac{1}{f(r)}}\right) & r < 3\,. \end{cases} \tag{56}$$

5.4. *Symmetry restoration?*

Because of the low dimensionality of the system that we are discussing, the symmetry breaking which occurs in the de-confined phase could be destroyed by quantum fluctuations. In fact, it would generally be the case in theories with local interactions.

For example, if N is finite, symmetry breaking is not possible. The phase transition that we have discussed here can only occur when N is infinite. Mathematically, we can think of large N as the analog of a large volume limit in a statistical mechanical system. If N is large but not infinite, the symmetry is not broken in a mathematical sense but the decay rate of a non-symmetric state is exponentially suppressed in the volume, in this case $\sim e^{-N\cdots}$.

The deconfined solution has a spectral density $\rho(\theta)$. Because of symmetry of the problem under replacing θ by θ+constant, there would be a zero mode of the linear equation for the fluctuations of $\rho(\theta)$, with wave-function $\psi(\theta) \sim \frac{d}{d\theta}\rho(\theta)$. This mode would provide the motion which would restore the symmetry. However, in the case of the semi-circle distribution, because of the square-root singularity at the edge of the distribution, this function is not square-integrable, and therefore not normalizable.

6. A stack of 5-branes

The fluctuation spectrum of the matrices about a stack of k 5-branes is known. If, rather than the trivial vacuum, we had chosen the one where the k-dimensional representation of SU(2) is repeated N/k times, the residual gauge invariance would be $SU(N/k)$. The gauge field would have a classical solution with $k\times k$ unit matrices times gauge fields A_a representing the whole blocks. The spectrum is known [32, 33] and we can again get an estimate of the Hagedorn temperature [12]

$$T_H(k=\infty) = .073815...\mu \ .$$

We see that the temperature is reduced only slightly in this case. We take this as evidence that the Hagedorn temperature in this weak coupling limit is insensitive to the number of 5-branes. This seemingly contradicts the k-dependence of the Hagedorn phase transition of little string theory which has been computed using holography and which behaves like $T_C \sim \sqrt{k}$.

An explanation for this contradiction can be found in the limit that we are using. It is a large N 'tHooft limit and a further expansion in weak 't Hooft coupling. In this limit, if we were to translate the parameters of our model to those that would describe the NS five-brane in type II string theory, the radius of the spherical five-brane would be

$$\frac{r^2}{\alpha'} \sim \lambda^{1/4} \ .$$

This means that we are expanding about a small, highly curved five-brane, whereas the usual holographic result for the Hagedorn temperature is for a large flat 5-brane. This is a similar difficulty as the one which appears in the AdS/CFT correspondence in general. There, the analog of the matrix model, which is maximally supersymmetric Yang-Mills theory, can be readily analyzed only in the weak coupling limit using an asymptotic expansion in λ. Supergravity and holography give tools which compute the strong coupling limit, where λ is large. Thermodynamic quantities like the free energy in particular were computed in both theories and they do not agree with each other for this reason.

7. Discussion

The phase transition in the matrix model is a peculiar one. Closer analysis reveals that it is of first order. It is easy to see from (45) that the expectation value of the Polyakov loop operator is zero in the symmetric phase but is non-vanishing in the high temperature phase and approaches a non-zero value there even as the critical point is approached from above. However, at this critical point, there is no co-existence region where one phase is meta-stable and the other is stable, as is normally the case for first order phase transitions. To see this, note that the potential the phase transition occurs where the potential becomes unstable to perturbations. Normally, in a first order behavior, there are two competing vacua, both of which are perturbatively stable in the transition region and as parameters are varied one or the other gets lower free energy and is preferred. Then they cross there is a phase transition. In the present case, the perturbative instability occurs at the same place as the phase transition.

There is a question as to whether this is an artifact of the approximation that we have done here, i.e. if we expanded the effective action for eigenvalues to higher loop order the phase transition might be more conventional.

Indeed, in other unitary matrix models applied to non-Abelian Coulomb gases [34, 35], where the eigenvalues live on a higher dimensional space (a $D = 1$ or even $D > 1$ unitary matrix model) there is a coexistence region.

A similar problem afflicts weakly coupled four-dimensional supersymmetric Yang-Mills theory [15]. In that case, Witten argued using AdS/CFT that, in the strong coupling limit of planar Yang-Mills theory, there should be a de-confining phase transition, identified with the Hawking-Page phase transition of supergravity on asymptotically anti-de Sitter space [36]. Hawking-Page is a normal first order phase transition where there is the possibility of metastable phases. At weak coupling, the analysis looks very similar to what we have done here for the matrix model and the phase transition has the same nature. Aharony et.al. have conjectured that the effect of higher loop corrections (3-loops) to the effective action in Yang-Mills theory would indeed change the phase transition to a more conventional one.

Finally, in the matrix model that we have analyzed, there is the question of whether the phase transition that we have found has anything to do with collapse to black holes. The subject of black holes on pp-wave backgrounds is a murky one which is presently being sorted out. In any case, the physics described by classical gravity should appear at a strong coupling limit, rather than the weak coupling limit that we have analyzed. It is tempting to conjecture that, if there were black holes, collapse to black holes at finite temperature is what our phase transition would describe if it persists at strong coupling.

There is the further question of whether our phase transition is related to the Hagedorn behavior that is seen in the string spectrum on pp-wave backgrounds [37–43]. This behavior of course occurs in weakly string theory which is a limit of the M-theory. It there were such a relationship, it would be interesting to ask whether it is related to the formation of black holes in ten dimensional supergravity near a pp-wave background [44, 45].

Of course of the conjectured relationship with the 5-brane is valid, then, indirectly, the Hagedorn phase transition of little string theory is related to horizon formation in the 5-brane geometry. In that case, our phase transition and its thermodynamics indeed describes the black hole, again in a limit which is far away from previous analysis of such objects. Note that we do not analyze the relative stability of membranes and five-branes. This question has been addressed in recent interesting papers [46, 47]

It is also interesting to ask whether the phase transition that we have identified is related to recent work which studied phase transitions for statistical systems of random walks on discrete groups like the permutation group. [48, 49]

Acknowledgments

The work reviewed in this paper was done in collaboration with Ehud Schreiber and Kazayuki Furuuchi. I have benefitted from conversations with Ofer Aharony, Shiraz Minwalla and Mark Van Raamsdonk. Many of the ideas here derive from an earlier work done in collaboration with Jan Ambjorn and Yuri Makeenko. And, of course Ian Kogan, who was the first one who interested me in questions about strings and black holes.

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