SMALL INSTANTONS, LITTLE STRINGS AND FREE FERMIONS

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We present new evidence for the conjecture that BPS correlation functions in $\mathcal{N}=2$ supersymmetric gauge theories are described by an auxiliary two dimensional conformal field theory. We study deformations of the $\mathcal{N}=2$ supersymmetric gauge theory by all gauge-invariant chiral operators. We calculate the partition function of the $\mathcal{N}=2$ theory on $\mathbb{R}^4$ with appropriately twisted boundary conditions. For the $U(1)$ theory with instantons (either noncommutative, or D-instantons, depending on the construction) the partition function has a representation in terms of the theory of free fermions on a sphere, and coincides with the tau-function of the Toda lattice hierarchy. Using this result we prove to all orders in the string loop expansion that the effective prepotential (for $U(1)$ with all chiral couplings included) is given by the free energy of the topological string on $\mathbb{CP}^1$. Gravitational descendants play an important rôle in the gauge fields/string correspondence. The dual string is identified with the little string bound to the five-brane wrapped on the two-sphere. We also discuss the theory with fundamental matter hypermultiplets.

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We dedicate this paper to the memory of Ian Kogan. We miss you, Ian

1. Introduction

The Holy Grail of theoretical physics is the nonperturbative theory which includes quantum gravity, sometimes called M-theory [1]. The current wisdom says there is no fundamental coupling constant. Whatever (string) perturbation theory is used depends on the particular solution one expands about. The expansion parameter is one of the geometric characteristics of the background. It is obviously interesting to look for simplified string and field theoretic models, which have a string loop expansion and where the string coupling constant has a geometric interpretation.

String expansion in gauge theory

Large N gauge theories are the most popular, and the most elusive, models with a string representation. In the gauge/string duality [2,3] one matches the connected correlation functions of the gauge theory observables with the partition function of the string theory in the bulk. The closed string dual has $\frac{1}{N}$ as the string coupling constant. Advances in studies of the type II string compactifications on Calabi-Yau manifolds led to another class of models, which in the low-energy limit reduce to $\mathcal{N}=2$ supersymmetric gauge theories, with a novel type of string loop expansion. Namely, certain couplings $\mathcal{F}_g$ in the low-energy effective action are given by the genus g partition function of the topologically twisted string on Calabi-Yau. The gauge group of the $\mathcal{N}=2$ theory does not have to be $U(N)$ with large N. It is determined by the geometry of the Calabi-Yau manifold [4–6]. The rôle of effective string coupling is played by the vev of the graviphoton field strength [7, 8], which is usually assumed to be constant [9].

Generalized Scherk-Schwarz construction

In this paper we shall explain that there exists another, natural from the gauge theory point of view, way to flesh out these couplings. The idea is to put the theory in a nontrivial geometric background, which we presently describe. Namely, consider any Lorentz-invariant field theory in d dimensions. Suppose the theory can be obtained by Kaluza-Klein reduction from some theory in $d+1$ dimensions. In addition, suppose the theory in $d+1$ dimensions had a global symmetry group H. Now compactify the $d+1$ dimensional theory on a circle $\mathbb{S}^1$ of circumference r, with a twist, so that in going around the circle, the space-time $\mathbb{R}^d$ experiences a Lorentz rotation, by an element $\exp(r\Omega)$, and in addition a Wilson line in the group H, $\exp(r\mathbb{A})$ is turned on. The resulting theory can be now considered in the $r \to 0$ limit, where for finite $\Omega, \mathbb{A}$ we find extra couplings in the d-dimensional Lagrangian. This is the background we shall use extensively. More specifically we shall be mostly interested in the four dimensional $\mathcal{N} = 2$ theories. They can all be viewed as dimensional reductions of $\mathcal{N} = 1$ susy gauge theories from six or five dimensions. The global symmetry group H in six dimensions is $SU(2)$ (R-symmetry).

These considerations lead to powerful results concerning exact non-perturbative calculations in the supersymmetric gauge theories. In particular, one arrives [10] at the technique of deriving effective prepotentials of the $\mathcal{N} = 2$ susy gauge theories with the gauge groups $U(N_1)\times\ldots\times U(N_k)$ (based on [11–15], see also Ref. 16). Previously, the effective low-energy action and the corresponding prepotential $\mathcal{F}^{SW}$ was determined using the constraints of holomorphy and electro-magnetic duality [17–19].

Higher Casimirs in gauge theory

One of the goals of the present paper is to extend the method of Ref. 10 to get the correlation functions of $\mathcal{N} = 2$ *chiral* operators. This is equivalent to solving for the effective prepotential of the $\mathcal{N} = 2$ theory whose microscopic prepotential (see Ref. 17 for introduction in $\mathcal{N} = 2$ susy) is given by

$$\mathcal{F}^{UV} = \tau_0 \mathrm{Tr}\,\Phi^2 + \sum_{\vec{n}} \tau_{\vec{n}} \prod_{J=1}^{\infty} \frac{1}{n_J!}\left(\frac{1}{J}\,\mathrm{Tr}\,\Phi^J\right)^{n_J}, \tag{1}$$

where $\vec{n} = (n_1, n_2, \ldots)$ label all possible gauge-invariant polynomials in the adjoint Higgs field Φ (note that $\tau_{0,1,0,\ldots}$ shifts τ_0). Let $\vec{\rho} = (1, 2, 3, \ldots)$, $|\vec{n}| = \sum_J n_J$, and $\vec{n}\cdot\vec{\rho} = \sum_J J n_J$.

In order for the theory defined by Eq. (1) to avoid the vanishing of second derivatives of the prepotential at large (quasiclassical) values of the Higgs field,

$$\langle\Phi\rangle_a \sim a \gg \Lambda \sim e^{2\pi i \tau_0}\,, \tag{2}$$

and not to run into a strong coupling singularity, the couplings $\tau_{\vec{n}}$ should be treated formally. One could also worry about the nonrenormalizabilty of the perturbation (1). This is actually not a problem, provided the conjugate prepotential $\bar{\mathcal{F}}$ is kept classical $\bar{\tau}_0 \mathrm{Tr}\,\bar{\Phi}^2$. The action is no longer real and the effective dimensions of the fields Φ and $\bar{\Phi}$ become 0 and 2, thereby justifying an infinite number of terms in Eq. (1).

We should note that there are relations between the deformations generated by derivatives w.r.t. $\tau_{\vec{n}}$, which originate in the fact that there are polynomial relations between the single-trace operators $\mathrm{Tr}\,\Phi^J$ for $J > N$ and the multiple-trace operators. When instantons are included these classical relations are modified. It seems convenient to keep all $\tau_{\vec{n}}$ as independent couplings. The classical prepotential then obeys additional constraints; the N-independent non-linear ones,

$$\frac{\partial \mathcal{F}^{UV}}{\partial \tau_{\vec{n}}} = \frac{\partial \mathcal{F}^{UV}}{\partial \tau_{\vec{n}_1}} \cdots \frac{\partial \mathcal{F}^{UV}}{\partial \tau_{\vec{n}_k}}, \qquad \vec{n} = \vec{n}_1 + \ldots \vec{n}_k \,, \tag{3}$$

and the N-dependent linear ones,

$$\sum_{\vec{n}:\ \vec{n}\cdot\vec{\rho}=N+k} (-1)^{|\vec{n}|} \frac{\partial}{\partial \tau_{\vec{n}}} \mathcal{F}^{UV} = 0 \,, \qquad k > 0 \,. \tag{4}$$

The quantum effective prepotential obeys instanton corrected constraints [13], which we implicitly determine in this paper.

Contact terms

The constraints of holomorphy and electro-magnetic duality are powerful enough to determine the effective low-energy prepotential $\mathcal{F}^{IR}$ (see Ref. 13), up to a diffeomorphism of the couplings $\tau_{\vec{n}}$, i.e. up to contact terms. In order to fix the precise mapping between the microscopic couplings (which we also call "times", in accordance with the terminology adopted in integrable systems) and the macroscopic ones, one needs more refined methods (see Ref. 15 for the discussion of the contact terms and their relation to the topology of the compactifications of the moduli spaces). As we shall explain in this paper, the direct instanton calculus is powerful enough to solve for $\mathcal{F}^{IR}$ in the form

$$\mathcal{F}^{IR}(a, \tau_{\vec{n}}) = \mathcal{F}^{SW}(a; \tau_0) + \sum_{\vec{n}} \tau_{\vec{n}} \mathcal{O}_{\vec{n}}(a) + \sum_{\vec{n},\vec{m}} \tau_{\vec{n}} \tau_{\vec{m}} \mathcal{O}_{\vec{n}\vec{m}}(a) + \ldots \,, \tag{5}$$

where

$$\mathcal{O}_{\vec{n}}(a) = \left\langle \prod_{J=1}^{\infty} \frac{1}{n_J!} \left(\frac{1}{J} \mathrm{Tr}\,\Phi^J \right)^{n_J} \right\rangle_a \,, \tag{6}$$

and $\mathcal{O}_{\vec{n}\vec{m}}$ are the expectation values of the contact terms between $\mathcal{O}_{\vec{n}}$ and $\mathcal{O}_{\vec{m}}$ [13, 20, 21].

Dual/little string theories

We shall argue that the prepotential generalized in this way (5), which is also a generating function of the correlators of chiral observables, is encoded in a certain *stringy* partition function. We shall demonstrate that the generating function of the expectation values of the chiral observables in the special $\mathcal{N}=2$ supergravity background is given by the exponential of the all-genus partition function of the topological string. For the "$U(1)$" theory the dual string lives on $\mathbb{CP}^1$ (A-model). To prove this we shall use the recent results of A. Okounkov and R. Pandharipande who related the partition function of the topological string on $\mathbb{CP}^1$ with the tau-function of the Toda lattice hierarchy. The expression of the generating function of the chiral operators through the tau-function of an integrable system is a straightforward generalization of the experimentally well-known relation between the Seiberg-Witten prepotentials and quasiclassical tau-functions Ref. 22 (see also Ref. 23 and references therein). For the tau-function giving the generating function for the correlators of chiral operators we will present a natural representation in terms of free fermionic or bosonic system. We think this is a substantial step towards understanding the physical origin of the results of Ref. 22.

One may think that this result is yet another example of local mirror symmetry Ref. 6. We should stress here that this is by no means obvious. Indeed, a powerful method to embed $\mathcal{N}=2$ gauge theories into string theory is by considering the type II string on local Calabi-Yau manifolds. Almost all of the results obtained in this way can be viewed as a degeneration of the theory which exists for global, compact Calabi-Yau manifolds. In other words, one assumes that the gauge theory decouples from gravity and excited string modes, when the Calabi-Yau is about to develop some singularity, and the global structure of Calabi-Yau is not relevant; but in this way one cannot really discuss the higher Casimir deformations (1). However, the main claim is there; the prepotential of the gauge theory, as well as the higher couplings $\mathcal{F}_g$, is given by the topological string amplitudes on the local Calabi-Yau.

If the local Calabi-Yau can be viewed as a degeneration of the compact Calabi-Yau then one simply takes the limit of the corresponding topological string amplitudes (effectively all irrelevant Kähler classes in the A-model are sent to infinity, and the worldsheet instantons do not know about them; however, one has to renormalize the zero instanton term). In this case one can, in principle, take the mirror theory, the B-model on a dual Calabi-Yau manifold, and try to perform the analogous degeneration there as in Ref. 6; this way even leads to some equations for the $\mathcal{F}_g$'s, see Ref. 24.

However, the situation here is still unsatisfactory. For the global Calabi-Yau's the whole sum $\sum_g \mathcal{F}_g \hbar^{2g-2}$ is identified with the logarithm of the partition function of the effective "closed string field theory" – the Kodaira-Spencer theory [7] on the B-side Calabi-Yau manifold. Nothing of this sort is known for the degenerations corresponding to local Calabi-Yau's on the A-side, for $g > 0$. For genus zero amplitudes the pair (mirror Calabi-Yau manifold, a holomorphic three-form) is replaced by the pair (an effective curve, a meromorphic 1-form) which captures correctly the relevant periods. In Ref. 5 these curves (which are Seiberg-Witten curves of the gauge theory) were identified in the following way. One views the degeneration of the mirror Calabi-Yau as an ALE fibration over a base $\mathbb{CP}^1$. To this fibration one can associate a finite cover of $\mathbb{CP}^1$ associated with the monodromy group of the $H_2(ALE, \mathbb{Z})$ bundle over the $\mathbb{CP}^1 \backslash$ degeneration locus. This finite cover is the Seiberg-Witten curve. In Ref. 10 it was conjectured that the Kodaira-Spencer theory should become a single free fermion theory on this curve. In the case of the Calabi-Yau being an elliptic curve this conjecture was studied a long time ago [7, 25, 26] with applications to the string theory of two dimensional Yang-Mills [27] in mind.

Our contribution to the subject is the identification of the analogue of the Kodaira-Spencer theory, at least in the specific context we focus on in this paper. This is, we claim, the free fermionic (or free bosonic) theory on a Riemann surface (a sphere for $U(1)$ gauge group), in some specific $\mathbb{W}$-background (i.e. with the higher spin chiral operators turned on), see Ref. 28. Note that in the conventional approach to Kodaira-Spencer theory via type B topological strings, the $\mathbb{W}$-deformations are not considered (except for $\mathbb{W}_2$, corresponding to the complex structure deformations). This mirrors the fact that on the A-side one does not usually take into account the contribution of the gravitational descendents.

Thus, we also have something new on the A-side. This is of course not the first time that Fano varieties have appeared in the context of local mirror symmetry. However, the topological string amplitudes, corresponding to the local Calabi-Yau do not coincide with those for Fano, even if the actual worldsheet instantons land on the Fano subvariety in the local Calabi-Yau. For example, the resolved conifold is the $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ bundle over $\mathbb{CP}^1$, all worldsheet instantons land in $\mathbb{CP}^1$ (which is Fano), yet the topological string amplitude is affected by the zero modes of the fermions, corresponding to the normal directions. These zero modes make the contributions of all positive ghost number observables of the topological string on the Fano variety vanish when it is embedded into the Calabi-Yau.

In our case, however, we literally get strings on $\mathbb{CP}^1$. This model is much richer than strings on the conifold. In particular, as we show, the gravitational descendants of the Kähler class of $\mathbb{CP}^1$ are dual to the higher Casimirs in the gauge theory.

It goes without saying that embedding our picture in the general story of local mirror symmetry will be beneficial for both. In particular, [7, 8] explain how the topological string amplitudes arise as the physical string amplitudes with the insertion of $2g$ powers of the sugra Weyl multiplet $\mathcal{W}$, the vertex operators for $\mathcal{W}$ effectively twisting the worldsheet theory. We claim that the topological string with gravitational descendants (which are constructed with the help of the fields of topological gravity) has direct and clear physical meaning on the gauge theory side. We do not know at the moment how to embed them in the framework of Ref. 7. However we shall make a suggestion.

Organization of the paper

The paper is organized as follows. Section 2 discusses instanton calculus in the $\mathcal{N}=2$ susy gauge theories from the physical point of view. The mathematical aspects, related to the equivariant cohomology of the moduli spaces and the equivariant methods which lead to the evaluation of the integrals one encounters in the gauge theory are described in the appendix **A**. As a result of these calculations one arrives at the generating function of the expectation values of the chiral operators, which is expressed as a partition function of a certain auxiliary statistical model on the Young diagrams. Section 3 specifies these results for the gauge group $U(1)$ and explains their interpretation from the point of view of the little string theory, which we claim is equivalent in this case to the topological string on $\mathbb{P}^1$, with the gravitational descendendents of the Kähler form $\sigma_k(\omega)$ lifted to the action. This section also introduces the formalism of free fermions which is very efficient in packaging the sums over partitions. Section 4 identifies the partition function with a simple correlator of free fermions, and also with the tau-function of the Toda lattice. Section 5 discusses the theory with fundamental matter, and its free field realization.

2. $\mathcal{N}=2$ Theory

2.1. *Gauge theory realizations*

We start our exposition with the case of pure $\mathcal{N}=2$ supersymmetric Yang-Mills theory with the gauge group $U(N)$ and its maximal torus $\mathbf{T}=U(1)^N$.

The field content of the theory is given by the vector multiplet $\mathbf{\Phi}$, whose components are the complex scalar Φ, two gluinos λ^i_α, $i = 1,2$; $\alpha = 1,2$ their conjugates $\bar{\lambda}_{\dot{\alpha}i}$, and the gauge field A_μ – all fields are in the adjoint representation of $U(N)$. The action is given by

$$S \propto \int d^4x \left(\int d^4\theta \mathcal{F}(\mathbf{\Phi}) + \int d^4\bar{\theta}\bar{\mathcal{F}}(\bar{\mathbf{\Phi}}) \right) , \tag{7}$$

where θ^i_α, $\alpha = 1,2$; $i = 1,2$ are the chiral Grassmann coordinates on the superspace, $\mathbf{\Phi} = \Phi + \theta\lambda + \theta\theta F^- + \ldots$ is the $\mathcal{N} = 2$ vector superfield, and $\mathcal{F}$ is the prepotential (locally, a holomorphic gauge invariant function of Φ). Classical supersymmetric Yang-Mills theory has

$$\mathcal{F}(\Phi) = \tau_0 \mathrm{Tr}\, \Phi^2 , \tag{8}$$

where τ_0 is a complex constant whose real and imaginary parts give the theta angle and the inverse square of the gauge coupling respectively,

$$\tau_0 = \frac{\vartheta_0}{2\pi} + \frac{4\pi i}{g_0^2} . \tag{9}$$

The subscript 0 reminds us that these are bare quantities defined at some high energy scale μ_{UV}. It is well-known that $\mathcal{N} = 2$ gauge theory has a moduli space of vacua, characterized by the expectation value of the complex scalar Φ in the adjoint representation. In the vacuum $[\Phi, \bar{\Phi}] = 0$, due to the potential term $\mathrm{Tr}\,[\Phi, \bar{\Phi}]^2$ in the action of the theory. Thus, one can gauge rotate Φ to the Cartan subalgebra of $\mathfrak{g}$: $\langle\Phi\rangle = a \in \mathbf{t} = Lie(\mathbf{T})$. We are studying the gauge theory on Euclidean space $\mathbb{R}^4$, and impose the boundary condition $\Phi(x) \to a$, for $x \to \infty$. It is also convenient to accompany the fixing of the asymptotics of the Higgs field by the fixing the allowed gauge transformations to approach unity at infinity.

The $\mathcal{N} = 2$ gauge theory in four dimensions is a dimensional reduction of the $\mathcal{N} = 1$ five dimensional theory. The latter theory needs an ultraviolet completion to be well-defined. However, some features of its low-energy behavior are robust [29].

In particular, the effective gauge coupling runs because of the one-loop vacuum polarization by the BPS particles. These particles are W-bosons (for nonabelian theory), four dimensional instantons, viewed as solitons in five dimensional theory, and the bound states thereof.

To calculate the effective couplings we need to know the multiplicities, the masses, the charges, and the spins of the BPS particles present in the spectrum of the theory [12, 30]. This can be done, in principle, by careful quantization of the moduli space of collective coordinates of the soliton solutions (which are four dimensional gauge instantons). Now suppose the theory is compactified on a circle. Then the one-loop effect of a given particle consists of a bulk term, present in the five dimensional theory, and a new finite-size effect, having to do with the loops wrapping the circle in space-time [12]. If in addition the noncompact part of the space-time in going around the circle is rotated then the loops wrapping the circle would have to be localized near the origin in the space-time. This localization is at the core of the method we are employing. Its mathematical implementation is discussed in the next section. Physically, the multiplicities of the BPS states are accounted for by the supersymmetric character-valued index [30],

$$\sum_{solitons} \mathrm{Tr}_{\mathcal{H}}(-)^F e^{-r\mathbb{P}_5} e^{r\Omega\cdot\mathbb{M}} e^{r\mathbf{A}\cdot\mathbb{I}},$$

where $\mathbb{P}_5$ is the momentum in the fifth direction, $\mathbb{M}$ is the generator of the Lorentz rotations, $\mathbb{I}$ is the generator of the R-symmetry rotations, and r is the circumference of the fifth circle. Under certain conditions on Ω and $\mathbf{A}$ this trace has some supersymmetry which enables us to evaluate it. In the process one gets some integrals over the instanton collective coordinates, as in [14, 31–33]. As in Ref. 14, these integrals are exactly calculable, thanks to equivariant localization, described in appendix.

Another point of view on our method is that by appropriately deforming the theory (in a controllable way) we arrange that the path integral has isolated saddle points, and thanks to the supersymmetry is exactly given by the WKB approximation. The final answer is then the sum over these critical points of the ratio of bosonic and fermionic determinants. This sum is shown to be equal to the partition function of an auxiliary statistical model, describing the random growth of Young diagrams. We describe this model in detail in the section 2.7.

We now conclude our discussion of the reduction of the five dimensional theory down to four dimensions. Actually, we can be more general, and discuss the reduction from six dimensions.

Consider lifting the $\mathcal{N} = 2$ four dimensional theory to $\mathcal{N} = (1, 0)$ six dimensional theory, and then compactifying on a two-torus with twisted boundary conditions (along both A and B cycles), such that as we go around a non-contractible loop $\ell \sim nA + mB$, the space-time and the fields of the gauge theory charged under the R-symmetry group $SU(2)_I$ are rotated by the element $(e^{i(na_1+mb_1)\sigma_3},\ e^{i(na_2+mb_2)\sigma_3},\ e^{i(na_2+mb_2)\sigma_3}) \in SU(2)_L \times SU(2)_R \times SU(2)_I = Spin(4) \times SU(2)_I$. In other words, we compactify the six dimensional $\mathcal{N} = 1$ susy gauge theory on the manifold with the topology $\mathbb{T}^2 \times \mathbb{R}^4$ with the metric and the R-symmetry gauge field Wilson line,

$$ds^2 = r^2 dz d\bar{z} + (dx^\mu + \Omega^\mu_\nu x^\nu dz + \bar{\Omega}^\mu_\nu x^\nu d\bar{z})^2 \,, \tag{10}$$

$$\mathbf{A}^a = (\Omega^{\mu\nu} dz + \bar{\Omega}^{\mu\nu} d\bar{z})\eta^a_{\mu\nu} \,, \quad \mu = 1,2,3,4 \,, \quad a = 1,2,3 \,,$$

where η is the anti-self-dual 't Hooft symbol. It is convenient to combine $a_{1,2}$ and $b_{1,2}$ into two complex parameters $\epsilon_{1,2}$ so that

$$\epsilon_1 - \epsilon_2 = 2(a_1 + ib_1) \,, \qquad \epsilon_1 + \epsilon_2 \,. = 2(a_2 + ib_2) \tag{11}$$

The antisymmetric matrices $\Omega, \bar{\Omega}$ are given by:

$$\Omega = \frac{\epsilon_1 + \epsilon_2}{2} \, J^L_3 + \frac{\epsilon_1 - \epsilon_2}{2} \, J^R_3, \qquad \bar{\Omega} = \frac{\bar{\epsilon}_1 + \bar{\epsilon}_2}{2} \, J^L_3 + \frac{\bar{\epsilon}_1 - \bar{\epsilon}_2}{2} \, J^R_3 \tag{12}$$

where we have used the decomposition $SU(2)_L \times SU(2)_R$ of the rotation group $Spin(4)$. Clearly, $[\Omega, \bar{\Omega}] = 0$. In the limit $r \to 0$ we get four dimensional gauge theory. We could also take the limit to the five dimensional theory, by considering the degenerate torus $\mathbb{T}^2$. We note in passing that the complex structure of the torus $\mathbb{T}^2$ could be kept finite. The resulting four dimensional theory (for gauge group $SU(2)$) is related to the theory of the so-called E-strings [34, 35]. The instanton contributions to the correlation functions of the chiral operators in this theory are related to the elliptic genera of the instanton moduli space [36] and could be summed up, giving rise to the Seiberg-Witten curves for these theories. However, in this paper we shall neither discuss elliptic, nor trigonometric limits, even though they lead to interesting integrable systems [37].

The action of the four dimensional theory in the limit $r \to 0$ is not that of the pure supersymmetric Yang-Mills theory on $\mathbb{R}^4$. Rather, it is a deformation of the latter by the Ω, $\bar{\Omega}$-dependent terms. We shall write down here only the terms with bosonic fields (for simplicity, we have set $\vartheta_0 = 0$) which are

$$S(\Omega)^{bos} = -\frac{1}{2g_0^2} \mathrm{Tr} \left(\frac{1}{2} F^2_{\mu\nu} + (D_\mu \Phi - \Omega^\nu_\lambda x^\lambda F_{\mu\nu})(D_\mu \bar{\Phi} - \bar{\Omega}^\nu_\lambda x^\lambda F_{\mu\nu}) + [\Phi, \bar{\Phi}]^2 \right) . \tag{13}$$

We shall call the theory (13) an $\mathcal{N} = 2$ theory in the Ω-background. It is amusing that this deformation indeed can be described as a superspace-dependent bare coupling τ_0 given by

$$\tau_0(x, \theta; \mu_{UV}) = \tau_0(\mu_{UV}) + \bar{\Omega}^- \theta\theta + \Omega_{\mu\nu} \bar{\Omega}_{\mu\lambda} x^\nu x^\lambda \,. \tag{14}$$

We are going to study the correlation functions of chiral observables. These observables are gauge invariant holomorphic functions of the superfield $\mathbf{\Phi}$. Viewed as a function on the superspace, every such observable $\mathcal{O}$ can be decomposed into

$$\mathcal{O}[\mathbf{\Phi}(x,\theta)] = \mathcal{O}^{(0)} + \mathcal{O}^{(1)}\theta + \ldots + \mathcal{O}^{(4)}\theta^4 \,. \tag{15}$$

The component $\mathcal{O}^{(4)}$ can be used to deform the action of the theory, this deformation is equivalent to the addition of $\mathcal{O}$ to the bare prepotential.

It is a nice property of the chiral observables that their correlation functions are independent of the anti-chiral deformations of the theory, in particular of $\bar{\tau}_0$.[a] We can, therefore, consider the limit $\bar{\tau}_0 \to \infty$. In this limit the term

$$\bar{\tau}_0 \, \|F^+\|^2$$

in the action localizes the path integral onto the instanton configurations. In addition, the Ω-background further localizes the measure on the instantons, invariant under rotations. Finally, the vev of the Higgs field shrinks these instantons to points, thus eliminating all integrations, reducing them to a single sum over the point-like invariant instantons.

Now we want to pause to discuss other physical realizations of our $\mathcal{N} = 2$ theories.

2.2. *String theory realizations*

The $\mathcal{N} = 2$ theory can arise as the low energy limit of the theory on a stack of D-branes in type II gauge theory. A stack of N parallel D3 branes in IIB theory in flat $\mathbb{R}^{1,9}$ carries $\mathcal{N} = 4$ supersymmetric Yang-Mills theory [38]. A stack of parallel D4 branes in IIA theory in flat $\mathbb{R}^{1,9}$ carries $\mathcal{N} = 2$ supersymmetric Yang-Mills theory in five dimensions. Upon compactification on a circle the latter theory reduces to the former in the limit of zero radius.

Now consider the stack of N D4 branes in the geometry $\mathbb{S}^1 \times \mathbb{R}^{1,8}$ with the metric

$$ds^2 = dx^\mu dx^\mu + r^2 d\varphi^2 + dv^2 + |dZ_1 + mrZ_1 d\varphi|^2 + |dZ_2 - mrZ_2 d\varphi|^2 \,. \tag{16}$$

Here x^μ denote the coordinates on the Minkowski space $\mathbb{R}^{1,3}$, φ is the periodic coordinate on the circle of circumference r, v is a real transverse direction, Z_1 and Z_2 are the holomorphic coordinates on the remaining $\mathbb{C}^2$. The worldvolume of the branes is $\mathbb{S}^1 \times \mathbb{R}^{1,3}$, which is located at $Z_1 = Z_2 = 0$,

[a] However, beware of the holomorphic anomaly.

and $v = v_l$, $l = 1, \ldots, N$. Together with the Wilson loop eigenvalues $e^{i\sigma_1}, \ldots, e^{i\sigma_N}$ around $\mathbb{S}^1$, the v_l's form N complex moduli $w_1, \ldots, w_N$, parameterizing the moduli space of vacua. In the limit $r \to 0$ the N complex moduli loose periodicity.

It is easy to check that the worldvolume theory has $\mathcal{N} = 2$ susy, with the massive hypermultiplet in the adjoint representation (of mass m). This realization is T-dual to the standard realization with the NS5 branes, as in Ref. 39.[b] Note that the background (10) is similar to (16). However, the D-branes are differently located, a fact which leads to very interesting geometries upon making a T-duality transformation and lifts to M-theory, providing (hopefully) another useful insight.

However, in our story we want to analyze the pure $\mathcal{N} = 2$ supersymmetric Yang-Mills theory. This can be achieved by taking the $m \to \infty$ limit, at the same time taking the weak string coupling limit. The resulting brane configuration can be described using two parallel NS5 branes and N D4 brane suspended between them, as in Ref. 39, or, alternatively, as a stack of N D3 (fractional) branes stuck at the $\mathbb{C}^2/\mathbb{Z}_2$ singularity, as in Ref. 40. In fact the precise form of the singularity is irrelevant, as long as it corresponds to a discrete subgroup of $SU(2)$, and all the fractional branes are of the same type. Note that for $mr = \frac{1}{K}$ the (Z^1, Z^2) part of the metric (16) in the limit $r \to 0$ looks like the metric on the orbifold $\mathbb{C}^2/\mathbb{Z}_K$. The relation between these two pictures is through the T-duality of the resolved $\mathbb{C}^2/\mathbb{Z}_2$ singularity. The fractional D3 branes blow up into D5 branes wrapping a non-contractible two-sphere. The resolved space $T^*\mathbb{CP}^1$ has a $U(1)$ isometry, with two fixed points (the North and South poles of the non-contractible two-sphere). Upon T-duality these turn into two NS5 branes. The D5 branes dualize to D4 branes suspended between NS5's.

The instanton effects in this theory are due to the fractional D(-1) instantons, which bind to the fractional D3 branes, in the IIB description. The "worldvolume" theory on these D(-1) instantons is the supersymmetric matrix integral, which we describe with the help of the ADHM construction below. In the IIA picture the instanton effects are due to Euclidean D0 branes, which "propagate" between two NS5 branes.

The IIB picture with the fractional branes corresponds to the metric (before Ω is turned on)

$$ds^2 = dx^\mu dx^\mu + dw d\bar{w} + ds^2_{\mathbb{C}^2/\mathbb{Z}_2}\,. \tag{17}$$

[b] NN thanks M. Douglas for an illuminating discussion on this point.

The singularity $\mathbb{C}^2/\mathbb{Z}_2$ has five moduli in IIB string theory: three parameters of the geometric resolution of the singularity, and the fluxes of the NSNS and RR 2-forms through the two-cycle which appears after blowup. The latter are responsible for the gauge couplings on the fractional D3 branes, as in Ref. 41,

$$\tau_0 = \int_{\mathbb{S}^2} B_{RR} + \tau_{IIB} \int_{\mathbb{S}^2} B_{NSNS} \ . \tag{18}$$

Our conjecture is that turning on the higher Casimirs, (and gravitational descendants on the dual closed string side) corresponds to a "holomorphic wave", where τ_0 depends holomorphically on w. From Ref. 42 this is known to be a solution of IIB supergravity.

We shall return to the fractional brane picture later on. Right now let us mention another stringy effect. By turning on the constant NSNS B-field along the worldvolume of the D3-branes we deform the super-Yang-Mills on $\mathbb{R}^4$ to the super-Yang-Mills on the noncommutative $\mathbb{R}^4_\Theta$, see [43–45]. On the worldvolume of the D(-1) instantons the noncommutativity acts as a Fayet-Illiopoulos term, deforming the ADHM equations, as in [46–48], and resolving the singularities of the instanton moduli space, as in Ref. 49. We shall use this deformation as a technical tool, so we shall not describe it in much detail. The necessary references can be found in Ref. 44.

At this point we remark that even for $N = 1$ the instantons are present in the D-brane picture. They become visible in the gauge theory when noncommutativity is turned on. Remarkably, the actual value of the non-commutativity parameter Θ does not affect the expectation values of the chiral observables, thus simplifying our life enormously.

So far we presented the D-brane realization of $\mathcal{N} = 2$ theory. There exists another useful realization, via local Calabi-Yau manifolds, introduced in Ref. 6. This realization, as we already explained in the introduction is useful in relating the prepotential to the topological string amplitudes. If the theory is embedded in the IIA string on local Calabi-Yau, then the interesting physics comes from the worldsheet instantons, wrapping some 2-cycles in the Calabi-Yau. In the mirror IIB description one gets a string without worldsheet instantons contributing to the prepotential, and effectively reducing to some field theory. This field theory is known in the case of global Calabi-Yau. But it is not known explicitly in the case of local Calabi-Yau. As we shall show, it can be sometimes identified with the free fermion theory on an auxiliary Riemann surface (cf. Ref. 25).

The relation with the geometrical engineering of Ref. 6 is also useful in making contact between our Ω-deformation and the sugra backgrounds with graviphoton field strength turned on. Indeed, our construction involved a lift to five or six dimensions. The first case embeds easily in IIA string theory where this corresponds to the lift to M-theory. To see the whole six dimensional picture (10) one should use IIB language and the lift to F-theory (one has to set $\bar{\Omega} = 0$, though).

Let us consider the five dimensional lift. We have M-theory on the 11-fold with the metric

$$ds^2 = (dx^\mu + \Omega^\mu_\nu x^\nu d\varphi)^2 + r^2 d\varphi^2 + ds^2_{CY} \,. \tag{19}$$

Here we assume, for simplicity, that $\epsilon_1 = -\epsilon_2$, so that $\Omega = \Omega^-$ generates an $SU(2)$ rotation, thus preserving half of susy. Now let us reduce on the circle $\mathbb{S}^1$ and interpret the background (19) in the type IIA string. Using Ref. 1 we arrive at the IIA background

$$g_s = \left(r^2 + \|\Omega \cdot x\|^2\right)^{\frac{3}{4}} \,, \tag{20}$$

$$A^{grav} = \frac{1}{r^2 + \|\Omega \cdot x\|^2}\, \Omega_{\mu\nu} x^\mu dx^\nu \,,$$

$$ds^2_{10} = \frac{1}{\sqrt{r^2 + \|\Omega \cdot x\|^2}} \left(r^2 dx^2 + \Omega^\mu_\nu \Omega^\lambda_\kappa \left(x^2 dx^2 \delta^{\nu\kappa} \delta_{\mu\lambda} - x^\nu x^\kappa dx^\mu dx^\lambda\right)\right) + \\ + \sqrt{r^2 + \|\Omega \cdot x\|^2}\, ds^2_{CY} \,,$$

where the graviphoton $U(1)$ field is turned on. The IIA string coupling becomes strong at $x \to \infty$. However, the effective coupling in the calculations of $\mathcal{F}_g$ is

$$\hbar \sim g_s \sqrt{\|dA^{grav}\|^2} \sim \left(r^2 + \|\Omega \cdot x\|^2\right)^{-\frac{1}{4}} \to 0, \qquad x \to \infty \,. \tag{21}$$

2.3. *The partition function*

Our next goal is the calculation of the partition function,

$$Z(\tau_{\vec{n}}; a, \Omega) = \int_{\phi(\infty) = a} D\Phi DA D\lambda \ldots e^{-S(\Omega)} \,, \tag{22}$$

of the $\mathcal{N} = 2$ susy gauge theory with all the higher couplings (1) on the background (10) with the fixed asymptotics of the Higgs field at infinity. We use the fact that the chiral deformations are not sensitive to the anti-chiral parameters (up to holomorphic anomaly). We take the limit $\bar{\tau}_0 \to \infty$, and the partition function becomes the sum over the instanton charges of the integrals over the moduli spaces $\mathfrak{M}$ of instantons of the measure, obtained by the developing the path integral perturbation expansion around instanton solutions.

On the other hand, if we take instead a low-energy limit, this calculation should reduce to that of the low-energy effective theory. In the Seiberg-Witten story [17] the low-energy theory is characterized by the complexified energy scale $\Lambda \sim \mu_{UV} e^{2\pi i \tau_0(\mu_{UV})}$. We now recall (14). In our setup the low-energy scale is (x, θ)-dependent,

$$\Lambda(x,\theta) = \mu_{UV} e^{2\pi i \tau_0(x,\theta;\mu_{UV})} = \Lambda e^{2\pi i \bar{\Omega}^- \theta^2 - \|\Omega \cdot x\|^2} . \tag{23}$$

Near $x = 0$ it is finite, while at $x \to \infty$ the theory becomes infinitely weakly coupled. With (14) in mind we can easily relate the partition function to the prepotential (5) (cf. Ref. 10) by

$$\begin{aligned} Z &= Z^{pert} \exp \left[\int d^4x d^4\theta \mathcal{F}^{inst} \left(a; \tau_{\vec{n}}; \Lambda(x,\theta)\right) + \text{higher derivatives} \right] \\ &= \exp \frac{1}{\epsilon_1 \epsilon_2} \Big[\mathcal{F}(a, \tau_{\vec{n}}; \Lambda) + O(\epsilon_1, \epsilon_2) \Big] , \end{aligned} \tag{24}$$

where $\mathcal{F}^{inst}$ is the sum of all instanton corrections to the prepotential, and Z^{pert} is the result of the perturbative calculation on the Ω-background. The corrections in $\epsilon_{1,2}$ come from the ignored higher derivative terms.

2.4. *Perturbative part*

The perturbative part is given by the one-loop contribution from W-bosons, as well as non-zero angular momentum modes of the abelian photons (we shall comment on this below). Recall that in the Ω-background one can integrate out all non-zero modes, as Ω lifts all massless fields. Because of the reduced supersymmetry the determinants do not quite cancel. The simplest way to calculate them is to go to the basis of normalizable spherical harmonics where

$$\Phi = \sum_{l,m=1}^{N} T_{lm} \sum_{i,j,\bar{i},\bar{j} \geq 1} \phi^{lm}_{ij\bar{i}\bar{j}} \; z_1^{i-1} z_2^{j-1} \bar{z}_1^{\bar{i}-1} \bar{z}_2^{\bar{j}-1} \; e^{-|z_1|^2 - |z_2|^2} \tag{25}$$

and similarly for the components of the gauge fields and so on. Here the terms with $l \neq m$ correspond to the W-bosons, massive components of the Higgs field, and the massive components of the gluinos, while $l = m$ represent the abelian part. We are doing the WKB calculation around the trivial gauge field $A = 0$; the unbroken susy guarantees that there are no further corrections. The integral over the bosonic and fermionic fluctuations formally becomes a ratio of the determinants,

$$\prod_{l,m=1}^{N}\prod_{i,j=1}^{\infty}\prod_{\bar{i},\bar{j}=1}^{\infty}\frac{(a_{lm}+\epsilon_1(i-\bar{i})+\epsilon_2(j-\bar{j}))(a_{lm}+\epsilon_1(i-\bar{i}-1)+\epsilon_2(j-\bar{j}-1))}{(a_{lm}+\epsilon_1(i-\bar{i}-1)+\epsilon_2(j-\bar{j}))(a_{lm}+\epsilon_1(i-\bar{i})+\epsilon_2(j-\bar{j}-1))}\,. \tag{26}$$

(Recall that the "weight" of $A_\mu(z,\bar{z})$ has in addition to its "orbital" weight, which comes from the $(z,\bar{z})$-dependence, a spin $(-\epsilon_1)$ weight, similarly for $A_{\bar{2}}$ we have an extra $(-\epsilon_2)$, for $F^{0,2}$ extra $(-\epsilon_1-\epsilon_2)$.) In the product over $\bar{i},\bar{j}$ only the term with $\bar{i}=\bar{j}=1$ is not canceled, giving rise to

$$Z^{pert}=\prod_{l,m;i,j\geq 1}{}^{\prime}\,(a_l-a_m+\epsilon_1(i-1)+\epsilon_2(j-1)) \tag{27}$$

times the conjugate term, which depends on $\bar{a}$. We shall ultimately take $\bar{a}\to\infty$, so we ignore this term – at any rate, it cancels out in the correlation functions of the chiral observables. The symbol $\prod'$ in Eq.(27) means that the contribution of the abelian zero angular momentum modes $l=m$, $i=j=1$ to the product is omitted (this has to do with our boundary conditions). We shall always understand Eq.(27) in the sense of ζ-regularization. After regularization one can analytically continue to $\epsilon_1+\epsilon_2=0$.

In fact, for $\epsilon_1=-\epsilon_2=\hbar$ one can expand

$$Z(\tau_{\vec{n}};a,\Omega)=\exp\left(-\sum_{g=0}^{\infty}\hbar^{2g-2}\mathcal{F}_g(a;\tau_{\vec{n}};\Lambda)\right)\,. \tag{28}$$

The higher "prepotentials" $\mathcal{F}_g$ will turn out later to be related to the higher genus string amplitudes.

2.5. *Mathematical realization of $\mathcal{N}=2$ theory*

The mathematical realization of the gauge theory we are studying is the following (details are in the appendix A). Consider the space $\mathcal{Y}$ of all gauge fields on $\mathbb{R}^4$ with finite Yang-Mills action. There are three groups of symmetries acting on this space which we shall study. The first group, $\mathcal{G}_\infty$, is the group of gauge transformations which are trivial at infinity, $g(x)\to 1$, $x\to\infty$. The second, G is the group of constant, global, gauge transformations. The group of all gauge transformations $\mathcal{G}$ is the extension of $\mathcal{G}_\infty$ by G, such that $G=\mathcal{G}/\mathcal{G}_\infty$. The third group $K=Spin(4)$, is the covering group of the group of Euclidean rotations about some fixed point $x=0$. Over the space $\mathcal{Y}$ we consider the $\mathcal{G}\ltimes K$-equivariant vector bundle $\mathcal{V}$ of the self-dual two-forms on $\mathbb{R}^4$ with the values in the adjoint representation of the gauge group $\mathcal{G}$. For a gauge field $A \in \mathcal{Y}$ the self-dual projection of its curvature F_A^+ defines a section of $\mathcal{V}$.

The path integral measure of the supersymmetric gauge theory with the extra Ω-couplings is nothing but the Mathai-Quillen representative of the Euler class of $\mathcal{V}$, written using the section F^+, and working with $\mathcal{G} \ltimes K$ equivariantly. Calculating the path integral corresponds to the pushforward onto the quotient by the group $\mathcal{G}_\infty$ and its further localization w.r.t the remaining groups $G \times K$. The result is given by the sum over the fixed points of the $G \times K$ action on the moduli space of instantons $\mathfrak{M}$, i.e. solutions to $F^+ = 0$.

The chiral observables translate to the equivariant Chern classes of some natural bundles (sheaves) over the moduli space $\mathfrak{M}$. Their calculation is more or less standard and is presented in the next section.

2.6. *Nonperturbative part*

We now proceed with the calculation of the nonperturbative contribution to the partition function (22). There are two ways of determining it. One way is the direct analysis of the saddle points of the path integral measure. This is a nice exercise, but it relies on very explicit knowledge of the deformed instanton solutions [48, 50, 51], invariant under the action of the group K of rotations [52]. Instead, we shall choose slightly less explicit, but more general route.

The general property of the chiral observables in $\mathcal{N} = 2$ theories, which is a direct consequence of the analysis in [53], is the cohomological nature of their correlation functions. Namely, in the limit $\bar{\tau}_0 \to 0$ these become the integrals over the instanton moduli space $\mathfrak{M}$. The chiral observables, evaluated on the instanton collective coordinates, become closed differential forms. Thus, if the moduli space $\mathfrak{M}$ was compact and smooth, one could choose some convenient representatives of their cohomology classes to evaluate their integrals. Moreover, a generalization of the arguments in [53] allows us to consider the $\mathcal{N} = 2$ theory in the Ω-background. In this case the differential forms on $\mathfrak{M}$ become K-equivariantly closed. Even though the space $\mathfrak{M}$ is not compact, the space of K-fixed points is, and this is good enough for the evaluation of the integrals of the K-equivariant integrals.

The final bit of information which makes the calculation of the chiral observables constructed out of the higher Casimirs possible, is the identification of the K-equivariantly closed differential forms on $\mathfrak{M}$ that they represent with the densities of the equivariant Chern classes of some natural bundles over $\mathfrak{M}$. We now proceed with the explicit description of $\mathfrak{M}$, these natural bundles, and finally the chiral observables.

ADHM construction

To get a handle on these fixed point sets and to calculate the characteristic numbers of the various bundles we have defined above, we need to recall a few facts about the actual construction of $\mathfrak{M}$, the so-called ADHM construction, see [49,54]. In this construction one starts with two Hermitian vector spaces W and V. One then looks for four Hermitian operators $\mathbb{X}^\mu : V \to V$, $\mu = 1,2,3,4$ and two complex operators $\lambda_\alpha : W \to V$, $\alpha = 1,2$ (and $\bar{\lambda}_{\dot\alpha} = \lambda^\dagger_\alpha : V \to W$), which can be combined into a sequence

$$0 \to W \otimes S_- \longrightarrow V \oplus W \otimes S_+ \to 0\,, \tag{29}$$

where the non-trivial map is given by

$$\mathcal{D}^+ = \lambda \oplus \mathbb{X}^\mu \sigma_\mu\,.$$

The ADHM equation requires that $\mathcal{D}\mathcal{D}^+$ commutes with the Pauli matrices σ_μ acting in S_-. In addition, one requires that $\mathcal{D}\mathcal{D}^+$ has a maximal rank. The moduli space $\mathfrak{M}$ is then identified with the space of such $\mathbb{X}, \lambda$ up to the action of the group $U(k)$ of unitary transformations in V. The group $G = U(N)$ acts on $\mathfrak{M}$ by the natural action, descending from that on λ ($\mathbb{X}$ are neutral). The group $K \approx Spin(4)$ acts on $\mathfrak{M}$ by rotating $\mathbb{X}$ in the vector representation and λ in the appropriate chiral spinor representation.

D-brane picture, again

The ADHM construction becomes very natural when the gauge theory is realized with the help of D-branes. The space V is the Chan-Paton space for the D(-1) branes, while W is the Chan-Paton space for the D3 branes. The matrices $\mathbb{X}$ are the ground states of the $(-1,-1)$ strings, while $\lambda_\alpha, \bar{\lambda}_{\dot a}$ are those of $(-1,3)$, $(3,-1)$. The ADHM equations are the conditions for unbroken susy. Their solutions describe the Higgs branch of the D(-1) instanton theory.[c] The D(-1) instantons also carry a multiplet responsible for the $U(V)$ "gauge" group. In particular, quantization of $(-1,-1)$ strings in addition to $\mathbb{X}$ gives rise to a matrix ϕ (not to be confused with Φ in the

[c] To make these statements literally true one should consider a D2-D6 system instead of D(-1)-D3 (in order to avoid off-shell string amplitudes, and the non-existence of moduli spaces of vacua for field theories in less than three dimensions).

adjoint of $U(N)$!) in the adjoint of $U(k)$, which represents the motion of D(-1) instantons in the directions, transverse to D3 branes.

Tangent and universal bundles

Here we recall some standard constructions. The problem considered here is typical in soliton physics. One finds some moduli space of solutions (collective coordinates) which should be quantized. The supersymmetric theories lead to supersymmetric quantum mechanics on the moduli spaces. If the gauge symmetry is present the collective coordinates are defined with the help of some gauge fixing procedure, which leads to the complications described below.

The tangent space to the instanton moduli space $\mathfrak{M}$ at the point m can be described as follows. Pick a gauge field A which corresponds to $m \in \mathfrak{M}$, $F^+(A) = 0$. Any two such choices differ by a gauge transformation. Now consider deforming A,

$$A \to A + \delta A \,,$$

so that the new gauge field also obeys the instanton equation $F^+(A+\delta A) = 0$. In other words, δA obeys the linear equations

$$D_A^+ \delta A = 0 \,, \tag{30}$$

$$D_A^* \delta A = 0 \,,$$

where the first equation is the linearized anti-self-duality equation, while the second is the gauge choice to project out the trivial deformations $\delta A \sim D_A \varepsilon$. Let us choose some basis in the (finite-dimensional) vector space of solutions to Eq. (30): $\delta A = a_\mu^K dx^\mu \zeta_K$, where a^K obey Eq. (30), and, say, are orthonormal with respect to the natural metric $\langle a^L | a^K \rangle \equiv \int_{\mathbb{R}^4} a^L \wedge \star a^K = \delta_{LK}$, $L, K = 1, \ldots, \dim \mathfrak{M}$. Now suppose we have a family of instanton gauge fields, parameterized by the points of $\mathfrak{M}$: $A_\mu(x; m)$, where $x \in \mathbb{R}^4$, $m \in \mathfrak{M}$. Let us differentiate A_μ w.r.t the moduli m. Clearly, one can expand

$$\frac{\partial A}{\partial m^L} = a^K \zeta_{LK} + D_A \varepsilon_L \,. \tag{31}$$

The compensating gauge transformations ε_L together with $A_\mu(m)$ form a connection $\mathcal{A} = A_\mu(x; m) dx^\mu + \varepsilon_L dm^L$ in the rank N vector bundle $\mathcal{E}$ over $\mathfrak{M} \times \mathbb{R}^4$. Now let us calculate its full curvature

$$\mathcal{F} = \mathbf{d}\mathcal{A} + [\mathcal{A}, \mathcal{A}], \qquad \mathbf{d} = d_{\mathfrak{M}} + d_{\mathbb{R}^4} \,, \tag{32}$$

$$\mathcal{F} = \Phi + \Psi + F\,, \tag{33}$$

where Φ is a two-form on $\mathfrak{M}$, Ψ is a one-form on $\mathfrak{M}$ and one-form on $\mathbb{R}^4$, and F is a two-form on $\mathbb{R}^4$. A straightforward calculation shows that Φ, Ψ, F solve the equations

$$\Delta_A \Phi = [\Psi, \star\Psi], \qquad D_A^+ \Psi = 0, \quad D_A^* \Psi = 0, \qquad F^+ = 0\,. \tag{34}$$

The equation for Φ is (up to Q-exact terms) identical to the equation for the adjoint Higgs field in the instanton background, while the equation for Ψ is (again, up to Q-exact terms) identical to that for gluon zero modes. This relation between $\mathcal{F}$ and the chiral observables (which are, after all, polynomials in Φ, Ψ, F, up to Q-exact terms) will prove extremely useful in what follows. In particular, we can write

$$\mathcal{O}_J^{(0)} = \frac{1}{J}\,\mathrm{Tr}\,\Phi^J, \qquad \ldots\ , \tag{35}$$

$$\begin{aligned}\mathcal{O}_J^{(4)} = &\sum_{l=0}^{J-2} \mathrm{Tr}\left(\Phi^l F \Phi^{J-2-l} F\right) + \sum_{l,n\geq 0, l+n\leq J-3} \mathrm{Tr}\left(\Phi^l F \Phi^n \Psi \Phi^{J-3-l-n}\Psi\right) \\ &+ \sum_{l,k,n\geq 0, l+k+n\leq J-4} \mathrm{Tr}\left(\Phi^l \Psi \Phi^k \Psi \Phi^n \Psi \Phi^{J-4-k-l-n}\Psi\right)\,,\end{aligned} \tag{36}$$

where we substitute the expressions for Φ, Ψ, F from Eq. 33.

A mathematically oriented reader would object at this point, as it *is well-known* that universal bundles together with nice connections do not exist over the compactified moduli spaces. We shall not pay attention to these (fully justified) remarks, as eventually there is a way around. We find it more straightforward to explain things as if such objects existed over the compactified moduli space of instantons. Let p denote the projection $\mathfrak{M} \times \mathbb{R}^4 \to \mathfrak{M}$. Suppose we know everything about $\mathcal{E}$. How would we reconstruct $T\mathfrak{M}$ from there? We know already that the tangent space to $\mathfrak{M}$ at a point m is spanned by the solutions to Eq. (30). Clearly these solutions should be identified with the cohomology of the Atiyah-Singer complex,

$$0 \longrightarrow \Omega^0(\mathbb{R}^4) \otimes \mathfrak{g} \longrightarrow \Omega^1(\mathbb{R}^4) \otimes \mathfrak{g} \longrightarrow \Omega^{2,+}(\mathbb{R}^4) \otimes \mathfrak{g} \longrightarrow 0\,, \tag{37}$$

where the first non-trivial arrow is the infinitesimal gauge transformation $\varepsilon \mapsto D_A\varepsilon$ and the second is $\delta A \mapsto D_A^+ \delta A$. Thanks to $F_A^+ = 0$ this is indeed a complex, i.e. $D_A^+ D_A = 0$. The spaces $\Omega^k \otimes \mathfrak{g}$ can be viewed as the bundles over $\mathfrak{M} \times \mathbb{R}^4$; e.g. for $G = U(N)$,

$$\Omega^k(\mathbb{R}^4) \otimes \mathfrak{g} = \mathcal{E} \otimes \mathcal{E}^* \otimes \Lambda^k T^* \mathbb{R}^4\,. \tag{38}$$

Generically the complex (37) has only H^1 cohomology. We are thus led to identify the K-classes $T\mathfrak{M} = H^1 - H^0 - H^2$.

Framing and Dirac bundles

We shall need two more natural bundles over $\mathfrak{M}$. As $\mathfrak{M}$ is defined by the quotient w.r.t. the group of gauge transformations, trivial at infinity, we have a bundle W over $\mathfrak{M}$ whose fiber is the fiber of the original $U(N)$ bundle over $\mathbb{R}^4$ at infinity. Another important bundle is the bundle V of Dirac zero modes. Its fiber over the point $m \in \mathfrak{M}$ is the space of normalizable solutions to the Dirac equation in the fundamental representation in the background of the instanton gauge field, corresponding to m. In $K(\mathfrak{M})$,

$$W = \lim_{x \to \infty} \mathcal{E}|_x \,, \qquad\qquad V = p_* \mathcal{E} \,. \tag{39}$$

The pushforward p_* is defined here in the L^2 sense. In what follows we shall need its equivariant analogue. Finally, let $S_\pm$ denote the bundles of positive and negative chirality spinors over $\mathbb{R}^4$. These bundles are trivial topologically. However they are nontrivial as K-equivariant bundles.

Relations among bundles

We arrive at the relations

$$\mathcal{E} = W \oplus V \otimes (S_+ - S_-) \,, \tag{40}$$

$$T\,\mathfrak{M} = -p_* \left(\mathcal{E} \otimes \mathcal{E}^* \right) \,.$$

among the virtual bundles. The chiral operators $\mathcal{O}_{\vec{n}}$ we discussed in the introduction are now in one-to-one correspondence with the characteristic classes of the $U(N)$ bundles. A convenient basis in the space of such classes is given by the skew Schur functions, labeled by the partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \lambda_N \geq 0)$,

$$\mathbf{ch}_\lambda = \mathrm{Det} \| ch_{\lambda_i - i + j} \| \,. \tag{41}$$

Another basis is labeled by finite sequences $n_1, n_2, \ldots, n_k$ of non-negative integers,

$$\mathcal{O}_{\vec{n}} = \prod_{J=1}^{\infty} \frac{1}{n_J!} \left(\frac{ch_J}{J} \right)^{n_J} \,. \tag{42}$$

It is this basis that we used in Eq. (1).

The Eqs. (40) imply some relations among the Chern classes, which we shall use. It is convenient to discuss the Chern characters first. Recall that we always work $G \times K$-equivariantly.

We get

$$Ch(\mathcal{E}) = Ch(W) + Ch(V) \prod_{i=1}^{2} \left(e^{\frac{x_i}{2}} - e^{-\frac{x_i}{2}} \right) , \tag{43}$$

$$Ch(T\mathfrak{M}) = - \int_{\mathbb{R}^4} Ch(\mathcal{E}) Ch(\mathcal{E}^*) \prod_{i=1}^{2} \left(\frac{x_i}{e^{\frac{x_i}{2}} - e^{-\frac{x_i}{2}}} \right) .$$

Here x_1, x_2 are the equivariant Chern roots of the tangent bundle to $\mathbb{R}^4$,

$$x_i = \epsilon_i + \mathcal{R}_i \, , \tag{44}$$

where $\mathcal{R}_i = \frac{1}{2\pi i} \delta^2(z_i) dz_i \wedge d\bar{z}_i$ is a curvature two-form[d] on $\mathbb{R}^4$. As everything is K-equivariant, the integral over $\mathbb{R}^4$ localizes onto the K-fixed point, the origin, as is also clear from the explicit formula Eq. (44)

$$Ch(T\mathfrak{M}) = - \left[Ch(\mathcal{E}) Ch(\mathcal{E}^*) \right]_0 \prod_{i=1}^{2} \left(\frac{1}{e^{\frac{\epsilon_i}{2}} - e^{-\frac{\epsilon_i}{2}}} \right) , \tag{45}$$

where $\left[Ch(\mathcal{E}) Ch(\mathcal{E}^*) \right]_0$ is the evaluation of the product of the Chern characters at the origin of $\mathbb{R}^4$.

Integration over $\mathfrak{M}$

Now we want to integrate over $\mathfrak{M}$. Suppose the integrand is the $\mathbb{G} = G \times K$-equivariant differential form (see appendix A for definitions) $\Omega_{\mathcal{O}}[\varphi]$, $\varphi \in Lie(\mathbb{G})$. Such integrals can be computed using localization. In plain words this means that they are given by the sums over the fixed points of the action of the one-parametric subgroup $\exp(t\mathfrak{a})$, $t \in \mathbb{R}$, of $\mathbb{G}$, $\mathfrak{a} \in Lie(\mathbb{G})$. The contribution of each fixed point $P \in \mathfrak{M}$ (assuming it is isolated and $\mathfrak{M}$ is smooth at this point) is given by the ratio

$$Z_P = \frac{\Omega_{\mathcal{O}}[\mathfrak{a}]^{(0)}|_P}{c(T\mathfrak{M})[\mathfrak{a}]^{(0)}|_P} \, , \tag{46}$$

where $\omega^{(0)}$ denotes the scalar component of the inhomogeneous differential form corresponding to the equivariant differential form ω, and $c(T\mathfrak{M})$ is the

[d] For those worried by the singular form of Eq. (44), here is a nonsingular representative. Choose a smooth function $f(r)$ which is equal to 1 for sufficiently large r, and vanishes at $r = 0$. Then x_i is K-equivariantly cohomologous to $\epsilon_i f(|z_i|^2) + \frac{1}{2\pi} f'(|z_i|^2) dz_i \wedge d\bar{z}_i$.

equivariant Chern polynomial of $T\mathfrak{M}$. It is defined as follows. As $T\mathfrak{M}$ is $\mathbb{G}$-equivariant, with respect to the maximal torus $\mathbb{T}$ it splits into a direct sum of the line bundles, $T\mathfrak{M} = \bigoplus_i L_i$, on which $\mathfrak{t}$ acts with some weight w_i (a linear function on $\mathfrak{t}$). The equivariant Chern polynomial is defined simply by

$$c(T\mathfrak{M})[\mathfrak{a}] = \prod_i \left(c_1(L_i) + w_i(\mathfrak{a})\right) . \tag{47}$$

Physicists are familiar with the Duistermaat-Heckmann formulae like Eq. (46) [55] in the context of two-dimensional Yang-Mills theory, and in the (perhaps less known) context of sigma models [56].

In order to proceed we need to calculate the numerator and the denominator of Eq. (46) and to sum over the points P. We need first the equivariant Chern polynomial $c(T\mathfrak{M})$. We already have an expression Eq. (45) for the equivariant Chern character of $T\mathfrak{M}$. To use it we recall that, in terms of L_i's,

$$Ch(T\mathfrak{M}) = \sum_i e^{c_1(L_i)+w_i(\mathfrak{a})} \tag{48}$$

so that if (48) is known then so is (47). Moreover, if the fixed points P are isolated (and they will be), the actual first Chern classes of L_i will never contribute (they are two-forms and we simply want to evaluate Eqs. (48, 47) at a point P), so we only need to find the w_i's – the weights.

Now, what about $\Omega_{\mathcal{O}}$? Well, we construct it using the descendents of the Casimirs $\mathrm{Tr}\,\Phi^J$ and their multi-trace products. As we explained above, these become polynomials in the traces of the powers of the universal curvature $\mathcal{F}$ as in Eq. (35). That is to say, they are cohomologous to the Chern classes of the universal bundle $\mathcal{E}$.

We are mostly interested in the correlators of the 4-descendents $\mathcal{O}^{(4)}$ of the invariant polynomials $\mathcal{P}(\Phi)$ on $Lie(G)$. On the moduli space $\mathfrak{M}$ these are cohomologous to the integrals over $\mathbb{R}^4$ of the polynomials in the Chern classes $ch_k(\mathcal{E})$ of the universal bundle. Again, thanks to $\mathbb{G}$-equivariance, these integrals are simply given by localization at the origin in $\mathbb{R}^4$,

$$\mathcal{O}^{(4)}_{\mathcal{P}} = \frac{[\mathcal{P}(\mathcal{F})]_0}{\epsilon_1\epsilon_2} . \tag{49}$$

For $\mathcal{P}_k(\Phi) = \frac{1}{(2\pi i)^k\ k!}\mathrm{Tr}\,\Phi^k$, $\mathcal{P}_k(\mathcal{F}) = ch_k(\mathcal{E})$. Any other invariant polynomial is a polynomial in these $\mathcal{P}_k$.

Evaluation of Chern classes at fixed points

So, we see that everything reduces to the enumeration of the fixed points P, and the evaluation of the Chern classes of $\mathcal{E}$ at these points. Moreover, thanks to Eq. (48) it is sufficient to evaluate the restriction of $Ch(W)$ and $Ch(V)$.

These problems were solved in Ref. 10 for any N using the results of Ref. 49 for $N = 1$. The result is the following. The fixed points are in one-to-one correspondence with the N-tuples of partitions, $\vec{\mathfrak{k}} = (\mathfrak{k}_1, \ldots, \mathfrak{k}_N)$, where

$$\mathfrak{k}_l = (k_{l1} \geq k_{l2} \geq k_{l3} \geq \ldots k_{l\ n_l} > k_{l\ n_l+1} = 0 \ldots) \ . \tag{50}$$

At the fixed point $P_{\vec{\mathfrak{k}}}$ corresponding to such an N-tuple, the Chern characters of the bundles W and V evaluate to

$$[Ch(W)]_{P_{\vec{\mathfrak{k}}}} = \sum_{l=1}^{N} e^{a_l} \,, \tag{51}$$

$$[Ch(V)]_{P_{\vec{\mathfrak{k}}}} = \sum_{l=1}^{N} \sum_{i=1}^{n_l} \sum_{j=1}^{k_{li}} e^{a_l + \epsilon_1(i-1) + \epsilon_2(j-1)} \,.$$

From this we derive expressions for $Ch(\mathcal{E})$ and $c(T\,\mathfrak{M})$.

D-brane picture of partitions

It is useful to recall here the D-brane interpretation of the partitions $\mathfrak{k}$. In this picture, the fractional D3-branes are separated in the w direction, and are located at $w = a_l$, $l = 1, \ldots, N$. To the l'th D3 brane k_l D(-1) instantons ($k_l = \sum_i k_{li}$) are attached. In the noncommutative theory with the noncommutativity parameter Θ,

$$[x^1, x^2] = [x^3, x^4] = i\Theta \,,$$

these D(-1) instantons are located near the origin $(z_1, z_2) \sim 0$, where $z_1 = x^1 + ix^2, z_2 = x^3 + ix^4$. Different partitions correspond to the different 0-dimensional "submanifolds" (in the algebraic geometry sense) of $\mathbb{C}^2$. If we denote by $\mathcal{I}_l$ the algebra of holomorphic functions (polynomials) on $\mathbb{C}^2$ which vanish on the D(-1) instantons, stuck to the l'th D3-brane, then it can be identified with the ideal in the ring of polynomials $\mathbb{C}[z_1, z_2]$ such that the quotient $\mathbb{C}[z_1, z_2]/\mathcal{I}_l$ is spanned by the monomials

$$z_1^{i-1} z_2^{j-1}, \qquad 1 \leq j \leq k_{li} \,.$$

Remark on Planck constant

In what follows we set $\epsilon_1 = -\epsilon_2 = \hbar$. Note, that this Planck constant has nothing to do with the coupling constant of the gauge theory, where it appears as the parameter of the geometric background (10). It corresponds however exactly to the loop counting in the dual string theory, while the gauge theory Planck constant in string theory picture arises as a *worldsheet* parameter, according to the relation between the world-sheet and gauge theory instantons, described below.

2.7. *Correlation functions of the chiral operators*

Now we are ready to attack the correlation function in Eq. (5). First of all, using the unbroken supercharges one argues that this correlation function is independent of the coefficient in front of the term $|F^+|^2 + \ldots$ which is $\{Q, \ldots\}$. Therefore, one can go to the weak coupling regime (with the theta angle appropriately adjusted, so that τ_0 is finite, while $\bar{\tau}_0 \to \infty$) in which the expression in Eq. (5) is saturated by instantons (cf. Ref. 57).

In this limit the descendants of the chiral operators become the Chern classes of the universal bundle, "integrated" (in the equivariant sense), over $\mathbb{R}^4$. Here is the table of equivariant integrals Ref. 55, cf. Eq. (24),

$$\int_{\mathbb{R}^4} \Omega^{(4)} = \frac{\Omega^{(0)}(0)}{\epsilon_1 \epsilon_2}\,. \tag{52}$$

We should then integrate these classes over $\mathfrak{M}$. But then again, we use equivariant localization, this time on the fixed points in $\mathfrak{M}$. These fixed points are labeled by partitions $\mathfrak{k}$. The calculation of the expectation values of the chiral operators becomes equivalent to the calculation of the expectation values of some operators in the statistical mechanical model, where the basic variables are the N-tuples of partitions (50). In this statistical model, the operator $\mathcal{O}_J^{(0)} = \frac{1}{J}\mathrm{Tr}\,\Phi^J$ in the gauge theory translates to the operator $(a_l = \hbar M_l)$

$$\begin{aligned} \mathcal{O}_J[\vec{\mathfrak{k}}] \quad &\equiv \quad \left[\int_{\mathbb{R}^4} \mathcal{O}_J^{(4)}\right]_{P_{\vec{\mathfrak{k}}}} = \frac{\hbar^J}{J}\sum_{l=1}^{N}\Bigg[M_l^J + \sum_{i=1}^{\infty}\Big((M_l + k_{li} - i + 1)^J \\ &\qquad -(M_l + k_{li} - i)^J - (M_l + 1 - i)^J + (M_l - i)^J\Big)\Bigg] \\ &\overset{\text{formally}}{=} \frac{1}{J}\sum_{l,i}\left[((a_l + \hbar(k_{li} + 1 - i))^J - (a_l + \hbar(k_{li} - i))^J\right]\,. \end{aligned} \tag{53}$$

This is a straightforward consequence of Eq. (51) for $\epsilon_1 = -\epsilon_2 = \hbar$.

Given the single-trace operators $\mathcal{O}_J$ we build arbitrary gauge-invariant operators $\mathcal{O}_{\vec{n}}$ as in Eqs. (1, 6). After that one can integrate their $\mathcal{N} = 2$ descendants $\mathcal{O}^{(4)}_{\vec{n}}$ using the table of equivariant integrals (52).

The gauge theory generating function of the correlators of the chiral operators becomes the statistical model partition function with all the integrated operators $\int_{\mathbb{R}^4} \mathcal{O}^{(4)}_{\vec{n}}$ added to the Hamiltonian. In other words, we sum over the partitions $\{\mathbf{k}_l\} = \{k_{li}\}$ the Bolzmann weights $\exp\left(-\frac{1}{\hbar^2}\sum_{\vec{n}} t_{\vec{n}}\mathcal{O}_{\vec{n}}\right)$, and the measure on the partitions is given by the square of the regularized discretized Vandermonde determinant,

$$\mu_{\vec{\mathfrak{k}}} = \prod_{(li)\neq(mj)} (\lambda_{li} - \lambda_{mj}) \ , \qquad \lambda_{li} = a_l + \hbar(k_{li} - i) \, . \tag{54}$$

The product in Eq. (54) is taken over all pairs $(li) \neq (mj)$ which is short for $\{(l \neq m); \text{or } (l = m, i \neq j);\}$ and can be understood with the help of ζ-regularization in which

$$\mu_{\vec{\mathfrak{k}}} = \exp\left(-\frac{d}{ds}\frac{1}{\Gamma(s)}\int_0^\infty dt\ t^{s-1} \sum_{(li)\ \neq(mj)} e^{-t(\lambda_{li}-\lambda_{mj})}\Big|_{s=0} \right) . \tag{55}$$

The sum in Eq. (55) is defined by analytic continuation, as the sum over (l, i) converges for $\mathrm{Re}(\hbar t) < 0$, while the sum over (m, j) converges for $\mathrm{Re}(\hbar t) > 0$.

3. Abelian Theory

3.1. *A little string that could*

Now suppose we take $N = 1$. In the pure $\mathcal{N} = 2$ gauge theory this is not the most interesting case, since neither perturbative, nor non-perturbative corrections affect the low-energy prepotential. Imagine, then, that we embed the $N = 1$, $\mathcal{N} = 2$, theory in the theory with instantons. One possibility is the noncommutative gauge theory, another possibility is the theory on the D-brane, e.g. fractional D3-brane at the ADE-singularity, or the D5/NS5 brane wrapping a $\mathbb{CP}^1$ in $\mathbf{K3}$. In this setup the theory has non-perturbative effects, coming from noncommutative instantons, or fractional D(-1) branes, or the worldsheet instantons of D1 strings bound to D5, or the elementary string worldsheet instantons in the background of NS5 brane, or an $SL_2(\mathbb{Z})$ transform thereof. In either case, we shall get the instanton contributions to the effective prepotential. Let us calculate them.

We shall slightly change the notation for the times $\tau_{\vec{n}}$ as in this case there is no need to distinguish between $\mathrm{Tr}\,\Phi^J$ and $(\mathrm{Tr}\,\Phi)^J$. We set

$$\sum_{\vec{n}} \tau_{\vec{n}} \prod_{J=1}^{\infty} \frac{x^{Jn_J}}{n_J!(J)^{n_J}} = \sum_{J=1}^{\infty} t_J \frac{x^{J+1}}{(J+1)!} \tag{56}$$

and consider the partition function as a function of the times t_J.

First, let us turn off the higher order Casimirs. Then, we are to calculate

$$e^{-t_1 \frac{a^2}{2\hbar^2}} \sum_{\mathfrak{k}} \mu_{\mathfrak{k}} e^{t_1 |\mathfrak{k}|} \,. \tag{57}$$

Partitions and representations

As it is well-known, the partitions $\mathfrak{k} = (k_1 \geq k_2 \geq k_3 \geq \dots k_n)$ are in one-to-one correspondence with the irreducible representations $R_{\mathfrak{k}}$ of the symmetric group $\mathcal{S}_k$, $k = |\mathfrak{k}|$. Moreover, in the case $N = 1$, one gets from Eq. (54)

$$\mu_{\mathfrak{k}} = \prod_{i \neq j}^{\infty} \frac{\hbar(k_i - k_j + j - i)}{\hbar(j - i)} \,.$$

Using the relation between partitions $\mathfrak{k}$ and Young diagrams $Y_{\mathfrak{k}}$, whose i'th row contains $k_i > 0$ boxes, $1 \leq i \leq n$, corresponding to the irreducible representation $R_{\mathfrak{k}}$ of the symmetric group $\mathcal{S}_k$ (and to the irreducible representation $\mathcal{R}_{\mathfrak{k}}$ of the group $U(\hat{N})$, for any $\hat{N} \geq n$), this can be rewritten as

$$\mu_{\mathfrak{k}} = (-1)^k \left[\prod_{i<j}^{n} \left(\hbar\left(k_i - k_j + j - i\right)\right) \prod_{i=1}^{n} \frac{1}{\hbar^{k_i+n-i}(k_i+n-i)!} \right]^2 = (-1)^k \left[\frac{\mathrm{dim} R_{\mathfrak{k}}}{\hbar^k \; k!} \right]^2 ,$$

where we employ the rule $\frac{l\cdot(l+1)\cdot(l+2)\dots}{1\cdot 2\cdot 3 \dots l\cdot(l+1)\cdot(l+2)\dots} = \frac{1}{l!}$. Hence

$$\mu_{\mathfrak{k}} = \left(\frac{\mathrm{dim} R_{\mathfrak{k}}}{k!} \right)^2 (-\hbar^2)^{-k} \,. \tag{58}$$

The measure (58) on the partitions is the so-called Plancherel measure, introduced by A.M. Vershik, and studied extensively by himself and S.V. Kerov, see Ref. 58. Our immediate problem is rather simple, however. The summation over $\mathfrak{k}$ is trivial thanks to Burnside's theorem, and we conclude that

$$Z = \exp\left[-\frac{1}{\hbar^2} \left(t_1 \frac{a^2}{2} + e^{t_1} \right) \right] . \tag{59}$$

We see that the gauge theory prepotential or the free energy of our statistical model coincides with the Gromov-Witten prepotential of the $\mathbb{CP}^1$ topological sigma model.

Back to fractional branes and to little strings

At this point it is fair to ask where this $\mathbb{CP}^1$ came from. After all, in conventional physical applications of topological strings the target space should be a Calabi-Yau manifold, and $\mathbb{CP}^1$ is definitely. One can imagine the topological string on a local Calabi-Yau, which is a resolved conifold, i.e. the total space of the $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ bundle over $\mathbb{CP}^1$. One can then tune the so-called twisted masses μ_1, μ_2, or, more mathematically speaking, equivariant parameters with respect to the rotations of the fiber of the vector bundle. In the limit $\mu_{1,2} \to 0$ the sigma model is localized onto the maps into $\mathbb{CP}^1$ proper. Is this the way to embed our model in a full-fledged string compactification? We doubt this is the case.

Rather, we think the proper model should be that of little string theory [59] compactified on $\mathbb{CP}^1$. Indeed, the discussion at the beginning of this section suggests a realization of the abelian gauge theory with instantons by means of the D5 brane wrapping a $\mathbb{CP}^1$ inside the Eguchi-Hanson space $T^*\mathbb{CP}^1$, which is the resolution of the $\mathbb{C}^2/\mathbb{Z}_2$ singularity. The wrapped D5 brane is a blown-up fractional D3 brane stuck at the singularity. It supports an $\mathcal{N} = 2$ gauge theory with a single abelian vector multiplet. In addition, it has instantons, coming from fractional D(-1) branes, or, after resolution, D1 string worldsheet instantons. These are bound to the D5 brane worldvolume. After S-duality and appropriate decoupling limits these turn into the so-called *little strings*, about which very little is known. In particular, much debate has been devoted to the issue of the tunable coupling constant in these theories. Our results strongly suggest such a possibility.

3.2. *Free fermions*

Now let us turn on the higher order Casimirs in the gauge theory. To facilitate the calculus it is convenient to introduce the formalism of free fermions. Consider the theory of a single free complex fermion on a two-sphere in which

$$\psi(z) = \sum_{r \in \mathbb{Z}+\frac{1}{2}} \psi_r \, z^{-r} \left(\frac{dz}{z}\right)^{\frac{1}{2}}, \qquad \widetilde{\psi}(z) = \sum_{r \in \mathbb{Z}+\frac{1}{2}} \widetilde{\psi}_r \, z^{r} \left(\frac{dz}{z}\right)^{\frac{1}{2}},$$

$$\{\psi_r, \widetilde{\psi}_s\} = \delta_{rs} \,. \tag{60}$$

The fermionic Fock space is constructed with the help of the charge M vacuum state [e]

$$|M\rangle = \psi_{M+\frac{1}{2}}\psi_{M+\frac{3}{2}}\psi_{M+\frac{5}{2}}\ \cdots\ , \tag{61}$$

$$\psi_r|M\rangle = 0, \qquad r > M\,,$$

$$\widetilde{\psi}_r|M\rangle = 0, \qquad r < M\,.$$

It is also convenient to use the basis of the so-called partition states (see, e.g. [60, 61]). For each partition $\mathfrak{k} = (k_1 \geq k_2 \geq \ldots)$ we introduce the state

$$|M;\mathfrak{k}\rangle = \psi_{M+\frac{1}{2}-k_1}\psi_{M+\frac{3}{2}-k_2}\cdots\,. \tag{62}$$

One defines the $U(1)$ current as

$$J = :\widetilde{\psi}\psi: = \sum_{n\in\mathbb{Z}} J_n z^{-n}\frac{dz}{z}\,, \tag{63}$$

$$J_n = \sum_{r<n}\widetilde{\psi}_r\psi_{n-r} - \sum_{r>n}\psi_{n-r}\widetilde{\psi}_r\,.$$

Recall the bosonization rules

$$\psi =: e^{i\phi}:\,, \quad \tilde{\psi} =: e^{-i\phi}:\,, \quad J = i\partial\phi\,, \tag{64}$$

and a useful fact from $U(\hat{N})$ group theory; the famous Weyl correspondence states that

$$(\mathbb{C}^{\hat{N}})^{\otimes k} = \bigoplus_{\mathfrak{k},|\mathfrak{k}|=k} R_{\mathfrak{k}} \otimes \mathcal{R}_{\mathfrak{k}} \tag{65}$$

is an $\mathcal{S}_k \times U(\hat{N})$ representation. Now let $U = \mathrm{diag}\left(u_1,\ldots,u_{\hat{N}}\right)$ be a $U(\hat{N})$ matrix. Then one can easily show using the Weyl character formula and the standard bosonization rules that

$$\mathrm{Tr}\,_{\mathcal{R}_{\mathfrak{k}}}U = \langle \hat{N};\mathfrak{k}|\,:e^{i\sum_{n=1}^{\hat{N}}\phi(u_n)}:\,:e^{-i\hat{N}\phi(0)}:|0\rangle\,. \tag{66}$$

From this formula one derives

$$e^{\frac{J_{-1}}{\hbar}}\,|M\rangle = \sum_{\mathfrak{k}}\frac{\dim R_{\mathfrak{k}}}{\hbar^k\ k!}\,|M;\mathfrak{k}\rangle\,. \tag{67}$$

[e] Any M is good for building the space.

4. Integrable System and $\mathbb{CP}^1$ Sigma Model

The importance of the fermions is justified by the following statement. The generating function with turned on higher Casimirs equals the the correlation function

$$Z = \langle M | e^{\frac{J_1}{\hbar}} \exp\left[\sum_{p=1}^{\infty} \hat{t}_p W_{p+1} \right] e^{-\frac{J_{-1}}{\hbar}} | M \rangle \,. \tag{68}$$

Here

$$\sum_{p=1}^{\infty} \hat{t}_p \, x^p = \sum_{p=1}^{\infty} \frac{1}{(p+1)!} \, t_p \, \frac{(x+\frac{\hbar}{2})^{p+1} - (x-\frac{\hbar}{2})^{p+1}}{\hbar} \,, \tag{69}$$

and

$$W_{p+1} = \frac{1}{\hbar} \oint \, : \, \widetilde{\psi} \, (\hbar D)^p \, \psi : \,, \qquad D = z\partial_z \ . \tag{70}$$

If only $t_1 \neq 0$ the correlator in Eq. (68) is trivially computed and gives Eq. (59) with $a = \hbar M$. Comparison of Eq. (68) with the results of Ref. 60 shows that the generating function (68), as a function of times $\hat{t}_p$, is a tau-function of the Toda lattice hierarchy. Note that the fermionic matrix element (68) is very different from the standard representation for the Toda tau-function, cf. Ref. 62. In our case the "times" are coupled to the "zero modes" of higher W-generators, while usually they couple to the components (63) of the $U(1)$ current.

The free fermionic representation, Eq. (68), is useful in several respects. One of them is the remarkable mapping of the gauge theory correlation function to the amplitudes of a (topological type A) string propagating on $\mathbb{CP}^1$. Indeed, using the results of Ref. 60 one can show that

$$\left\langle \exp \int_{\mathbb{R}^4} \sum_{J=1}^{\infty} t_J \, \mathcal{O}^{(4)}_{J+1} \right\rangle^{\text{gauge theory}}_{a,\hbar} = \exp \sum_{g=0}^{\infty} \hbar^{2g-2} \langle\langle \exp \int_{\Sigma_g} a \cdot \mathbf{1} + \sum_{p=1}^{\infty} \hat{t}_p \sigma_{p-1}(\omega) \rangle\rangle^{\text{string}}_g \,. \tag{71}$$

Here $\langle\langle \ldots \rangle\rangle_g$ stands for the genus g connected partition function.

It is tempting to speculate that a similar relation holds for nonabelian gauge theories. The left hand side of Eq. (71) is known for the gauge group $U(N)$ (we essentially described it by the Eqs. (53, 54), see also Ref. 10) but the right hand side is not, although there are strong indications that the free fermion representation and relation to the $\mathbb{CP}^1$ sigma model holds in this case too, Ref. 63.

The Eq. (71) is the content of our gauge theory/string theory correspondence. We have an explicit mapping between the gauge theory operators and the string theory vertex operators. In this mapping the higher Casimirs map to gravitational descendents of the Kähler form.

Full duality?

The topological string on $\mathbb{CP}^1$ actually has even more observables then the ones presented in Eq. (71). Indeed, we are missing all the gravitational descendants of the puncture operator $\sigma_k(\mathbf{1}), k > 0$. We conjecture, that their gauge theory dual, by analogy with AdS/CFT correspondence, Ref. 64, is the shift of vevs of the operators $\mathrm{Tr}\,\phi^J$, for $\sigma_{J-1}(\mathbf{1})$. For $J = 1$ we are talking about shifting a, the vev of ϕ. This is indeed the case. When all these couplings are taken into account we would expect to see the full two-dimensional Toda hierarchy [62].

Chiral ring

Another application of Eq. 68 is the calculation of the expectation values of $\mathcal{O}_J$. This exercise is interesting in relation to the recent matrix model/gauge theory correspondence of R. Dijkgraaf and C. Vafa [65] which predicts, according to Ref. 66,

$$\langle \mathrm{Tr}\,\phi^J\rangle = \oint x^J \frac{dz}{z}\,, \qquad z+\frac{\Lambda^{2N}}{z} = P_N(x) = x^N + u_1 x^{N-1} + u_2 x^{N-2} + \ldots + u_N\,, \tag{72}$$

in agreement with the formulae from Ref. 21 obtained in the context of the Seiberg-Witten theory.

To compute the expectation values of $\mathcal{O}_J$ in our approach (for $N = 1$) it suffices to calculate $-\hbar^2\,\frac{1}{Z}\partial_{t_{J-1}}Z$ at $t_2 = t_3 = \ldots = 0$ and then send $\hbar \to 0$ (as Ref. 66 did not look at the equivariance with respect to the space-time rotations),

$$\langle \mathcal{O}_J\rangle_{a,0} = \tag{73}$$

$$= \lim_{\hbar\to 0} \hbar^J \frac{\langle M| e^{\frac{1}{\hbar}\oint :\tilde{\psi} z \psi:} \oint \; : \tilde{\psi}\left((D+\frac{1}{2})^J - (D-\frac{1}{2})^J\right)\psi : \; e^{-\frac{\Lambda^2}{\hbar}\oint :\tilde{\psi} z^{-1}\psi:} |M\rangle}{\langle M| e^{\frac{1}{\hbar}\oint :\tilde{\psi} z \psi:} \; e^{-\frac{\Lambda^2}{\hbar}\oint :\tilde{\psi} z^{-1}\psi:} |M\rangle} =$$

$$= \oint \left(a + z + \frac{\Lambda^2}{z}\right)^J \frac{dz}{z}\,, \qquad \Lambda^2 = e^{t_1}, \qquad a = \hbar M\,.$$

The last relation is proved by bosonization. This reproduces Eq. (72) for $N = 1$.

5. Theories with Matter

In this section we shall discuss theories with matter in the fundamental representation. We shall again consider only the $U(1)$ case, but as above we shall be, in general, interested in turning on higher Casimirs. To avoid confusion, we shall use capital letters T_p for the couplings of the theory with matter.

5.1. *4d and 2d field theory*

The famous condition of asymptotic freedom, $N_f \leq 2N_c$, if extrapolated to the case $N_c = 1$, suggests that we could add up to two fundamental hypermultiplets. It is a straightforward exercise to extend the fixed point calculus to incorporate the effect of the charged matter. Let us briefly recall the important steps. Susy equations in the presence of a matter hypermultiplet $M = (\tilde{Q}, Q)$ change from $F^+ = 0$ to $F^+ + \bar{M}\Gamma M = 0$, $\not{D}\ M = 0$. The moduli space of solutions to these equations looks near the $M = 0$ locus like a vector bundle over $\mathfrak{M}$ – the instanton moduli, whose fiber is the bundle of Dirac zero modes.

It can be shown that the instanton measure gets an extra factor, the equivariant Euler class of the Dirac bundle (see Ref. 13 for more details and more references). The localization formulae still work, but now each partition $\mathfrak{k}$ has an extra weight, see Ref. 10. The contribution of the fixed point to the path integral in the presence of the matter fields is (54) multiplied by the extra factor (the content polynomial, see Ref. 67)

$$\tilde{\mu}_{\mathfrak{k}}(a, m) = Z^{pert}(a, m) \times \prod_{f=1}^{2} \prod_{i=1}^{\infty} (a + m_f + \hbar(1 - i)) \ldots (a + m_f + \hbar(k_i - i)) \ , \tag{74}$$

where

$$Z^{pert}(a, m) = \prod_{f} \prod_{i=1}^{\infty} \Gamma\left(\frac{a + m_f}{\hbar} + 1 - i\right) \sim \exp \int_0^\infty \frac{\mathrm{d}t}{t} \sum_f \frac{e^{-t(a+m_f)}}{\sinh^2\left(\frac{\hbar t}{2}\right)} =$$
$$= \sum_f \left[\frac{(a+m_f)^2}{2\hbar^2} \log(a+m_f) + \frac{1}{12} \log(a+m_f) + \sum_{g>1} \frac{B_{2g}}{2g(2g-2)} \left(\frac{\hbar}{a+m_f}\right)^{2g-2} \right] . \tag{75}$$

The bosonization rule Eq. (66) leads to

$$Z^{inst} = \left\langle e^{i\frac{a+m_2}{\hbar}\phi(\infty)} e^{-i\frac{m_2}{\hbar}\phi(1)} e^{\sum_p T_p W_{p+1}} e^{i\frac{m_1}{\hbar}\phi(1)} e^{-i\frac{a+m_1}{\hbar}\phi(0)} \right\rangle . \tag{76}$$

It can be shown that the full partition function $Z^{pert}Z^{inst}$ has a CFT interpretation and also obeys the Toda lattice equations. We shall discuss this in a future publication.

5.2. *Relation to geometric engineering*

Now let us turn off the higher Casimirs, i.e. set $T_p = 0$, for $p > 1$. Then Eqs. (76, 74) lead to

$$\mathcal{F}_0 = \frac{1}{2}T_1 a^2 - m_1 m_2 \log(1 - e^{T_1}) + \sum_f \frac{1}{2}(a + m_f)^2 \log(a + m_f) \ , \quad (77)$$

$$\mathcal{F}_1 = \frac{1}{12}\log(a + m_1)(a + m_2) \, ,$$

$$\mathcal{F}_g = \frac{B_{2g}}{2g(2g-2)} \sum_f \frac{\hbar^{2g-2}}{(a + m_f)^{2g-2}} \, .$$

We remark that Eq. (77) is a limit of the all-genus topological string prepotential in the geometry described in Ref. 68 (Fig. 12, Eq. (7.34)). The specific limit is to take t_1, t_2, g_s in their notation to zero, as $t_f = \beta(a + m_f), g_s = \beta\hbar$, $\beta \to 0$, while $-r'$ (their notation) $= T_1$ (our notation) is finite. The prepotential in Ref. 68 actually describes the five dimensional susy gauge theory compactified on a circle of circumference β. The limit $\beta \to 0$ actually takes us to the four dimensional theory, which is what we were studying in this paper. It is clear from Ref. 68 (Fig. 12c) that the geometry corresponds to the $U(1)$ gauge theory with two fundamental hypermultiplets (two D-branes pulling on the sides).

Our results are, however, stronger. Indeed, we were able to calculate the prepotential and $\mathcal{F}_g$'s with arbitrary higher Casimirs turned on. In the limit

$$m_1, m_2 \to \infty, \quad e^{T_1} \to 0, \quad (78)$$

so that $\Lambda^2 = m_1 m_2 e^{T_1} = e^{t_1}$ in our old notation is finite, we get back the pure $U(1)$ theory, which we identified with the topological string on $\mathbb{CP}^1$, Eq. (71). Note that this was not $\mathbb{CP}^1$ embedded into Calabi-Yau, as in the latter case no gravitational descendants ever showed up. We are led, therefore, to the conclusion that the topological string on the geometry of Fig. 12 of Ref. 68 has a deformation allowing gravitational descendants and flowing, in the limit (78), to the pure $\mathbb{CP}^1$ model. This fascinating prediction certainly deserves further study.

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Appendix A. Equivariant integration and localization

Let $\mathcal{Y}$ be a manifold with an action of a Lie group $\mathcal{G}$, and let $\mathcal{X}$ be a $\mathcal{G}$-invariant submanifold. Moreover, let $\mathcal{X}$ be a zero locus of a section s of a $\mathcal{G}$-equivariant vector bundle $\mathcal{V}$ over $\mathcal{Y}$.

Suppose that we need to develop an integration theory on the quotient $\mathcal{X}/\mathcal{G}$. It is sometimes convenient to work $\mathcal{G}$-equivariantly on $\mathcal{Y}$, and use the so-called Mathai-Quillen representative of the Euler class of the bundle $\mathcal{V}$.

The equivariant cohomology classes are represented with the help of the equivariant forms. These are functions on $\mathbf{g} = Lie(\mathcal{G})$ with the values in the de Rham complex of $\mathcal{Y}$. In addition, these functions are required to be $\mathcal{G}$-equivariant, i.e. the adjoint action of $\mathcal{G}$ on $\mathbf{g}$ must commute with the action of $\mathcal{G}$ on the differential forms on $\mathcal{Y}$.

Let us denote the local coordinates on $\mathcal{Y}$ by y^μ, and their exterior differentials dy^μ by ψ^μ. The equivariant differential is the operator

$$Q = \psi^\mu \frac{\partial}{\partial y^\mu} + \phi^a V_a^\mu(y) \frac{\partial}{\partial \psi^\mu} \ , \tag{A.1}$$

where ϕ^a are the linear coordinates on $\mathbf{g}$, and $V_a = V_a^\mu \partial_\mu$ are the vector fields on $\mathcal{Y}$ generating the action of $\mathcal{G}$. The operator Q raises the so-called ghost number by one,

$$gh = \psi \frac{\partial}{\partial \psi} + 2\phi \frac{\partial}{\partial \phi} \ . \tag{A.2}$$

The equivariant differential forms can be now written as $\mathcal{G}$-invariant functions of (y, ψ, ϕ).

In the applications one uses a more refined (Dolbeault) version of the equivariant cohomology. There, one multiplies $\mathcal{Y}$ by $\mathfrak{g}$, and extends the action of $\mathcal{G}$ by the adjoint action on $\mathbf{g}$. The coordinate on this copy of $\mathfrak{g}$ is conventionally denoted by $\bar{\phi}$, and its differential by η. The equivariant differential on $\mathcal{Y} \times \mathbf{g}$ acts, obviously, as

$$Q = \psi^\mu \frac{\partial}{\partial y^\mu} + \phi^a V_a^\mu(y) \frac{\partial}{\partial \psi^\mu} + \eta \frac{\partial}{\partial \bar{\phi}} + [\phi, \bar{\phi}] \frac{\partial}{\partial \eta} \ . \tag{A.3}$$

However, the ghost number is defined not as in Eq. (A.2) but rather with a shift (in some papers this shift is reflected by the notation $\mathfrak{g}[-2]$),

$$gh = \psi \frac{\partial}{\partial \psi} + 2\phi \frac{\partial}{\partial \phi} - 2\bar{\phi} \frac{\partial}{\partial \bar{\phi}} - \eta \frac{\partial}{\partial \eta} \ . \tag{A.4}$$

Suppose $\mathcal{O}(y, \psi, \bar{\phi}, \eta, \phi)$ is $\mathbb{G}$ invariant and annihilated by Q. Suppose in addition that the integral

$$\mathcal{I}_{\mathcal{O}}(\bar{\phi}, \phi, \eta) = \int dy d\psi \ \mathcal{O}(y, \psi, \bar{\phi}, \phi, \eta) \tag{A.5}$$

makes sense. Then $\mathcal{I}_{\mathcal{O}}$ is $\mathbb{G}$-equivariant on $\mathbf{g}$. One can integrate it over $\bar{\phi}, \eta$, and ϕ against any $\mathcal{G}$ equivariant form.

Amplitudes

In particular, one can simply integrate $\mathcal{I}_{\mathcal{O}}$ over all of $\mathbf{g}$ to get

$$\mathcal{I}_{\mathcal{O}}^{top} = \int d\bar{\phi} d\eta \frac{d\phi}{\mathrm{Vol}(\mathcal{G})} \ \mathcal{I}_{\mathcal{O}}(\bar{\phi}, \phi, \eta) \, . \tag{A.6}$$

The more general construction proceeds by picking a normal subgroup $H \subset \mathcal{G}$, and integrating over $Lie(\mathcal{G}/H)$, with the extra measure,

$$\mathcal{I}_{\mathcal{O}}^H(\varphi)_\kappa = \int \frac{d\bar{\phi}^\perp d\eta^\perp d\phi^\perp}{\mathrm{Vol}(\mathcal{G}/H)} \ \mathcal{I}_{\mathcal{O}}(\bar{\phi}^\perp + \bar{\varphi}, \eta^\perp, \phi^\perp + \varphi) \tag{A.7}$$
$$\times e^{-\frac{1}{\kappa}\left(\| [\bar{\phi}^\perp + \bar{\varphi}, \phi^\perp + \varphi] \|^2 - \langle [\eta^\perp, \phi^\perp + \varphi], \eta^\perp \rangle \right)} ,$$

where $\phi^\perp \in Lie(\mathcal{G}/H)$ etc., $\varphi, \bar{\varphi} \in Lie(H)$, and as long as $[\varphi, \bar{\varphi}] = 0$ the left hand side of $\mathcal{I}_{\mathcal{O}}^H(\varphi)$ does not depend on $\bar{\varphi}$, as a consequence of Q-symmetry. Clearly $\mathcal{I}^{top} = \mathcal{I}^1 \big|_{\kappa = \infty}$.

Now let us make our construction a little bit more sophisticated. Recall that we had a vector bundle $\mathcal{V}$ over $\mathcal{Y}$, with the section $s = (s^a(y))$. Suppose, in addition, that there is a $\mathbb{G}$-invariant metric g_{ab} on the fibers of $\mathcal{V}$, and let $\Gamma^b_{\mu a} dy^\mu$ denote a connection on $\mathcal{V}$, compatible with g_{ab}. Then the integral

$$\mathcal{O}_{\mathcal{V}}(y,\psi,\phi,\bar{\phi},\eta) = \int d\chi dH e^{i\chi_a \psi^\mu (\partial_\mu + \Gamma_\mu) s^a + iH_a s^a - \frac{1}{2} g^{ab} \left[H_a H_b + \left(\mathcal{F}^c_{\mu\nu,a} \psi^\mu \psi^\nu + R(\phi,y)^c_a \right) \chi_c \chi_b \right]}, \tag{A.8}$$

is a Q-invariant form. Here $\mathcal{F} = d\Gamma + [\Gamma,\Gamma]$ is the curvature of Γ, and $R(\phi,y)$ is the representation of $\mathbf{g}$, acting on the sections of $\mathcal{V}$ (a Lie algebraic 1-cocycle).

Now, if we rescale the metric $g^{ab} \to t g^{ab}$ then the value of Eq. (A.8) should not change (the variation is Q-exact). In particular, in the limit $t \to 0$ the form $\mathcal{O}_{\mathcal{V}}$ is supported on the zeroes of the section s. In the opposite limit, $t \to \infty$ it becomes independent of s and turns into the form

$$\mathcal{O}_{\mathcal{V}} \sim \mathrm{Pf}\left(\mathcal{F} + R(\phi, y)\right) .$$

One can also consider more general variations of the metric g^{ab}.

Localization

Let us go back to Eq. (A.7). As we said, the answer is independent of $\bar{\varphi}$. Let us make good use of this fact. To this end, let us multiply $\mathcal{O}$ in Eq. (A.5) by the extra factor

$$\mathcal{O} e^{-Q(\bar{\phi}^a V^\mu_a g_{\mu\nu} \psi^\nu)},$$

where $g_{\mu\nu}$ is any $\mathcal{G}$-invariant metric on $\mathcal{Y}$. Explicitly, we have in the exponential

$$g(V_a, V_b)\phi^b \bar{\phi}^a + fermions\,.$$

Now let us take the limit $\bar{\varphi} \to \infty$. The measure will be localized near the zeroes of the vector field $V_a \bar{\varphi}^a$. This is the source of equivariant localization. Say, take $H = \mathcal{G}$ (more general case can be easily worked out). Then

$$\mathcal{I}^{\mathcal{G}}_{\kappa}(\varphi) = \sum_{p \in F} \frac{\mathcal{O}(p,\varphi)}{\prod_i w_i(\varphi)}\,, \tag{A.9}$$

where F is the set of points on $\mathcal{Y}$ at which $V_a \varphi^a$ vanishes, and $w_i(\varphi)$ are the weights of the action of $\mathcal{G}$ on the tangent space to $\mathcal{Y}$ at p.

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