ANOMALY CANCELLATIONS ON LOWER-DIMENSIONAL HYPERSURFACES BY INFLOW FROM THE BULK

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Lower-dimensional (hyper)surfaces that can carry gauge or gauge/gravitational anomalies occur in many areas of physics: one-plus-one-dimensional boundaries or two-dimensional defect surfaces in condensed matter systems, four-dimensional brane-worlds in higher-dimensional cosmologies or various branes and orbifold planes in string or M-theory. In all cases we may have (quantum) anomalies localized on these hypersurfaces that are only cancelled by "anomaly inflow" from certain topological interactions in the bulk. Proper cancellation between these anomaly contributions of different origin requires a careful treatment of factors and signs. We review in some detail how these contributions occur and discuss applications in condensed matter (Quantum Hall Effect) and M-theory (five-branes and orbifold planes).

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This work is dedicated to the memory of my friend and collaborator Ian Kogan. His interest in physics spanned almost all of theoretical physics. I hope the present contribution goes a little bit in this direction.

1. Introduction

Quantum field theories involving chiral fields coupled to gauge fields and/or gravity may have anomalies. These anomalies are a breakdown of gauge, or local Lorentz or diffeomorphism invariance respectively, at the one-loop level. More specifically, the (one-loop) quantum effective action lacks these invariances necessary for renormalizability and unitarity. In a consistent theory all these anomalies must cancel.[a] Absence of anomalies has been much used as a criterion for models in high energy physics, from the prediction of the charmed quark [1] to the choice of gauge group of the superstring [2].

Pure gravitational anomalies can only occur in $2, 6, 10, \ldots$ dimensions while gauge or mixed gauge-gravitational anomalies are possible in all *even* dimensions [3]. On the other hand, quantum field theories in odd dimensions cannot be anomalous. Nevertheless, there are many interesting *odd*-dimensional theories that possess *even*-dimensional hypersurfaces with chiral matter localized on these surfaces. Typically the chirality originates either from an orbifold-like projection, or it is the property of a given solution (gravity background) with the corresponding "anti-solution" having opposite chirality.

Standard examples in eleven-dimensional M-theory [4] are the ten-dimensional orbifold planes arising from the $\mathbf{Z}_2$-projection in the Horava–Witten realization of the $E_8 \times E_8$ heterotic string [5] and the (six-dimensional) five-brane carrying a chiral tensor multiplet, while the anti five-brane carries the same multiplet but of opposite chirality [6]. One could also mention the G_2 compactifications with conical singularities treated as boundaries of the eleven-dimensional space-time [7]. Other examples are $3+1$-dimensional brane-world cosmologies with chiral matter in a 5-dimensional supergravity theory. A well-known example in condensed matter is the treatment of the $1+1$-dimensional chiral edge currents in the $2+1$-dimensional Quantum Hall Effect [8]. One might also consider chiral vortices, again in $2+1$ dimensions, or two-dimensional defect surfaces in three-dimensional (Euclidean) systems.

Typically, these even-dimensional chiral "subsystems" possess one-loop

[a] Of course, anomalies of *global* symmetries need not cancel and may even be welcome as they allow transitions otherwise forbidden by the symmetry.

anomalies. This does not contradict the fact that the original odd-dimensional theory is anomaly-free. Consider for example the effective action of eleven-dimensional M-theory. When computing the functional integral one has to sum the contributions of five-branes and of anti-five-branes and, of course, the anomalous contributions, being opposite, cancel. However, we rather like to think of M-theory within a given background with some five-branes in certain places and anti five-branes in others, maybe far away, and require *local* anomaly cancellation, i.e. on each (anti) five-brane separately, rather than just global cancellation. Remarkably, such local cancellation is indeed achieved by a so-called "anomaly inflow" from the bulk; M-theory has eleven-dimensional Chern–Simons like terms that are invariant in the bulk but have a anomalous variations on five-branes or on boundaries, precisely cancelling the one-loop anomalies locally [9, 10].

In this paper we will explain in some generality such anomaly inflow from the bulk and how it can and does cancel the gauge and gravitational anomalies on the even-dimensional hypersurfaces. We will discuss why anomaly inflow always originates from topological terms (in odd dimensions). Usually, when discussing anomaly cancellation between different chiral fields one need not be very careful about overall common factors. Here, however, we want to consider cancellations between anomaly contributions of very different origin and special attention has to be paid to all factors and signs (see [4]). To this end, we also discuss in some detail the continuation between Euclidean and Minkowski signature, which again sheds some new light on why it must be the topological terms that lead to anomaly inflow.

In the next section, we begin by a general discussion of anomalies and anomaly inflow from the bulk, spending some time and space on the subtle continuation between Euclidean and Minkowski signature. Part of this section is just a recollection of standard results on one-loop anomalies [11] with special attention to conventions and signs. We explain how anomaly inflow uses the descent equations and why it necessarily originates from a manifold of higher dimension than the one on which the anomalous theory lives. Section 3 describes an elementary application to the (integer) Quantum Hall Effect where the effective bulk theory is a Chern–Simons theory and the boundary degrees of freedom are the chiral edge currents; anomaly cancellation by inflow from the bulk correctly explains the quantized Hall conductance. In Section 4, we describe two examples of anomaly cancellation by inflow in M-theory in quite some detail: on five-branes and on the $\mathbf{Z}_2$-orbifold planes. For the five-branes, in order to get all signs and coefficients consistent, we rederive everything from scratch: the solution itself, the modified Bianchi identity, the zero-modes and their chirality, the one-loop anomaly and the FHMM [10] mechanism for the inflow. For the orbifold planes we insist on the correct normalization of the Bianchi identity and describe the modification of the Chern–Simons term obtained in [4] necessary to get the correct inflow. Finally, in Section 5, we briefly mention analogous cancellations in brane-world scenarios.

2. One-Loop Anomalies and Anomaly Inflow

2.1. *Conventions*

We begin by carefully defining our conventions. They are the same as in [4]. In Minkowskian space we always use signature $(-,+,\ldots,+)$ and label the coordinates x^μ, $\mu=0,\ldots D-1$. We always choose a *right*-handed coordinate system such that

$$\int \sqrt{|g|}\,\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\ldots\wedge\mathrm{d}x^{D-1}=+\int\sqrt{|g|}\,\mathrm{d}^Dx\geq 0\ . \tag{2.1}$$

(With x^0 being time and for even D, this is a non-trivial statement. In particular, for even D, if we relabelled time as $x^0\to x^D$ then $x^1,\ldots x^D$ would be a left-handed coordinate system!) We define the ϵ-tensor as

$$\epsilon_{01\ldots(D-1)}=+\sqrt{|g|}\quad\Leftrightarrow\quad\epsilon^{01\ldots(D-1)}=-\frac{1}{\sqrt{|g|}}\ . \tag{2.2}$$

Then

$$\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_D}=-\epsilon^{\mu_1\ldots\mu_D}\sqrt{|g|}\,\mathrm{d}^Dx\ . \tag{2.3}$$

A p-form ω and its components are related as

$$\omega=\frac{1}{p!}\,\omega_{\mu_1\ldots\mu_p}\,\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_p} \tag{2.4}$$

and its dual is

$${}^*\omega=\frac{1}{p!(D-p)!}\,\omega_{\mu_1\ldots\mu_p}\,\epsilon^{\mu_1\ldots\mu_p}{}_{\mu_{p+1}\ldots\mu_D}\,\mathrm{d}x^{\mu_{p+1}}\wedge\ldots\wedge\mathrm{d}x^{\mu_D}\ . \tag{2.5}$$

We have ${}^*({}^*\omega)=(-)^{p(D-p)+1}\,\omega$ and

$$\omega\wedge{}^*\omega=\frac{1}{p!}\,\omega_{\mu_1\ldots\mu_p}\,\omega^{\mu_1\ldots\mu_p}\,\sqrt{|g|}\,\mathrm{d}^Dx\ . \tag{2.6}$$

Finally we note that the components of the $(p+1)$-form $\xi=\mathrm{d}\omega$ are given by

$$\xi_{\mu_1\ldots\mu_{p+1}}=(p+1)\,\partial_{[\mu_1}\omega_{\mu_2\ldots\mu_{p+1}]} \tag{2.7}$$

(where the brackets denote anti-symmetrization with total weight one) and that the divergence of a p-form is expressed as

$$^*\mathrm{d}^*\omega = \frac{(-)^{D(p-1)+1}}{(p-1)!}\,\nabla^\nu \omega_{\nu\mu_1\ldots\mu_{p-1}}\,\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_{p-1}}\ . \tag{2.8}$$

We define the curvature 2-form $R^{ab} = \frac{1}{2}R^{ab}{}_{\nu\sigma}\,\mathrm{d}x^\nu\wedge\mathrm{d}x^\sigma$ in terms of the spin-connection ω^{ab} as $R^{ab} = \mathrm{d}\omega^{ab} + \omega^a{}_c\wedge\omega^{cb}$. Here $a,b,c = 0,\ldots D-1$ are "flat" indices, related to the "curved" ones by the D-bein e^a_μ. The torsion is $T^a = \mathrm{d}e^a + \omega^a{}_b\wedge e^b$. The Riemann tensor $R^{\mu\rho}{}_{\nu\sigma}$ is related to the curvature 2-form via $R^{ab}{}_{\nu\sigma} = e^a_\mu e^b_\rho R^{\mu\rho}{}_{\nu\sigma}$, and the Ricci tensor is $\mathcal{R}^\mu{}_\nu = R^{\mu\rho}{}_{\nu\rho}$ while the Ricci scalar $\mathcal{R}$ is given by $\mathcal{R} = \mathcal{R}^\mu{}_\mu$. With this sign convention, (space-like) spheres have $\mathcal{R} > 0$.

For gauge theory, the gauge fields, field strength and gauge variation are given by

$$\begin{aligned} &A = A_\mu\,\mathrm{d}z^\mu\ ,\quad A_\mu = A^\alpha_\mu\,\lambda^\alpha\ ,\quad (\lambda^\alpha)^\dagger = -\lambda^\alpha\ ,\\ &F = \mathrm{d}A + A^2 \equiv \mathrm{d}A + A\wedge A\ ,\quad \delta_v A = \mathrm{D}v = \mathrm{d}v + [A,v]\ . \end{aligned}\tag{2.9}$$

Thus F is anti-hermitian and differs by an i from a hermitian field strength used by certain authors.[b] For gravity, one considers the spin connection $\omega^a{}_b$ as an $SO(2n)$-matrix valued 1-form. Similarly, the parameters $\epsilon^a{}_b$ of local Lorentz transformations (with $\epsilon^{ab} = -\epsilon^{ba}$) are considered as an $SO(2n)$-matrix. Then

$$R = \mathrm{d}\omega + \omega^2\ ,\quad \delta_\epsilon e^a = -\epsilon^a{}_b e^b\ ,\quad \delta_\epsilon\omega = \mathrm{D}\epsilon = \mathrm{d}\epsilon + [\omega,\epsilon]\ . \tag{2.10}$$

For spin-$\frac{1}{2}$ fermions the relevant Dirac operator is (E^μ_a is the inverse $2n$-bein)

$$D\!\!\!\!/ = E^\mu_a\gamma^a\left(\partial_\mu + A_\mu + \frac{1}{4}\omega_{cd,\mu}\gamma^{cd}\right)\ ,\quad \gamma^{cd} = \frac{1}{2}[\gamma^c,\gamma^d]\ . \tag{2.11}$$

2.2. *Continuation Between Minkowski and Euclidean Signature*

We now turn to the continuation to Euclidean signature. While the Minkowskian functional integral contains e^{iS_M}, the Euclidean one contains e^{-S_E}. This implies

$$S_\mathrm{M} = i\,S_\mathrm{E}\quad ,\quad x^0 = -i\,x^0_\mathrm{E}\ . \tag{2.12}$$

[b] For $U(1)$-gauge theories, the usual definition of the covariant derivative is $\partial_\mu + iq\mathcal{A}_\mu$, with q being the charge, and hence $A \simeq iq\mathcal{A}$ and $F \simeq iq\mathcal{F}$ where $\mathcal{F} = \mathrm{d}\mathcal{A}$.

However, for a Euclidean manifold $M_{\rm E}$ it is natural to index the coordinates from 1 to D, not from 0 to $D-1$. One could, of course, simply write $ix^0 = x^0_{\rm E} \equiv x^D_{\rm E}$. The problem then is for even $D=2n$ that $\mathrm{d}x^0_{\rm E} \wedge \mathrm{d}x^1 \wedge \ldots \mathrm{d}x^{2n-1} = -\,\mathrm{d}x^1 \wedge \ldots \mathrm{d}x^{2n-1} \wedge \mathrm{d}x^{2n}_{\rm E}$ and if $(x^0_E, \ldots x^{2n-1})$ was a right-handed coordinate system then $(x^1, \ldots x^{2n}_E)$ is a left-handed one. This problem is solved by shifting the indices of the coordinates as

$$i\,x^0 = x^0_{\rm E} = z^1 \ , \quad x^1 = z^2 \ , \quad \ldots \ , \quad x^{D-1} = z^D \ . \tag{2.13}$$

This is equivalent to a specific choice of an orientation on the Euclidean manifold $M_{\rm E}$. In particular, we impose

$$\int \sqrt{g}\, \mathrm{d}z^1 \wedge \ldots \wedge \mathrm{d}z^D = + \int \sqrt{g}\, \mathrm{d}^D z \geq 0 \ . \tag{2.14}$$

Then, of course, for any tensor we similarly shift the indices, e.g. $C_{157} = C^{\rm E}_{268}$ and $C_{034} = i\, C^{\rm E}_{145}$. We have $G_{\mu\nu\rho\sigma}\, G^{\mu\nu\rho\sigma} = G^{\rm E}_{jklm}\, G_{\rm E}^{jklm}$ as usual, and for a p-form

$$\omega = \frac{1}{p!}\, \omega_{\mu_1 \ldots \mu_p}\, \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p} = \frac{1}{p!}\, \omega^{\rm E}_{j_1 \ldots j_p}\, \mathrm{d}z^{j_1} \wedge \ldots \wedge \mathrm{d}z^{j_p} = \omega^{\rm E} \ . \tag{2.15}$$

In particular, we have for $p = D$

$$\int_{M_{\rm M}} \omega = \int_{M_{\rm E}} \omega^{\rm E} \ , \tag{2.16}$$

which will be most important below. Finally, note that the Minkowski relations (2.2) and (2.3) become

$$\mathrm{d}z^{j_1} \wedge \ldots \wedge \mathrm{d}z^{j_D} = +\, \epsilon_{\rm E}^{j_1 \ldots j_D}\, \sqrt{g}\, \mathrm{d}^D z \quad \text{with} \quad \epsilon_{\rm E}^{1 \ldots D} = \frac{1}{\sqrt{g}} \ . \tag{2.17}$$

The dual of a p-form $\omega^{\rm E}$ is defined as in (2.5) but using $\epsilon_{\rm E}$. It then follows that ${}^*({}^*\omega_{\rm E}) = (-)^{p(D-p)}\, \omega_{\rm E}$ (with an additional minus sign with respect to the Minkowski relation) and, as in the Minkowskian case, $\omega_{\rm E} \wedge {}^*\omega_{\rm E} = \frac{1}{p!}\, \omega^{\rm E}_{j_1 \ldots j_p}\, \omega_{\rm E}^{j_1 \ldots j_p}\, \sqrt{g}\ \mathrm{d}^D z$.

It follows from the preceding discussion that the Euclidean action is not always real, not even its bosonic part. The original (real) Minkowskian action can contain two types of (locally) Lorentz invariant terms, terms that involve the metric like such as

$$S^{(1)}_{\rm M} = \frac{1}{2} \int\ \mathrm{tr}\, F \wedge {}^*F = \frac{1}{4} \int\ \mathrm{tr}\, F_{\mu\nu} F^{\mu\nu} \sqrt{|g|}\, \mathrm{d}^D x \tag{2.18}$$

and topological (Chern–Simons type) terms that do not involve the metric such as (if D is odd)

$$S_{\rm M}^{(2)} = \int \operatorname{tr} A \wedge F \wedge \ldots \wedge F \ . \tag{2.19}$$

It follows from (2.12), (2.13) and (2.16) that the Euclidean continuations of these two terms are

$$S_{\rm E}^{(1)} = -\frac{1}{2}\int \operatorname{tr} F_{\rm E} \wedge {}^*F_{\rm E} = -\frac{1}{4}\int \operatorname{tr} F^{\rm E}_{\mu\nu} F_{\rm E}^{\mu\nu} \sqrt{g}\, {\rm d}^D z \tag{2.20}$$

and

$$S_{\rm E}^{(2)} = -i\int \operatorname{tr} A_{\rm E} \wedge F_{\rm E} \wedge \ldots \wedge F_{\rm E} \ . \tag{2.21}$$

Hence the imaginary part of the Euclidean bosonic action is given by the topological terms. Note that from now on we will not write the wedge products explicitly, but $\operatorname{tr} AF^2$ will be short-hand for $\operatorname{tr} A \wedge F \wedge F$, etc.

There is a further subtlety that needs to be settled when discussing the relation between the Minkowskian and Euclidean forms of the anomalies. One has to know how the chirality matrix γ is continued from the Euclidean to the Minkowskian and vice versa. This will be relevant for the $2n$-dimensional submanifolds. The continuation of the γ-matrices is dictated by the continuation of the coordinates we have adopted (cf (2.13)):

$$i\,\gamma_{\rm M}^0 = \gamma_{\rm E}^1 \ , \quad \gamma_{\rm M}^1 = \gamma_{\rm E}^2 \ , \quad \ldots \quad \gamma_{\rm M}^{2n-1} = \gamma_{\rm E}^{2n} \ . \tag{2.22}$$

In accordance with Ref. [11] we define the Minkowskian and Euclidean chirality matrices $\gamma_{\rm M}$ and $\gamma_{\rm E}$ in $2n$ dimensions as

$$\gamma_{\rm M} = i^{n-1}\gamma_{\rm M}^0 \ldots \gamma_{\rm M}^{2n-1} \quad , \quad \gamma_{\rm E} = i^n \gamma_{\rm E}^1 \ldots \gamma_{\rm E}^{2n} \ . \tag{2.23}$$

Both $\gamma_{\rm M}$ and $\gamma_{\rm E}$ are hermitian. Taking into account (2.22) this leads to

$$\gamma_{\rm M} = -\gamma_{\rm E} \ , \tag{2.24}$$

i.e. what we call positive chirality in Minkowski space is called negative chirality in Euclidean space and vice versa. This relative minus sign is somewhat unfortunate, but it is necessary to define self-dual n-forms from a pair of positive chirality spinors, both in Minkowskian space (with our convention for the ϵ-tensor) and in Euclidean space (with the conventions of [11]). [c]

[c] Since we will take [11] as the standard reference for computing anomalies in Euclidean space, we certainly want to use the same convention for $\gamma_{\rm E}$. On the other hand, we have somewhat more freedom to choose a sign convention for $\gamma_{\rm M}$. The definition (2.23) of $\gamma_{\rm M}$ has the further advantage

Indeed, as is well-known, in $2n = 4k+2$ dimensions, from a pair of spinors of the same chirality one can always construct the components of an n-form H by sandwiching n (different) γ-matrices between the two spinors. In Minkowskian space we call such an n-form $H^{\rm M}$ self-dual if

$$H^{\rm M}_{\mu_1\ldots\mu_n} = +\frac{1}{n!}\,\epsilon_{\mu_1\ldots\mu_{2n}} H_{\rm M}^{\mu_{n+1}\ldots\mu_{2n}} \tag{2.25}$$

(with ϵ given by (2.2)) and it is obtained from 2 spinors ψ_I $(I = 1,2)$ satisfying $\gamma_{\rm M}\,\psi_I = +\psi_I$. In Euclidean space $H^{\rm E}$ is called self-dual if (cf [11])

$$H^{\rm E}_{j_1\ldots j_n} = +\frac{i}{n!}\,\epsilon^{\rm E}_{j_1\ldots j_{2n}} H_{\rm E}^{j_{n+1}\ldots j_{2n}} \tag{2.26}$$

(with $\epsilon^{\rm E}$ given by (2.17)) and it is obtained from 2 spinors χ_I $(I = 1,2)$ satisfying $\gamma_{\rm E}\,\chi_I = +\chi_I$. With these conventions a self-dual n-form in Minkowski space continues to an *anti*-self-dual n-form in Euclidean space, and vice versa, consistent with the fact that positive chirality in Minkowski space continues to negative chirality in Euclidean space. The situation is summarized in Table 1 where each of the four entries corresponds to any of the 3 others.

Table 1. Correspondences between the (anti-) self-duality of n-forms in $2n = 4k+2$ dimensions and the chirality of the corresponding pair of spinors are given, as well as their Euclidean, resp. Minkowskian continuations.

	Minkowskian	Euclidean
spinors	positive chirality	negative chirality
n-form	self-dual	anti-self-dual

As we will recall below, the anomalies are given by topological terms $\int_{M^{2n}_{\rm M}} D^{(2n)}_{\rm M}$ whose continuation is simply $\int_{M^{2n}_{\rm E}} D^{(2n)}_{\rm E}$ (cf Eq. (2.16)) where $D^{(2n)}_{\rm M}$ is the anomaly expression obtained by continuation from $D^{(2n)}_{\rm E}$ with *the chiralities corresponding as discussed above*. One also has to remember that the continuation of the effective action Γ includes an extra factor i according to Eq. (2.12). In conclusion, the anomaly of a positive chirality spinor (or a self-dual n-form) in Minkowski space is given by $\delta\Gamma_M = \int_{M^{2n}_M} \hat I^1_{2n}$

that in $D = 10$, $\gamma_{\rm M} = \gamma^0_{\rm M}\ldots\gamma^9_{\rm M}$ which is the usual convention used in string theory [12]. Our $\gamma_{\rm M}$ also agrees with the definition of [13] in $D = 2$, 6 and 10 (but differs from it by a sign in $D = 4$ and 8).

if in Euclidean space the anomaly of a negative chirality spinor (or an anti-self-dual n-form) is given by $\delta\Gamma_E = -i\int_{M_E^{2n}} \hat{I}^1_{2n}$. This will be discussed in more detail in the next subsection.

2.3. *The One-Loop Anomalies*

This subsection is a summary of the results of [11] where the anomalies for various chiral fields in Euclidean space were related to index theory. This whole subsection will be in Euclidean space of even dimension $2n$. We first give the different relevant indices. The simplest index is that of a positive chirality spin-$\frac{1}{2}$ field. Here positive chirality means positive Euclidean chirality as defined above. For spin-$\frac{1}{2}$ fermions the relevant Euclidean Dirac operator is (cf. (2.11)

$$\not{D} = E_a^j \gamma^a \Big(\partial_j + A_j + \frac{1}{4}\,\omega_{cd,j}\gamma^{cd}\Big)\,, \quad \gamma^{cd} = \frac{1}{2}\,[\gamma^c,\gamma^d]\,. \tag{2.27}$$

Define $\not{D}_{\frac{1}{2}} = \not{D}\,\frac{1+\gamma}{2}$ and the index as

$$\begin{aligned}\mathrm{ind}\big(i\not{D}_{\frac{1}{2}}\big) = &\text{ number of zero modes of } i\not{D}_{\frac{1}{2}}\\ &-\text{number of zero modes of } \big(i\not{D}_{\frac{1}{2}}\big)^\dagger\,.\end{aligned} \tag{2.28}$$

Then by the Atiyah–Singer index theorem

$$\mathrm{ind}\big(i\not{D}_{\frac{1}{2}}\big) = \int_{M_{2n}} [\hat{A}(M_{2n})\ \mathrm{ch}(F)]_{2n}\,, \tag{2.29}$$

where $\mathrm{ch}(F) =$ tr exp $\big(\frac{i}{2\pi}F\big)$ is the Chern character and $\hat{A}(M_{2n})$ is the Dirac genus of the manifold, given below. The subscript $2n$ indicates to pick only the part which is a $2n$-form. Note that if the gauge group is $\prod_k G_k$, then $\mathrm{ch}(F)$ is replaced by $\prod_k \mathrm{ch}(F_k)$.

Another important index is that of a positive chirality spin-$\frac{3}{2}$ field. Such a field is obtained from a positive chirality spin-$\frac{1}{2}$ field with an extra vector index by subtracting the spin-$\frac{1}{2}$ part. An extra vector index leads to an additional factor for the index density,

$$\mathrm{tr}\ \exp\Big(\frac{i}{2\pi}\,\frac{1}{2}R_{ab}T^{ab}\Big) = \mathrm{tr}\ \exp\Big(\frac{i}{2\pi}R\Big)\,, \tag{2.30}$$

since the vector representation is $(T^{ab})_{cd} = \delta^a_c\delta^b_d - \delta^a_d\delta^b_c$. Hence

$$\mathrm{ind}\big(iD_{\frac{3}{2}}\big) = \int_{M_{2n}} \Big[\hat{A}(M_{2n})\Big(\ \mathrm{tr}\ \exp\Big(\frac{i}{2\pi}R\Big) - 1\Big)\ \mathrm{ch}(F)\Big]_{2n}\,. \tag{2.31}$$

The third type of field which leads to anomalies is a self-dual or anti-self-dual n-form H in $2n = 4k+2$ dimensions. Such antisymmetric tensor fields carry no charge w.r.t. the gauge group. As discussed above, a self-dual tensor can be constructed from a pair of positive chirality spinors. Correspondingly, the index is $\hat{A}(M_{2n})$ multiplied by $\mathrm{tr}\, \exp\left(\frac{i}{2\pi}\frac{1}{2}R_{ab}T^{ab}\right)$, where $T^{ab} = \frac{1}{2}\gamma^{ab}$ as appropriate for the spin-$\frac{1}{2}$ representation. Note that the trace over the spinor representation gives a factor 2^n in $2n$ dimensions. There is also an additional factor $\frac{1}{2}$ from the chirality projector of this second spinor and another factor $\frac{1}{2}$ from a reality constraint (H is real),

$$\mathrm{ind}(iD_A) = \frac{1}{4}\int_{M_{2n}} \left[\hat{A}(M_{2n})\, \mathrm{tr}\, \exp\left(\frac{i}{2\pi}\frac{1}{4}R_{ab}\gamma^{ab}\right)\right]_{2n} = \frac{1}{4}\int_{M_{2n}} [L(M)]_{2n}\, . \tag{2.32}$$

$L(M)$ is called the Hirzebruch polynomial, and the subscript on D_A stands for "antisymmetric tensor". (Note that, while $\hat{A}(M_{2n})\, \mathrm{tr}\, \exp\left(\frac{i}{2\pi}\frac{1}{4}R_{ab}\gamma^{ab}\right)$ carries an overall factor 2^n, $L(M_{2n})$ has a factor 2^k in front of each $2k$-form part. It is only for $k = n$ that they coincide.)

Of course, the index of a negative chirality (anti-self-dual) field is minus that of the corresponding positive chirality (self-dual) field. Explicitly one has:

$$\mathrm{ch}(F) = \mathrm{tr}\, \exp\left(\frac{i}{2\pi}F\right) = \mathrm{tr}\, \mathbf{1} + \frac{i}{2\pi}\,\mathrm{tr}\, F + \ldots + \frac{i^k}{k!(2\pi)^k}\,\mathrm{tr}\, F^k + \ldots\, , \tag{2.33}$$

$$\begin{aligned}\hat{A}(M_{2n}) &= 1 + \frac{1}{(4\pi)^2}\frac{1}{12}\,\mathrm{tr}\, R^2 + \frac{1}{(4\pi)^4}\left[\frac{1}{360}\,\mathrm{tr}\, R^4 + \frac{1}{288}(\mathrm{tr}\, R^2)^2\right] \\ &\quad + \frac{1}{(4\pi)^6}\left[\frac{1}{5670}\,\mathrm{tr}\, R^6 + \frac{1}{4320}\,\mathrm{tr}\, R^4\,\mathrm{tr}\, R^2 + \frac{1}{10368}(\mathrm{tr}\, R^2)^3\right] + \ldots\, ,\end{aligned} \tag{2.34}$$

$$\begin{aligned}\hat{A}(M_{2n})\left(\mathrm{tr}\, e^{\frac{i}{2\pi}R} - 1\right) &= (2n-1) + \frac{1}{(4\pi)^2}\frac{2n-25}{12}\,\mathrm{tr}\, R^2 \\ &\quad + \frac{1}{(4\pi)^4}\left[\frac{2n+239}{360}\,\mathrm{tr}\, R^4 + \frac{2n-49}{288}(\mathrm{tr}\, R^2)^2\right] \\ &\quad + \frac{1}{(4\pi)^6}\left[\frac{2n-505}{5670}\,\mathrm{tr}\, R^6 + \frac{2n+215}{4320}\,\mathrm{tr}\, R^4\,\mathrm{tr}\, R^2 + \frac{2n-73}{10368}(\mathrm{tr}\, R^2)^3\right] + \ldots\, ,\end{aligned} \tag{2.35}$$

$$\begin{aligned}L(M_{2n}) &= 1 - \frac{1}{(2\pi)^2}\frac{1}{6}\,\mathrm{tr}\, R^2 + \frac{1}{(2\pi)^4}\left[-\frac{7}{180}\,\mathrm{tr}\, R^4 + \frac{1}{72}(\mathrm{tr}\, R^2)^2\right] \\ &\quad + \frac{1}{(2\pi)^6}\left[-\frac{31}{2835}\,\mathrm{tr}\, R^6 + \frac{7}{1080}\,\mathrm{tr}\, R^4\,\mathrm{tr}\, R^2 - \frac{1}{1296}(\mathrm{tr}\, R^2)^3\right] + \ldots\, .\end{aligned} \tag{2.36}$$

To proceed, we need to define exactly what we mean by the anomaly. For the time being, we suppose that the classical action is invariant (no inflow), but that the Euclidean quantum effective action $\Gamma_E[A]$ has an anomalous variation under the gauge transformation (2.9) with parameter v of the form

$$\delta_v \Gamma_E[A] = \int \operatorname{tr} v\, \mathcal{G}(A)\,. \tag{2.37}$$

Local Lorentz anomalies are treated analogously. Note that

$$\delta_v \Gamma_E[A] = \int (D_\mu v)^\alpha \frac{\delta \Gamma_E[A]}{\delta A_\mu^\alpha} = -\int v^\alpha (D_\mu J^\mu)^\alpha \tag{2.38}$$

or [d]

$$\delta_v \Gamma_E[A] = \int \operatorname{tr} D_\mu v \frac{\delta \Gamma_E[A]}{\delta A_\mu} = -\int \operatorname{tr} v D_\mu \frac{\delta \Gamma_E[A]}{\delta A_\mu} \tag{2.39}$$

so that $\mathcal{G}(A)$ is identified with $-D_\mu \frac{\delta \Gamma_E[A]}{\delta A_\mu}$ or $\mathcal{G}(A)^\alpha$ with $-(D_\mu J^\mu)^\alpha$. To avoid these complications, we will simply refer to the anomalous variation of the effective action, $\delta_v \Gamma_E[A]$ as the anomaly. So our anomaly is the negative integrated divergence of the quantum current (multiplied with the variation parameter v).

A most important result of [11] is the precise relation between the anomaly in $2n$ dimensions and index theorems in $2n+2$ dimensions, which for the pure gauge anomaly of a positive chirality spin-$\frac{1}{2}$ field is (Eq. (3.35) of [11])

$$\delta_v \Gamma_E^{spin\frac{1}{2}}[A] = +\frac{i^n}{(2\pi)^n (n+1)!} \int Q_{2n}^1(v, A, F)\,. \tag{2.40}$$

The standard descent equations $\mathrm{d}Q_{2n}^1 = \delta_v Q_{2n+1}$ and $\mathrm{d}Q_{2n+1} = \operatorname{tr} F^{n+1}$ relate Q_{2n}^1 to the invariant polynomial $\operatorname{tr} F^{n+1}$. Comparing with (2.33) we see that the pure gauge anomaly is thus given by $\delta_v \Gamma_E^{spin\frac{1}{2}}[A] = \int I_{2n}^{1,gauge}$ with the descent equations $\mathrm{d}I_{2n}^{1,gauge} = \delta_v I_{2n+1}^{gauge}$ and $\mathrm{d}I_{2n+1}^{gauge} = I_{2n+2}^{gauge}$, where $I_{2n+2}^{gauge} = -2\pi i\, [\mathrm{ch}(F)]_{2n+2}$. This is immediately generalized to include all gauge and local Lorentz anomalies due to all three types of chiral fields

$$\delta \Gamma_E[A] = \int I_{2n}^1\,, \tag{2.41}$$

$$\mathrm{d}I_{2n}^1 = \delta I_{2n+1} \ ,\ \mathrm{d}I_{2n+1} = I_{2n+2}\,, \tag{2.42}$$

[d] Note that if $A = A^\alpha \lambda^\alpha$, $B = B^\beta \lambda^\beta$ and $\operatorname{tr} \lambda^\alpha \lambda^\beta = -\delta^{\alpha\beta}$ (the λ^α are anti-hermitian) then e.g. $\operatorname{tr} AB = -A^\alpha B^\alpha$ and $\frac{\delta}{\delta A^\alpha} \int \operatorname{tr} AB = -B^\alpha$. Hence one must define $\frac{\delta}{\delta A} = -\lambda^\alpha \frac{\delta}{\delta A^\alpha}$ so that $\frac{\delta}{\delta A} \int \operatorname{tr} AB = B$. Another way to see this minus sign in $\frac{\delta}{\delta A}$ is to note that $A^\alpha = -\operatorname{tr} \lambda^\alpha A$.

where I_{2n+2} equals $-2\pi i$ times the relevant index density appearing in the index theorem in $2n+2$ dimensions (corrected by a factor of $\left(-\frac{1}{2}\right)$ in the case of the antisymmetric tensor field, see below). This shows that the Euclidean anomaly is purely imaginary. It is thus convenient to introduce $\hat{I}$ as $I = -i\ \hat{I}$ so that

$$\delta\Gamma_E[A] = -i \int \hat{I}^1_{2n}\,, \tag{2.43}$$

$$\mathrm{d}\hat{I}^1_{2n} = \delta\hat{I}_{2n+1}\ \ ,\ \ \mathrm{d}\hat{I}_{2n+1} = \hat{I}_{2n+2}\,. \tag{2.44}$$

Explicitly we have (always for positive Euclidean chirality, respectively Euclidean self-dual forms)

$$\hat{I}^{spin\frac{1}{2}}_{2n+2} = 2\pi \left[\hat{A}(M_{2n})\ \mathrm{ch}(F)\right]_{2n+2}\,, \tag{2.45}$$

$$\hat{I}^{spin\frac{3}{2}}_{2n+2} = 2\pi \left[\hat{A}(M_{2n})\ \left(\mathrm{tr}\ \exp\left(\frac{i}{2\pi}R\right) - 1\right)\ \mathrm{ch}(F)\right]_{2n+2}\,, \tag{2.46}$$

$$\hat{I}^{A}_{2n+2} = 2\pi \left[\left(-\frac{1}{2}\right)\frac{1}{4}\ L(M_{2n})\right]_{2n+2}\,. \tag{2.47}$$

The last equation contains an extra factor $\left(-\frac{1}{2}\right)$ with respect to the index (2.32). The minus sign takes into account the Bose rather than Fermi statistics, and the $\frac{1}{2}$ corrects the 2^{n+1} to 2^n which is the appropriate dimension of the spinor representation on M_{2n} while the index is computed in $2n+2$ dimensions. Note that in the cases of interest, the spin-$\frac{3}{2}$ gravitino is not charged under the gauge group and in (2.46) the factor of $\mathrm{ch}(F)$ simply equals 1.

Equations (2.43)-(2.47) together with (2.33)-(2.36) give explicit expressions for the anomalous variation of the Euclidean effective action. In the previous subsection we carefully studied the continuation of topological terms like $\int \hat{I}^1_{2n}$ between Minkowski and Euclidean signature. It follows from equations (2.12), (2.16) and (2.43) that the anomalous variation of the Minkowskian effective action is given directly by $\hat{I}^1_{2n}$,

$$\delta\Gamma_M = \int_{M^M_{2n}} \hat{I}^1_{2n}\,. \tag{2.48}$$

However, one has to remember that (with our conventions for γ_M) the chiralities in Minkowski space and Euclidean space are opposite. While $\hat{I}^1_{2n}$ corresponds to positive chirality in the Euclidean, it corresponds to negative chirality in Minkowski space, i.e. *Eq. (2.48) is the anomaly for a negative chirality field in Minkowski space.*

Obviously, the anomaly of a positive chirality field in Minkowski space is just the opposite.

To facilitate comparison with references [12] (GSW) and [14] (FLO) we note that

$$I_{\rm GSW} = (2\pi)^n \hat{I}_{2n+2}\,, \quad I_{\rm FLO} = -\hat{I}_{2n+2}\,. \tag{2.49}$$

The flip of sign between $I_{\rm FLO}$ and $\hat{I}_{2n+2}$ is such that $\int I^1_{\rm FLO}$ directly gives the variation of the Minkowskian effective action for positive chirality spinors in Minkowskian space (with our definition of $\gamma_{\rm M}$).

Before we go on, it is perhaps useful to look at an explicit example in four dimensions. Consider the simple case of a spin-$\frac{1}{2}$ fermion of *negative* Minkowskian chirality coupled to $SU(N)$ gauge fields. In the Euclidean, this corresponds to positive chirality and hence the anomalous variation of the Minkowskian effective action is $\delta\Gamma_M = \int \hat{I}^1_4$, where $\hat{I}^1_4$ is related via the descent equations to $\hat{I}_6$ which is obtained from (2.45) as

$$\hat{I}_6 = -\frac{i}{6(2\pi)^2}\,{\rm tr}\ F^3\,. \tag{2.50}$$

Note that this is real since by (2.9) tr F^3 is purely imaginary. Also, there is no mixed gauge-gravitational anomaly since the relevant term $\sim$ ${\rm tr}\, R^2\, {\rm tr}\, F$ vanishes for $SU(N)$ gauge fields. It is only for $U(1)$ gauge fields that one can get a mixed gauge-gravitational anomaly in four dimensions. Using the descent equations one explicitly gets

$$\delta\Gamma_M = -\frac{i}{6(2\pi)^2}\int {\rm tr}\, v\ {\rm d}\Big(A{\rm d}A + \frac{1}{2}A^3\Big)\,. \tag{2.51}$$

It is important to note that we are only discussing the so-called consistent anomaly. Indeed, since our anomaly is defined as the variation of the effective action it automatically satisfies the Wess-Zumino consistency condition [15] and hence is the consistent anomaly. There is also another manifestation of the anomaly, the so-called covariant anomaly (which in the present example would be $-\frac{i}{2(2\pi)^2}\int\ {\rm tr}\, vF^2$). The latter is not relevant to us here and we will not discuss it further (see however Ref. [16]).

Finally, it is worth mentioning that the anomalies are "quantized" in the following sense: once we have normalized the gauge and gravitational fields in the usual way (so that $F = {\rm d}A + A^2$ and $R = {\rm d}\omega + \omega^2$) the anomalies have no explicit dependence on the gauge or gravitational coupling constants. In a given theory, the total anomaly is a sum of the fixed anomalies $\hat{I}^{spin\frac{1}{2}}$, $\hat{I}^{spin\frac{3}{2}}$ and $\hat{I}^A$ with coefficients that count the multiplicities of the corresponding fields, i.e. are integers. Of course, this came about from the relation with index theory, and there was just no place where any coupling constants could show up. Another way to see this is to recall that the anomalies are one-loop contributions to the effective action coming from exponentiating determinants. In the loop expansion of the effective action only the one-loop term is independent of the coupling constants.

2.4. *Anomaly Cancellation by Inflow*

We have seen that the anomalous variation of the one-loop quantum effective action is $\delta\Gamma_{\rm E} = -i\int_{M_E^{2n}} \hat{I}_{2n}^1$ in the Euclidean and $\delta\Gamma_{\rm M} = \int_{M_M^{2n}} \hat{I}_{2n}^1$ in the Minkowskian case. Now, we want to discuss the situation where M_{2n} is a $2n$-dimensional submanifold (on which live the chiral fields that give rise to the anomaly) embedded in a manifold of higher dimension D.

To appreciate the role of the higher-dimensional embedding, let us first remark that a (consistent) anomaly in $2n$ dimensions cannot be cancelled by adding to the classical invariant action a local non-invariant $2n$-dimensional "counterterm" $\Gamma_E^{(1)}[A,\omega] = -i\int \gamma[A,\omega]$ that depends on the gauge and gravitational fields only (as does $\hat{I}_{2n+1}$). Indeed, a consistent anomaly $\hat{I}_{2n}^1$, characterized by a non-vanishing $\hat{I}_{2n+2}$, is only defined up to the addition of such a local counterterm;[e] this is the essence of the descent equations (2.42) or (2.44). To see this, suppose one has the one-loop anomaly $\delta\Gamma_E = -i\int \hat{I}_{2n}^1$. Upon descent this leads to $\hat{I}_{2n+2}$. If one adds the counterterm $\Gamma_E^{(1)}[A,\omega]$ to the classical action the variation of the new effective action and the descent equations (2.44) are

$$\begin{aligned}
\delta\Gamma_E + \delta\Gamma_E^{(1)} &= -i\int\left(\hat{I}_{2n}^1 + \delta\gamma\right), \\
{\rm d}\left(\hat{I}_{2n}^1 + \delta\gamma\right) &= \delta\hat{I}_{2n+1} + \delta{\rm d}\gamma = \delta\left(\hat{I}_{2n+1} + {\rm d}\gamma\right), \\
{\rm d}\left(\hat{I}_{2n+1} + {\rm d}\gamma\right) &= \hat{I}_{2n+2} + 0
\end{aligned} \tag{2.52}$$

with the same $\hat{I}_{2n+2}$ as before; the invariant polynomial is insensitive to the addition of a local counterterm.

While addition of a local counterterm cannot eliminate the anomaly, it can be used to shift between two different expressions of the "same" anomaly. Consider as an example the mixed U(1) gauge-gravitational anomaly for a negative chirality spin-$\frac{1}{2}$ fermion in four Minkowskian dimensions character-

[e] One always has the freedom to add a local counterterm to the action. If this was enough to cancel the anomaly one could consistently quantize the theory without problems.

ized by the invariant 6-form

$$\hat{I}_6^{\rm mixed} = -\frac{q}{12(4\pi)^2}\,\mathcal{F}\,{\rm tr}\,R^2 \tag{2.53}$$

(recall that for U(1) gauge fields $A \simeq iq\mathcal{A}$, $F \simeq iq\mathcal{F}$ and $v \simeq iq\tilde{\epsilon}$). Upon descent, this gives $\hat{I}_4^{\rm mixed,1}$ either as

$$\hat{I}_4^{\rm mixed,1} = -\frac{q}{12(4\pi)^2}\,\tilde{\epsilon}\,{\rm tr}\,R^2 \quad \text{or} \quad \hat{I}_4^{\rm mixed,1} = -\frac{q}{12(4\pi)^2}\,\mathcal{F}\,{\rm tr}\,\epsilon{\rm d}\omega\ . \tag{2.54}$$

Addition of the counterterm

$$\Gamma^{(1)} = -\frac{q}{12(4\pi)^2}\int \mathcal{A}\,{\rm tr}\big(\omega{\rm d}\omega + \frac{2}{3}\omega^3\big) \tag{2.55}$$

allows interpolation between the two expressions of the anomaly since $\delta\Gamma^{(1)} = -\frac{q}{12(4\pi)^2}\left(\int \mathcal{F}\,{\rm tr}\,\epsilon{\rm d}\omega - \int \tilde{\epsilon}\,{\rm tr}\,R^2\right)$.

The preceding discussion shows that the anomaly cannot be cancelled by adding local terms defined on the same $2n$-dimensional manifold on which live the chiral fields responsible for the anomaly. Instead, we will consider local terms defined on a higher-dimensional manifold which contains the $2n$-dimensional one as a submanifold.

The simplest example is a 3-dimensional manifold M_3 whose boundary is a 2-dimensional manifold $M_2 = \partial M_3$. In practice, one has to pay attention to the orientations of ∂M_3 and M_2 and be careful whether what one calls M_2 is ∂M_3 or $-\partial M_3$, i.e. ∂M_3 with opposite orientation. Suppose that on M_2 lives a chiral spin-$\frac{1}{2}$ field coupled to a gauge field A. The gauge anomaly is (for positive Euclidean chirality) $\delta\Gamma_E = -i\int_{M_2}\hat{I}_2^1$ where $\hat{I}_2^1$ is obtained by the descent equations from $\hat{I}_4 = -\frac{1}{4\pi}\,{\rm tr}\,F^2$. Explicitly

$$\hat{I}_3 = -\frac{1}{4\pi}\,{\rm tr}\left(A{\rm d}A + \frac{2}{3}A^3\right) \equiv -\frac{1}{4\pi}\,Q_3^{\rm CS}\,, \tag{2.56}$$

$$\hat{I}_2^1 = -\frac{1}{4\pi}\,{\rm tr}\,v{\rm d}A \equiv -\frac{1}{4\pi}\,Q_2^{\rm CS,1} \tag{2.57}$$

where $Q_3^{\rm CS}$ is the usual Chern–Simons 3-form, obviously obeying

$$\delta Q_3^{\rm CS} = {\rm d}Q_2^{\rm CS,1}\ . \tag{2.58}$$

Now suppose that the 3-dimensional Euclidean action contains a Chern–Simons term

$$S_{\rm CS} = -\frac{i}{4\pi}\int_{M_3} Q_3^{\rm CS}\,. \tag{2.59}$$

As discussed in Section 2.1, this topological term needs to be purely imaginary in order to correspond to a real term in the Minkowskian action. On the other hand, being imaginary in the Euclidean case is exactly what is needed to match the anomalous part of the effective action, as we now proceed to show. Under a gauge variation, the Chern–Simons term transforms as

$$\delta S_{\rm CS} = -\frac{i}{4\pi}\int_{M_3} {\rm d}Q_2^{\rm CS,1} \ , \tag{2.60}$$

which would vanish if M_3 had no boundary. By Stoke's theorem we have

$$\delta S_{\rm CS} = -\frac{i}{4\pi}\int_{\partial M_3} Q_2^{\rm CS,1} = i\int_{M_2} \hat{I}_2^1 \, . \tag{2.61}$$

Thus the non-invariance of the Chern–Simons term is localized on the 2-dimensional boundary manifold M_2 and, with the coefficient chosen as above, it exactly cancels the one-loop anomaly. This is called anomaly cancellation by anomaly inflow from the bulk. This example is particularly simple as the Chern–Simons term is nothing but $S_{\rm CS} = i\int \hat{I}_3$ and the anomaly inflow is governed directly by the descent equations $\delta\hat{I}_3 = {\rm d}\hat{I}_2^1$.

As an example of a somewhat different type, consider a 5-dimensional Minkowskian theory involving a U(1)-gauge field and gravity and suppose it admits solutions that are analogous to magnetic monopoles in 4 dimensions. In 5 dimensions these are magnetically charged string or vortex like solutions. Their world-volume is a 2-dimensional manifold W_2. In the presence of such a solution, the Bianchi identity ${\rm d}\mathcal{F} = 0$ is modified as

$${\rm d}\mathcal{F} = \alpha\ \delta_{W_2}^{(3)} \ , \tag{2.62}$$

where α is some coefficient measuring the magnetic charge density on the string and $\delta_{W_2}^{(3)}$ is a Dirac distribution 3-form with support on the 2-dimensional world-volume W_2. It has the property that $\int_{M_5} \delta_{W_2}^{(3)}\xi = \int_{W_2}\xi$ for any 2-form ξ. Typically, on W_2 live some chiral fields. If we suppose that they carry no U(1)-charge and that there are n_+ positive and n_- negative (Minkowskian) chirality spin-$\frac{1}{2}$ fields, there will only be a gravitational anomaly in two dimensions equal to

$$\delta\Gamma_M = \frac{n_- - n_+}{96\pi}\int_{W_2} {\rm tr}\,\epsilon\,{\rm d}\omega \ . \tag{2.63}$$

This can again be cancelled by anomaly inflow from the bulk. Suppose there is a topological term in the 5-dimensional action involving $\mathcal{F}$ and the gravitational Chern–Simons 3-form,

$$S_{\rm top} = \beta \int_{M_5} \mathcal{F}\ {\rm tr}\left(\omega {\rm d}\omega + \frac{2}{3}\omega^3\right). \tag{2.64}$$

Its variation is (using again a descent relation analogous to (2.58))

$$\begin{aligned} \delta S_{\rm top} &= \beta \int_{M_5} \mathcal{F}\, {\rm d}\, {\rm tr}\, \epsilon\, {\rm d}\omega = -\beta \int_{M_5} {\rm d}\mathcal{F}\ {\rm tr}\, \epsilon {\rm d}\omega \\ &= -\alpha\beta \int_{M_5} \delta^{(3)}_{W_2}\ {\rm tr}\, \epsilon\, {\rm d}\omega = -\alpha\beta \int_{W_2} {\rm tr}\, \epsilon {\rm d}\omega\,, \end{aligned} \tag{2.65}$$

and, if $\alpha\beta = \frac{n_- - n_+}{96\pi}$, this cancels the gravitational anomaly on the two-dimensional world-volume. This second example is a very simplified version of the cancellation of the five-brane anomalies in M-theory, which will be discussed (with all its coefficients) in some detail below.

It is worthwhile to note a generic feature of anomaly inflow in the previous example. Suppose we decide to rescale the U(1)-gauge field by some factor η so that $\mathcal{F} \to \tilde{\mathcal{F}} = \eta\mathcal{F}$. Then, the coefficient α in the Bianchi identity also gets rescaled as $\alpha \to \tilde{\alpha} = \eta\alpha$ so that it still reads ${\rm d}\tilde{\mathcal{F}} = \tilde{\alpha}\, \delta^{(3)}_{W_2}$. The coefficient β in (2.64) obviously becomes $\beta \to \tilde{\beta} = \beta/\eta$, and $\tilde{\alpha}\tilde{\beta} = \alpha\beta$. We see that the anomaly cancelling condition $\alpha\beta = \frac{n_- - n_+}{96\pi}$ *is invariant under any rescalings* as it must be since the one-loop anomaly only depends on the integers n_+ and n_-.

It is clear from these examples that by some mechanism or another the variation of a $(D > 2n)$-dimensional topological term in the classical action gives rise to a $2n$-dimensional topological term

$$\delta S^{\rm cl}_{\rm M} = \int_{M^{2n}_{\rm M}} D^{(2n)}_{\rm M} \quad \Leftrightarrow \quad \delta S^{\rm cl}_{\rm E} = -i \int_{M^{2n}_{\rm E}} D^{(2n)}_{\rm E} \tag{2.66}$$

with $D^{(2n)}_{\rm M} = D^{(2n)}_{\rm E} \equiv D^{(2n)}$ according to (2.15). Thus the total variation of the $2n$-dimensional action including the one-loop anomaly is

$$\delta\Gamma_{\rm M} = \int_{M^{2n}_{\rm M}} \left(\hat{I}^1_{2n} + D^{(2n)}\right) \ \Leftrightarrow\ \delta\Gamma_{\rm E} = -i \int_{M^{2n}_{\rm E}} \left(\hat{I}^1_{2n} + D^{(2n)}\right) \tag{2.67}$$

(where now $\hat{I}^1_{2n}$ is meant to contain all the contributions to the one-loop anomaly, with all the relevant signs and factors to take into account the different chiralities and multiplicities). In any case, the condition for anomaly cancellation is the same in Euclidean and Minkowski signature,

$$\hat{I}^1_{2n} + D^{(2n)} = 0\,. \tag{2.68}$$

3. Anomaly Cancellation by Inflow in Condensed Matter: The Quantum Hall Effect

A most important example from condensed matter is the Quantum Hall Effect [8]. The relevant geometry of a Hall sample is two-dimensional with a one-dimensional boundary, e.g. an annulus. Typically, the boundary has two disconnected pieces (edges) like the inner and outer boundary of the annulus. Adding time, the physics is on a $2+1$ dimensional manifold with a 1+1 dimensional boundary.

A magnetic field is applied perpendicular to the Hall sample and an electric field is present along the sample (usually perpendicular to the edges) resulting in a voltage drop. All this is again described by a 2+1 dimensional electromagnetic field $\mathcal{F}$ with (recall that our signature is $(-++)$)

$$\mathcal{F}_{01} = -E_1 \ , \quad \mathcal{F}_{02} = -E_2 \ , \quad \mathcal{F}_{12} = B \ . \tag{3.1}$$

When the filling factor (controlled by the ratio of the electron density and the magnetic field) takes values in certain intervals, one observes a vanishing longitudinal resistivity. The conductivity matrix being the inverse of the resistivity matrix, the longitudinal conductivity also vanishes and the current and electric field are related as

$$j^a = \sigma^{ab} E_b = -\sigma^{ab} \mathcal{F}_{0b} \ , \quad a, b = 1, 2 \tag{3.2}$$

with $\sigma^{11} = \sigma^{22} = 0$ and $\sigma^{12} = -\sigma^{21} \equiv \sigma_H$ being the transverse or Hall conductivity. In the integer Quantum Hall Effect, this Hall conductivity σ_H is an integer multiple of e^2/h, or since we have set $\hbar = 1$,

$$\sigma_H = n \, \frac{e^2}{2\pi} \ , \quad n \in \mathbf{Z} \ , \tag{3.3}$$

$-e$ being the elementary charge of the electron. In the fractional Quantum Hall Effect, n is replaced by certain rational numbers.

The integer Quantum Hall Effect is quite well understood in terms of elementary quantum mechanics of electrons in a strong magnetic field, giving rise to the usual Landau levels, together with an important role played by disorder (impurities) in the sample, leading to localization (see e.g. [8]). The fractional Quantum Hall Effect is more intriguing and has given rise to a large literature (which I will not cite). In both cases, effective field theories of the Chern–Simons type have played an important role, see e.g. Refs. [17–22].

Here, we will only consider a simple field theoretic model neglecting most of the subtleties discussed in the above-mentioned references, as well as in others. Consider an effective field theory given by a 2+1 dimensional Chern–

Simons term of the electromagnetic vector potential $\mathcal{A}_\mu$ plus a coupling to the electromagnetic current j^μ,

$$S_{2+1} = \frac{\sigma}{2} \int_{M_{2+1}} \mathrm{d}^3 x \, \epsilon^{\mu\nu\rho} \mathcal{A}_\mu \partial_\nu \mathcal{A}_\rho + \int_{M_{2+1}} \mathrm{d}^3 x \, j^\mu \mathcal{A}_\mu \ . \tag{3.4}$$

(For simplicity we assume a trivial metric.) Varying this action with respect to $\mathcal{A}_\mu$ gives the equation of motion

$$j^\mu = -\frac{\sigma}{2} \, \epsilon^{\mu\nu\rho} \mathcal{F}_{\nu\rho} \ , \quad \mu = 0, 1, 2 \ . \tag{3.5}$$

Specializing to $\mu = 1, 2$ and using $\epsilon^{012} = -1$ (see Eq. (2.2)) we see that the effective action (3.4) correctly reproduces the Hall relation (3.2) with

$$\sigma_H = \sigma \ . \tag{3.6}$$

The action (3.4) can be rewritten using forms (cf (2.3)) as

$$S_{2+1} = -\frac{\sigma_H}{2} \int_{M_{2+1}} \mathcal{A} \wedge \mathrm{d}\mathcal{A} + \int_{M_{2+1}} {}^*j \wedge \mathcal{A} \ . \tag{3.7}$$

It is well-known in the integer Quantum Hall Effect that there are chiral massless excitations on the boundaries (edge currents). They can be viewed as excitations of the incompressible two-dimensional electron gas or resulting from an interruption of the semiclassical cyclotron trajectories by the edges [8]. In any case, they are 1+1 dimensional chiral degrees of freedom. In 1+1 dimensions it does not matter whether they are described as chiral bosons or as chiral fermions, both descriptions being related. Suppose there are n_k species of them on the edge k (we label the two edges as $k = 1, 2$). Note that all species on a given edge have the same chirality. These chiral fermions being charged have a one-loop U(1) gauge anomaly. Recall that for U(1) gauge fields we replace $A \simeq iq\mathcal{A}$ and similarly for the field strength $F \simeq iq\mathcal{F}$ and for the gauge variation parameter $v \simeq iq\tilde{\epsilon}$, where $q = -e$ is the (negative) electron charge. Then, $\mathrm{tr}\, v \, \mathrm{d}A \simeq -e^2 \, \tilde{\epsilon} \, \mathrm{d}\mathcal{A}$, and according to the general results of the previous section, the anomalous variation of the effective action on the $k^{\rm th}$ edge is

$$\delta_{\tilde{\epsilon}} \Gamma^{\mathrm{edge}\, k} = \pm n_k \int_{M_{1+1}^{(k)}} \hat{I}_2^{1, spin\frac{1}{2}} = \pm n_k \, \frac{e^2}{4\pi} \int_{M_{1+1}^k} \tilde{\epsilon} \, \mathrm{d}\mathcal{A} \ , \tag{3.8}$$

where the $\pm$ accounts for the (unspecified) chirality,[f] and we have used

[f] One should also be careful about the orientations of M_{2+1} and of the edges M_{1+1}^k to get the signs straight.

Eq. (2.57).

On the other hand, the bulk action S_{2+1} is also anomalous due to the boundary and it gives an anomaly inflow

$$\delta S_{2+1} = -\frac{\sigma_H}{2}\int_{M_{2+1}} \mathrm{d}(\tilde{\epsilon}\,\mathrm{d}\mathcal{A}) = \sum_{\text{edges } k}\left(-\frac{\sigma_H}{2}\right)\int_{M_{1+1}^{(k)}} \tilde{\epsilon}\,\mathrm{d}\mathcal{A}\ . \tag{3.9}$$

The quantum anomalies (3.8) and the anomaly inflow (3.9) cancel if and only if $\sigma_H = (\pm n_k)\,\frac{e^2}{2\pi}$. Since the anomaly should cancel on both edges k, this shows that $\pm n_1 = \pm n_2 \equiv n$ and

$$\sigma_H = n\,\frac{e^2}{2\pi}\ , \quad n \in \mathbf{Z} \tag{3.10}$$

in agreement with Eq. (3.3). Anomaly cancellation by inflow from the bulk forces the Hall conductivity to be correctly quantized!

As already noted, the fractional Quantum Hall Effect is more complicated and the edge excitations are described by more complicated quasiparticles involving exotic spins and statistics, so that our simple argument needs to be refined. Somewhat related arguments can be found in [20–22]. Other examples in 1+1 dimensional condensed matter where anomaly arguments play a role are quantum wires [22] and presumably vortices, as well as defect surfaces in 3-dimensional Euclidean statistical systems. Due to lack of competence, I will discuss none of them here.

4. Examples of Anomaly Cancellation by Inflow in M-Theory

4.1. *The low-energy effective action of M-Theory*

M-theory has emerged from a web of dualities between superstring theories. In its eleven-dimensional uncompactified version it can be considered as the strong-coupling limit of type IIA superstring theory. This tells us that its low-energy effective action is that of eleven-dimensional supergravity first written by Cremmer, Julia and Scherk [23]. In Minkowski space its bosonic part reads (using our conventions as exposed in Section 2.1)

$$S_{\mathrm{M}}^{\mathrm{CJS}} = \frac{1}{2\kappa^2}\left(\int \mathrm{d}^{11}x\sqrt{|g|}\ \mathcal{R} - \frac{1}{2}\int G\wedge {}^*G - \frac{1}{6}\int C\wedge G\wedge G\right)\ , \tag{4.1}$$

where $\kappa \equiv \kappa_{11}$ is the 11-dimensional gravitational constant, $\mathcal{R}$ is the Ricci scalar, and $G = \mathrm{d}C$. The coefficients of the second and third term in this action can be changed by rescaling the C-field. Also, some authors use a different relation between G and $\mathrm{d}C$. These issues have been extensively discussed in [4] where a table is given summarizing the conventions of various authors. Here, however, we will use the simple choice made in Eq. (4.1) which in the notation of Ref. [4] corresponds to $\alpha = \beta = 1$. Note that the third term is a topological term, usually referred to as the Chern–Simons term.

The C-field equation of motion is

$$\mathrm{d}^*G + \frac{1}{2}\, G \wedge G = 0\ , \tag{4.2}$$

or in components

$$\nabla_\mu G^{\mu\nu\rho\sigma} + \frac{1}{2\cdot 4!\cdot 4!}\ \epsilon^{\nu\rho\sigma\mu_1\ldots\mu_8}\, G_{\mu_1\ldots\mu_4} G_{\mu_5\ldots\mu_8} = 0\ , \tag{4.3}$$

and the Einstein equations are

$$\mathcal{R}_{\mu\nu} = \frac{1}{12}\Big(G_{\mu\rho\lambda\sigma}\, G_\nu^{\ \rho\lambda\sigma} - \frac{1}{12}\, g_{\mu\nu}\, G_{\rho\lambda\sigma\kappa}\, G^{\rho\lambda\sigma\kappa}\Big)\ . \tag{4.4}$$

Just as superstring theory possesses various D-branes, M-theory has two fundamental branes: membranes (2-branes) and 5-branes. Also, the low energy-effective action (4.1) certainly does receive higher-order corrections. Note that in eleven-dimensional supergravity there is no parameter besides the gravitational constant κ, and higher order necessarily means higher order in κ. The first such term is the famous Green–Schwarz term, initially inferred from considerations of anomaly cancellation on 5-branes by inflow [9,24]. It reads (in Minkowski space)

$$S_{\rm GS} = -\epsilon\, \frac{T_2}{2\pi} \int C \wedge X_8 = -\epsilon\, \frac{T_2}{2\pi} \int G \wedge X_7 \tag{4.5}$$

where we assumed that one can freely integrate by parts (no boundaries or singularities), and where

$$X_8 = \mathrm{d}X_7 = \frac{1}{(2\pi)^3\, 4!}\left(\frac{1}{8}\, \mathrm{tr}\, R^4 - \frac{1}{32} (\,\mathrm{tr}\, R^2)^2 \right) . \tag{4.6}$$

Here T_2 is shorthand for

$$T_2 = \left(\frac{2\pi^2}{\kappa^2} \right)^{1/3} \tag{4.7}$$

and is interpreted as the membrane tension. The parameter ϵ can be fixed by various considerations of anomaly cancellation as we will show below. Since there have been some ambiguities in the literature we will keep ϵ as a parameter and show that all anomalies considered below cancel if and only if

$$\epsilon = +1\ . \tag{4.8}$$

Note that adding the Green–Schwarz term to the action (4.1) modifies the equations of motion (4.2)-(4.4) by terms of order $\kappa^{4/3}$ which will be neglected below when looking for solutions of the "classical" equations of motion.

The Euclidean continuations of the action (4.1) and the Green–Schwarz term are

$$S_{\rm E}^{\rm CJS} = \frac{1}{2\kappa^2}\left(-\int {\rm d}^{11}z\,\sqrt{g}\,\mathcal{R}_{\rm E} + \frac{1}{2}\int G_{\rm E}\wedge {}^*G_{\rm E} + \frac{i}{6}\int C_{\rm E}\wedge G_{\rm E}\wedge G_{\rm E}\right) \tag{4.9}$$

and

$$S_{\rm GS}^{\rm E} = i\,\epsilon\,\frac{T_2}{2\pi}\int C_{\rm E}\wedge X_8^{\rm E} = i\,\epsilon\,\frac{T_2}{2\pi}\int G_{\rm E}\wedge X_7^{\rm E}\,. \tag{4.10}$$

4.2. *The M-Theory Five-Brane*

The 5-brane and anti-5-brane are solutions of 11-dimensional supergravity that preserve half of the 32 supersymmetries. The metric is a warped metric preserving Poincaré invariance on the $(5+1)$-dimensional world-volume (for flat 5-branes) and the 4-form G has a non-vanishing flux through any 4-sphere surrounding the world-volume. This is why the 5-branes are called "magnetic" sources. It will be enough for us to exhibit the bosonic fields only.

Although the original 11-dimensional supergravity is non-chiral, the 5-brane is a chiral solution; it carries a chiral $(5+1)$-dimensional supermultiplet which gives rise to anomalies. Of course, the anti-5-brane carries the supermultiplet of opposite chirality. As a result, when computing an "M-theory functional integral" one has to sum over classical solutions of opposite chirality and the overall result is correctly non-chiral. However, we like to adopt a more modest view and consider M-theory in a given background with some number of 5-branes somewhere and some other number of anti-5-branes somewhere else. Then the anomalies cannot cancel between the different branes and anomaly cancellation must occur for each 5-brane or anti-5-brane separately. This will be achieved by anomaly inflow from the two topological terms, the Chern–Simons and the Green–Schwarz term.

It is not too difficult to determine the nature of the chiral 6-dimensional supermultiplet living on the world-volume of a 5-brane [25]. What requires some more care is to correctly determine its chirality. We will see that the 5-brane acts as a "magnetic" source for the C-field leading to a modification of the Bianchi identity ${\rm d}G=0$. This is at the origin of anomaly inflow from the Green–Schwarz term [9] and similar to the mechanism outlined in Section 2.4 for the magnetic string. However, it was noticed [6] that there is a left-over "normal bundle" anomaly which is only canceled by further inflow from the (slightly modified) Chern–Simons term [10]. In principal, this should have fixed the coefficient ϵ of the Green–Schwarz term. In the literature one can find about as many times $\epsilon = +1$ as $\epsilon = -1$ (after eliminating the effect of using different conventions). This was the motivation in [4] to redo the whole computation from first principles. Here we will outline this computation again, with the result $\epsilon = +1$.

4.2.1. *The classical 5-brane solution*

We work in Minkowski space and split the coordinates into longitudinal ones $x^\alpha,\ \alpha = 0, \dots 5$ and transverse ones $x^m \equiv y^m,\ m = 6, \dots 10$. Then the metric is

$$\mathrm{d}s^2 = \Delta(r)^{-1/3}\,\eta_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta + \Delta(r)^{2/3}\,\delta_{mn}\mathrm{d}y^m\mathrm{d}y^n\,, \tag{4.11}$$

where

$$\Delta(r) = 1 + \frac{r_0^3}{r^3}\,, \qquad r = (\delta_{mn}y^m y^n)^{1/2}\,, \qquad r_0 \geq 0\,, \tag{4.12}$$

(with $\eta_{\alpha\beta} = \mathrm{diag}(-1, 1, \dots 1)$). From this one has to compute the Ricci tensor and finds that Einstein's equations (4.4) are solved by

$$G_{mnpq} = \pm 3\,\frac{r_0^3}{r^5}\,\widetilde{\epsilon}_{mnpqs}\,y^s\,, \qquad \text{all other } G_{\mu\nu\rho\sigma} = 0\,. \tag{4.13}$$

The other equation of motion (4.3) reduces to $\partial_m\left(\sqrt{|g|}\;G^{mnpq}\right) = 0$, which is automatically satisfied. The solution with the upper sign (+) is called a 5-brane and the one with the lower sign (−) an anti-5-brane. Details are given e.g. in [4], where one can also find a discussion of how things change under a rescaling of the C-field. The 4-form corresponding to (4.13) is

$$G = \pm\,\frac{r_0^3}{8}\,\widetilde{\epsilon}_{mnpqs}\,\frac{y^s}{r^5}\,\mathrm{d}y^m \wedge \mathrm{d}y^n \wedge \mathrm{d}y^p \wedge \mathrm{d}y^q \tag{4.14}$$

and for any 4-sphere in the transverse space surrounding the world-volume we have the "magnetic charge"

$$\int_{S^4} G \;=\; \pm\, 3r_0^3\mathrm{vol}(S^4) \;=\; \pm\, 8\pi^2 r_0^3\,. \tag{4.15}$$

Hence, for the 5-brane the flux of G is positive and for the anti-5-brane it is negative.

The parameter r_0 sets the scale for the (anti-) 5-brane solution. One can compute the energy per 5-volume of the brane, i.e. the 5-brane tension T_5 as a function of r_0. Using the Dirac quantization condition between membranes and 5-branes then relates the membrane tension T_2 and the 5-brane tension T_5 as $T_2 T_5 = \frac{2\pi}{2\kappa^2}$ so that in the end $8\pi^2 r_0^3 = \frac{2\pi}{T_2}$, see [4] for details. (Recall from (4.7) that $T_2 = (2\pi^2/\kappa^2)^{1/3}$.) It follows that Eq. (4.15) can be rewritten as

$$\int_{S^4} G \ = \ \pm \ \frac{2\pi}{T_2} = \pm \ (4\pi\kappa^2)^{1/3}. \tag{4.16}$$

This is equivalent to the modified Bianchi identity

$$\mathrm{d}G = \ \pm \ \frac{2\pi}{T_2} \delta_{W_6}^{(5)} \ = \ \pm \ (4\pi\kappa^2)^{1/3} \ \delta_{W_6}^{(5)} \tag{4.17}$$

where again the upper sign $(+)$ applies for a 5-brane and the lower sign $(-)$ for an anti-5-brane. $\delta_{W_6}^{(5)}$ is a 5-form Dirac distribution with support on the world-volume W_6 such that $\int_{M_{11}} \omega_{(6)} \wedge \delta_{W_6}^{(5)} = \int_{W_6} \omega_{(6)}$.

To summarize, the 5-brane and anti-5-brane solutions both have a metric given by (4.11). The 4-form G is given by (4.14) and satisfies the Bianchi identity (4.17). The upper sign always corresponds to 5-branes and the lower sign to anti-5-branes.

4.2.2. *The zero-modes*

The (massless) fields that live on a five-brane are the zero-modes of the equations of motion in the background of the 5-brane solution. Hence, to determine them, we will consider the zero-modes of the bosonic equations of motion in this 5-brane background. The fermionic zero-modes then are simply inferred from the completion of the supermultiplet. The anti-5-brane background can be treated similarly (flipping signs in appropriate places).

Apart from fluctuations describing the position of the 5-brane, there are zero-modes of the C-field. A zero-mode is a square-integrable fluctuation $\delta G = \mathrm{d}\delta C$ around the 5-brane solution G_0 (given by (4.13) or (4.14) with the upper sign) such that $G = G_0 + \delta G$ still is a solution of (4.3) or (4.2). Of course, G must also solve the Einstein equations to first order in δG. This will be the case with the same metric if the r.h.s. of (4.4) has no term linear in δG.

The linearization of Eq. (4.3) around the 5-brane solution (4.13) is

$$\nabla_\mu \delta G^{\mu\nu\rho\sigma} + \frac{1}{4!\,4!} \frac{3\,r_0^3}{r^5} \ \epsilon^{\nu\rho\sigma\mu_1\ldots\mu_4 mnpq} \ \widetilde{\epsilon}_{mnpqs}\, y^s \,\delta G_{\mu_1\ldots\mu_4} = 0\,. \tag{4.18}$$

Since there are only 5 transverse directions, the second term is non-vanishing only if exactly one of the indices $\nu\rho\sigma\mu_1\ldots\mu_4$ is transverse. It is not too difficult to see that the only solutions are such that all components of δG but $\delta G_{m\alpha\beta\gamma}$ vanish. This also ensures that δG cannot contribute linearly to the Einstein equations. We take the ansatz [25]

$$\delta G_{m\alpha\beta\gamma} = \Delta(r)^{-1-\zeta}\, r^{-5}\, y^m\, H_{\alpha\beta\gamma}\,, \quad \text{with } \partial_n H_{\alpha\beta\gamma} = 0\,, \tag{4.19}$$

and use $\sqrt{|g|} = \Delta(r)^{2/3}$, $g^{mn} = \Delta(r)^{-2/3}\,\delta^{mn}$, $g^{\alpha\beta} = \Delta(r)^{1/3}\,\eta^{\alpha\beta}$, as well as the convention that indices of $H_{\alpha\beta\gamma}$ are raised with $\eta^{\alpha\beta}$ and those of $\delta G_{m\alpha\beta\gamma}$ with g^{mn} and $g^{\alpha\beta}$. This means that $\delta G^{m\alpha\beta\gamma} = \Delta(r)^{-2/3-\zeta}\, r^{-5}\, y^m\, H^{\alpha\beta\gamma}$. We further need

$$\epsilon^{\alpha\beta\gamma t\delta\epsilon\varphi mnpq}\,\widetilde{\epsilon}_{mnpqs} = -\frac{4!}{\sqrt{|g|}}\,\delta^t_s\,\widetilde{\epsilon}^{\alpha\beta\gamma\delta\epsilon\varphi}\,, \tag{4.20}$$

with $\widetilde{\epsilon}^{\alpha\beta\gamma\delta\epsilon\varphi}$ completely antisymmetric and $\widetilde{\epsilon}^{012345} = -1$, i.e. $\widetilde{\epsilon}$ is exactly the ϵ-tensor (as defined in (2.2)) for the $(5+1)$-dimensional world-volume with metric $\eta_{\alpha\beta}$. Then, for $(\nu,\rho,\sigma) = (\alpha,\beta,\gamma)$, Eq. (4.18) becomes [g]

$$\partial_m\left(\Delta(r)^{-\zeta}\, r^{-5}\, y^m\right) H^{\alpha\beta\gamma} - \frac{r_0^3}{2}\,\widetilde{\epsilon}^{\alpha\beta\gamma\delta\epsilon\varphi}\Delta(r)^{-1-\zeta}\, r^{-8}\, H_{\delta\epsilon\varphi} = 0\,. \tag{4.21}$$

Since $\partial_m\left(\Delta(r)^{-\zeta}\, r^{-5}\, y^m\right) = +3\,\zeta\,\Delta(r)^{-\zeta-1}\, r_0^3\, r^{-8}$ we finally get

$$\zeta\; H^{\alpha\beta\gamma} = \frac{1}{6}\,\widetilde{\epsilon}^{\alpha\beta\gamma\delta\epsilon\varphi}\; H_{\delta\epsilon\varphi}\,. \tag{4.22}$$

Consistency of this equation requires either $\zeta = +1$ in which case H is self-dual (cf (2.25)) or $\zeta = -1$ in which case H is anti-self-dual.

As mentioned above, the zero-modes must be square-integrable,

$$\begin{aligned} \infty > &\int \mathrm{d}^{11}x\sqrt{|g|}\,\delta G_{m\alpha\beta\gamma}\,\delta G^{m\alpha\beta\gamma} \\ &= \frac{8\pi^2}{3}\int_0^\infty \mathrm{d}r\, r^{-4}\Delta(r)^{-1-2\zeta}\int_{W_6}\mathrm{d}^6x\, H_{\alpha\beta\gamma}\, H^{\alpha\beta\gamma}\,. \end{aligned} \tag{4.23}$$

The r-integral converges if and only if $\zeta > 0$. Thus square-integrability selects $\zeta = +1$ and, hence, $H = \mathrm{d}B$ is a self-dual 3-form on the world-volume.

To summarize, in Minkowski signature, on a 5-brane, there is a self-dual 3-form H (which continues to an anti-self-dual Euclidean 3-form H_{E}), while on an anti-5-brane the 3-form H is anti-self-dual (and continues to a self-dual Euclidean 3-form H_{E}). To complete the 6-dimensional supermultiplets, we know that the self-dual 3-form is accompanied by two spinors of positive

[g] For $(\nu,\rho,\sigma) = (m,\beta,\gamma)$ Eq. (4.18) gives $\partial_\alpha H^{\alpha\beta\gamma} = 0$, so that $H_{\alpha\beta\gamma} = 3\,\partial_{[\alpha}B_{\beta\gamma]}$, as expected.

chirality, and the anti-self-dual 3-form by two spinors of negative chirality. We note that the same discussion can be equally well carried out entirely in the Euclidean case (see [4]), with the same result, of course.

4.2.3. *The tangent and normal bundle anomalies*

Now that we have determined the nature and chiralities of the fields living on the 5-brane world-volume, it is easy to determine the one-loop anomaly, using the results of Section 2.3. For the Euclidean 5-brane we have an anti-self-dual 3-form and two negative chirality spinors. While the 3-form cannot couple to gauge fields, the spinors couple to the "$SO(5)$-gauge" fields of the normal bundle. This coupling occurs via

$$D_i = \partial_i + \frac{1}{4}\,\omega_{ab,i}\gamma^{ab} + \frac{1}{4}\,\omega_{pq,i}\gamma^{pq} \tag{4.24}$$

inherited from the eleven-dimensional spinor. Here a, b and i run from 1 to 6, while $p, q = 7, \dots 11$. Thus $\omega_{pq,i}$ behaves as an $SO(5)$-gauge field A_i^α with generators $\lambda^\alpha \sim \frac{1}{2}\gamma^{pq}$. We see that the relevant $SO(5)$ representation is the spin representation [6] and hence ($R_{pq} = d\omega_{pq} + \omega_{pr}\omega_{rq} \equiv R^\perp_{pq}$)

$$F = F^\alpha\lambda^\alpha \longleftrightarrow \frac{1}{4}R^\perp_{pq}\gamma^{pq} \tag{4.25}$$

$$\mathrm{ch}(F) \longleftrightarrow \mathrm{tr}\,\exp\left(\frac{i}{2\pi}\frac{1}{4}R^\perp_{pq}\gamma^{pq}\right) \equiv \mathrm{ch}(S(N))\,. \tag{4.26}$$

This trace appeared already in (2.32), except that there R_{ab} was the curvature on the manifold (i.e. on the tangent bundle). One has

$$\mathrm{ch}(S(N)) = 4\left[1 - \frac{1}{(4\pi)^2}\,\frac{1}{4}\,\mathrm{tr}\,R_\perp^2 + \frac{1}{(4\pi)^4}\left[-\frac{1}{24}\,\mathrm{tr}\,R_\perp^4 + \frac{1}{32}\,(\mathrm{tr}\,R_\perp^2)^2\right] + \dots\right]. \tag{4.27}$$

The relevant anomaly polynomial includes an extra factor $\frac{1}{2}$ from a chirality projector (as in (2.32)) as well as a minus sign for negative chirality. It is ($R = \tilde{R} + R_\perp$)

$$\begin{aligned}\left[-\frac{1}{2}\,\hat{A}(M_6)\,\mathrm{ch}(S(N))\right]_8 = -\frac{2}{(4\pi)^4}\bigg[&\frac{1}{360}\,\mathrm{tr}\,\tilde{R}^4 + \frac{1}{288}(\,\mathrm{tr}\,\tilde{R}^2)^2 \\ &-\frac{1}{24}\,\mathrm{tr}\,R_\perp^4 + \frac{1}{32}(\,\mathrm{tr}\,R_\perp^2)^2 - \frac{1}{48}\,\mathrm{tr}\,\tilde{R}^2\,\mathrm{tr}\,R_\perp^2\bigg]\,.\end{aligned} \tag{4.28}$$

The part not involving $R_\perp$ is just $-2[\hat{A}(M_6)]_8$ and can be interpreted as the contribution to the tangent bundle anomaly of the two negative chirality spinors on M_6. Adding the contribution of the anti-self-dual three-form, which is $\left[-\left(-\frac{1}{8}\right) L(M_6)\right]_8$ (evaluated using $\tilde{R}$) we get the anomaly on the Euclidean 5-brane as $\delta\Gamma_E = -i \int \hat{I}_6^{1,5-brane}$ with

$$\begin{aligned} \hat{I}_8^{5-brane} &= 2\pi \left[-\frac{1}{2} \hat{A}(M_6) \, \mathrm{ch}(S(N)) + \frac{1}{8} L(M_6) \right]_8 \\ &= -X_8(\tilde{R}) - \hat{I}_8^{normal} , \end{aligned} \tag{4.29}$$

where X_8 is given in (4.6) (now with $R \to \tilde{R}$) and

$$\hat{I}_8^{normal} = \frac{1}{(2\pi)^3 4!} \left[-\frac{1}{8} \operatorname{tr} R_\perp^4 + \frac{3}{32} (\operatorname{tr} R_\perp^2)^2 - \frac{1}{16} \operatorname{tr} \tilde{R}^2 \operatorname{tr} R_\perp^2 \right] . \tag{4.30}$$

The part $-X_8(\tilde{R})$ is called the tangent bundle anomaly and $-\hat{I}_8^{normal}$ the normal bundle anomaly.

4.2.4. *Anomaly inflow from the Green–Schwarz and Chern–Simons terms*

In this subsection we return to Minkowski space. As we have seen, the 5-brane has chiral zero-modes on its 6-dimensional world-volume with its Minkowski anomaly given by

$$\delta\Gamma_M^{1-loop} = \int_{W_6} \hat{I}_6^{1,5-brane} , \tag{4.31}$$

where $\hat{I}_6^{1,5-brane}$ is the descent of $\hat{I}_8^{5-brane}$ given in (4.29) and $I_8^{5-brane} = -X_8(\tilde{R}) - \hat{I}_8^{normal}$. The tangent bundle anomaly $-X_8(\tilde{R})$ is cancelled [9] through inflow from the Green–Schwarz term $\sim \int G \wedge X_7(R)$. The latter, however, gives $X_8(R) = X_8(\tilde{R} + R_\perp)$, not $X_8(\tilde{R})$. The difference, as well as the normal bundle anomaly is cancelled through inflow from the Chern–Simons term as was shown in [6, 10]. As a result, cancellation of the total 5-brane anomaly fixes both coefficients of the Green–Schwarz and Chern–Simons terms. In particular, it establishes a correlation between the two coefficients. Moreover, as we will see, cancellation can only occur if the sign of the anomaly due to the five-brane zero-modes is exactly as in (4.31), (4.29).

Let us first consider the simpler inflow from the Green–Schwarz term (4.5) in the form $S_{GS} = -\epsilon \frac{T_2}{2\pi} \int G \wedge X_7$. Using the Bianchi identity (4.17) we get

$$\delta S_{GS} = -\epsilon \frac{T_2}{2\pi} \int G \wedge \delta X_7 = -\epsilon \frac{T_2}{2\pi} \int G \wedge \mathrm{d}X_6^1$$
$$= \epsilon \frac{T_2}{2\pi} \int \mathrm{d}G \wedge X_6^1 = \epsilon \int \delta_{W_6}^{(5)} \wedge X_6^1 = \epsilon \int_{W_6} X_6^1 \ , \tag{4.32}$$

where, as already noted, X_6^1 is $X_6^1(R)$. This corresponds via descent to an invariant polynomial

$$\hat{I}_8^{GS} = \epsilon \ X_8(R). \tag{4.33}$$

Next, inflow from the Chern–Simons term is more subtle. We review the computation of [10], again paying particular attention to issues of signs and orientation. The two key points in [10] are: (i) the regularization

$$\delta_{W_6}^{(5)} \to \mathrm{d}\rho \wedge \frac{e_4}{2} \tag{4.34}$$

where $\rho(r)$ rises monotonically from -1 at $r=0$ to 0 at some finite distance $\tilde{r}$ from the 5-brane, and $e_4 = \mathrm{d}e_3$ is a certain angular form with $\int_{S^4} \frac{e_4}{2} = 1$; and (ii) a modification of the Chern–Simons term close to the 5-brane, where $G \neq \mathrm{d}C$.

The regularized Bianchi identity reads

$$\mathrm{d}G = \frac{2\pi}{T_2} \mathrm{d}\rho \wedge \frac{e_4}{2} \tag{4.35}$$

which is solved by (requiring regularity at $r=0$ where e_4 is singular)

$$G = \mathrm{d}C + \frac{\pi}{T_2}(2\mathrm{d}\rho \wedge \mathrm{d}B - \mathrm{d}\rho \wedge e_3)$$
$$= \frac{\pi}{T_2}\rho\, e_4 + \mathrm{d}\left(C - \frac{\pi}{T_2}(\rho\, e_3 + 2\mathrm{d}\rho \wedge B)\right)$$
$$\equiv \frac{\pi}{T_2}\rho\, e_4 + \mathrm{d}\,\widetilde{C} \ . \tag{4.36}$$

Under a local Lorentz transformation, $\delta e_3 = \mathrm{d}e_2^1$, and G is invariant if $\delta C = 0$ and $\delta B = \frac{1}{2}\, e_2^1$. Note that [10] include the $\mathrm{d}\rho \wedge B$-term in C and hence get a non-trivial transformation for C. If we let $\widetilde{G} = \mathrm{d}\widetilde{C}$ then the modified Chern–Simons term is

$$\widetilde{S}_{CS} = -\frac{1}{12\kappa^2} \lim_{\epsilon \to 0} \int_{M_{11} \backslash D_\epsilon W_6} \widetilde{C} \wedge \widetilde{G} \wedge \widetilde{G} \, , \tag{4.37}$$

where $M_{11} \backslash D_\epsilon W_6$ is M_{11} with a small "tubular" region of radius ϵ around the 5-brane world-volume cut out. (Of course, this radius ϵ should not be confused with the ϵ which is the coefficient of the Green–Schwarz term.) Its boundary is

$$\partial(M_{11} \backslash D_\epsilon W_6) = -S_\epsilon W_6 \tag{4.38}$$

where $S_\epsilon W_6$ is the 4-sphere bundle over W_6. Note the minus sign that appears since the orientation of the boundary is opposite to that of the sphere bundle.

Under a local Lorentz transformation G and hence $\widetilde{G}$ are invariant and

$$\delta\widetilde{C} = -\frac{\pi}{T_2}\,\mathrm{d}(\rho\, e_2^1)\,. \tag{4.39}$$

Inserting this variation into (4.37), and using $\mathrm{d}\widetilde{G} = 0$ one picks up a boundary contribution[h]

$$\delta\widetilde{S}_{CS} = -\frac{\pi}{12\kappa^2\,T_2}\,\lim_{\epsilon\to 0}\int_{S_\epsilon W_6} \rho e_2^1 \wedge \widetilde{G} \wedge \widetilde{G}\,. \tag{4.40}$$

In $\widetilde{G} = \mathrm{d}C - \frac{\pi}{T_2}(\mathrm{d}\rho \wedge e_3 + \rho\, e_4 - 2\mathrm{d}\rho \wedge \mathrm{d}B)$ the terms $\sim \mathrm{d}\rho$ cannot contribute to an integral over $S_\epsilon W_6$. Also the contribution of the $\mathrm{d}C$-terms vanishes in the limit $\epsilon \to 0$. Hence the only contribution comes from [10, 26]

$$\int_{S_\epsilon W_6} e_2^1 \wedge e_4 \wedge e_4 = 2\int_{W_6} p_2(NW_6)^1\,, \tag{4.41}$$

where $p_2(NW_6)^1$ is related via descent to the second Pontrjagin class $p_2(NW_6)$ of the normal bundle given below. Using $\rho(0) = -1$ and (4.7) we arrive at

$$\delta\widetilde{S}_{CS} = \frac{1}{6\kappa^2}\left(\frac{\pi}{T_2}\right)^3 \int_{W_6} p_2(NW_6)^1 = \frac{\pi}{12}\int_{W_6} p_2(NW_6)^1\,. \tag{4.42}$$

This corresponds to an invariant polynomial

$$\hat{I}_8^{CS} = \frac{\pi}{12}\,p_2(NW_6)\,. \tag{4.43}$$

[h] We get three minus signs, one from (4.37), (4.38) and (4.39) each. Apparently the one from (4.38) was overlooked in [10].

Using

$$\frac{\pi}{12}\, p_2(NW_6) = \frac{1}{(2\pi)^3 4!}\left(-\frac{1}{4}\,\mathrm{tr}\, R_\perp^4 + \frac{1}{8}(\,\mathrm{tr}\, R_\perp^2)^2\right)$$
$$X_8(R) = X_8(\widetilde{R}) + \frac{1}{(2\pi)^3 4!}\left(\frac{1}{8}\,\mathrm{tr}\, R_\perp^4 - \frac{1}{32}(\,\mathrm{tr}\, R_\perp^2)^2 - \frac{1}{16}\,\mathrm{tr}\, \widetilde{R}^2\,\mathrm{tr}\, R_\perp^2\right) \tag{4.44}$$

we find that the total inflow corresponds to

$$\hat{I}_8^{GS} + \hat{I}_8^{CS} = \epsilon\, X_8(\widetilde{R}) + \frac{1}{(2\pi)^3 4!}\Bigg[\left(\frac{\epsilon}{8} - \frac{1}{4}\right)\,\mathrm{tr}\, R_\perp^4 + \left(\frac{1}{8} - \frac{\epsilon}{32}\right)(\,\mathrm{tr}\, R_\perp^2)^2 - \frac{\epsilon}{16}\,\mathrm{tr}\, \widetilde{R}^2\,\mathrm{tr}\, R_\perp^2\Bigg]. \tag{4.45}$$

Now it is easy to study anomaly cancellation. Invariance of the full quantum effective action requires that the sum of (4.29) and (4.45) vanishes. This gives four equations

$$\epsilon = 1\,, \qquad \left(\frac{\epsilon}{8} - \frac{1}{4}\right) = -\frac{1}{8}\,,$$
$$\left(\frac{1}{8} - \frac{\epsilon}{32}\right) = \frac{3}{32}\,, \qquad -\frac{\epsilon}{16} = -\frac{1}{16}\;. \tag{4.46}$$

The first equation ensures the cancellation of the tangent bundle anomaly and the three other equations ensure the cancellation of the normal bundle anomaly. All four equations are solved by

$$\epsilon = +1\,. \tag{4.47}$$

It is quite amazing to see that anomaly cancellation requires four different terms to vanish, and they all do if the single coefficient ϵ is chosen as above. Note also that a rescaling of the C-field changes the coefficients of the Chern–Simons and Green–Schwarz terms, but cannot change the *relative* sign between them. The effect of such rescalings has been carefully traced through the computations in ref [4] where it can be seen that the resulting equations (4.46) are indeed invariant under these rescalings, as they should.

It is also interesting to note that the four conditions (4.46) for anomaly cancellation have enough structure to provide a check that we correctly computed the sign of the one-loop anomaly (if we believe that the anomaly must cancel). Suppose we replaced equation (4.29) by

$$\hat{I}_8^{5-brane}(\eta) = -\eta\,[X_8(\widetilde{R}) + \hat{I}_8^{normal}]\,, \qquad \eta = \pm 1\,. \tag{4.48}$$

Then equations (4.46) would get an extra factor $\eta = \pm 1$ on their right-hand sides. However, the four equations are enough to uniquely determine both $\epsilon = +1$ and $\eta = +1$. Said differently, a one-loop anomaly of opposite sign could not be cancelled through inflow from the Chern–Simons or Green–Schwarz terms even with their signs flipped. At first sight this might seem surprising. However, as we have seen, such a sign flip merely corresponds to a redefinition of the fields and obviously cannot yield a different inflow.

4.3. *M-Theory on S^1/Z_2: the Strongly-Coupled Heterotic String*

While compactification of M-theory on a circle S^1 leads to (strongly-coupled) type IIA superstring theory, compactification on an interval gives the strongly-coupled heterotic string [5]. There are two ways to view this latter compactification. On the one hand, one considers the compactification manifold as being ten-dimensional Minkowski space M_{10} times the interval so that the 11-dimensional space-time has two boundaries, each of which is a copy of M_{10}. This is called the "downstairs approach". On the other hand, the interval being $S^1/\mathbf{Z}_2$, one may start with the 11-dimensional manifold being $M_{10} \times S^1$ and then perform the $\mathbf{Z}_2$ orbifold projection. In this case there are no boundaries, but two orbifold fixed-planes, each of which is again a copy of M_{10}. This is called the "upstairs approach".

One may also consider more complicated compactifications on orbifolds like e.g. $T^5/\mathbf{Z}_2$ with many intersecting orbifold planes. The latter constructions have given rise to some model building, see e.g. [27].

Here we will work in the upstairs approach. As argued in [5] the orbifold projection eliminated half of the supersymmetry leaving only one chiral (ten-dimensional) gravitino on each of the ten-dimensional orbifold planes. This leads to a gravitational anomaly with an irreducible R^6 piece. The latter piece can be cancelled by adding E_8 gauge fields on each of the orbifold planes (interpreted as "twisted" matter). The total one-loop anomaly then no longer has this R^6 piece and, remarkably, has a factorized form on each of the planes, a necessary condition for anomaly cancellation by inflow from the Green–Schwarz and Chern–Simons terms. There has been a long series of papers discussing this cancellation that culminated with Ref. [28], each paper correcting some errors of the preceding ones. However, this was not the end of the story, since one of the authors of [28] realized that there was still an unnoticed numerical error, and to correctly obtain complete anomaly cancellation requires a slight modification of the Chern–Simons term in the vicinity of the orbifold planes, quite similar to what happened for the 5-brane as discussed above. This was reported in [4] and we will review these results in this subsection. The attitude taken in [4] was to show that anomaly cancellation in this case again determines the value of ϵ to be $+1$. Here, instead, we will consider that the coefficient of the Green–Schwarz-term is already fixed from the 5-brane anomaly cancellation and that with this value we correctly obtain anomaly cancellation also in the present case.

4.3.1. *The one-loop anomalies on the orbifold 10-planes*

As always in Minkowski signature, we label the coordinates as x^μ, $\mu = 0, \ldots 10$. Here we will distinguish the circle coordinate $x^{10} \in [-\pi r_0, \pi r_0]$ from the other $x^{\bar\mu}$, $\bar\mu = 0, \ldots 9$. The $\mathbf{Z}_2$-projection then acts as $x^{10} \to -x^{10}$. As one can see from the Chern–Simons term, $C_{\bar\mu\bar\nu\bar\rho}$ is $\mathbf{Z}_2$-odd and $C_{\bar\mu\bar\nu 10}$ is $\mathbf{Z}_2$-even ($\bar\mu, \bar\nu, \bar\rho = 0, \ldots 9$). The projection on $\mathbf{Z}_2$-even fields then implies e.g. that

$$C = \tilde{B} \wedge \mathrm{d}x^{10} \ , \tag{4.49}$$

and all other components of C projected out. Also, this $\mathbf{Z}_2$-projection only leaves half of the components of the eleven-dimensional gravitino [5]. What remains is a ten-dimensional gravitino of positive chirality (in Minkowskian space), together with one negative chirality spin-$\frac{1}{2}$ field. Of course, in Euclidean space, this corresponds to one negative chirality spin-$\frac{3}{2}$ and a positive chirality spin-$\frac{1}{2}$ fermion. The 1-loop anomaly due to the eleven-dimensional gravitino on each 10-plane M_{10}^A, $A = 1, 2$ is thus given by

$$\hat{I}_{12,A}^{gravitino} = \frac{1}{2} \cdot \frac{1}{2} \left(-\hat{I}_{12}^{spin\frac{3}{2}}(R_A) + I_{12}^{spin\frac{1}{2}}(R_A) \right) \ , \tag{4.50}$$

where one factor $\frac{1}{2}$ is due to the Majorana condition and the other factor $\frac{1}{2}$ due to the "splitting" of the anomaly between the two fixed planes [5]. R_A denotes the curvature two-form on M_{10}^A which simply is the eleven-dimensional curvature R with its components tangent to S^1 suppressed. As is well known, such a polynomial has a $\operatorname{tr} R^6$-piece, and one must add an E_8 vector multiplet in the adjoint representation ($\mathrm{Tr}\, \mathbf{1} = 248$) with positive chirality (Minkowskian) Majorana spinors on each 10-plane. Then on each plane M_{10}^A one has a 1-loop anomaly corresponding to

$$\begin{aligned} \hat{I}_{12,A} &= \frac{1}{4} \left(-\hat{I}_{12}^{spin\frac{3}{2}}(R_A) + I_{12}^{spin\frac{1}{2}}(R_A) \right) - \frac{1}{2} \hat{I}_{12}^{spin\frac{1}{2}}(R_A, F_A) \\ &= I_{4,A} \left[X_8(R_A) + \frac{\pi}{3} I_{4,A}^2 \right] , \end{aligned} \tag{4.51}$$

where we used $\mathrm{Tr}\, F_A^4 = \frac{1}{100}(\mathrm{Tr}\, F_A^2)^2$, $\mathrm{Tr}\, F_A^6 = \frac{1}{7200}(\mathrm{Tr}\, F_A^2)^3$ and defined [i]

$$I_{4,A} = \frac{1}{(4\pi)^2}\left(\frac{1}{30}\,\mathrm{Tr}\, F_A^2 - \frac{1}{2}\,\mathrm{tr}\, R_A^2\right) \equiv \frac{1}{(4\pi)^2}\left(\mathrm{tr}\, F_A^2 - \frac{1}{2}\,\mathrm{tr}\, R_A^2\right). \quad (4.52)$$

Note that in the small radius limit with $R_1 = R_2 = R$ one has

$$\begin{aligned}\left[\hat I_{12,1} + \hat I_{12,2}\right]\Big|_{R_1=R_2=R} &= (I_{4,1} + I_{4,2})\left[X_8(R) + \frac{\pi}{3}\left(I_{4,1}^2 + I_{4,2}^2 - I_{4,1}I_{4,2}\right)\right] \\ &\equiv (I_{4,1} + I_{4,2})\ \widehat X_8(R, F_1, F_2)\,, \end{aligned} \quad (4.53)$$

thanks to the algebraic identity $a^3 + b^3 = (a+b)(a^2+b^2-ab)$. Here $\widehat X_8$ is the relevant 8-form that appears in the anomaly-cancelling term of the heterotic string,

$$\begin{aligned}\widehat X_8(R, F_1, F_2) = \frac{1}{(2\pi)^3 4!}\Bigg(&\frac{1}{8}\,\mathrm{tr}\, R^4 + \frac{1}{32}(\,\mathrm{tr}\, R^2)^2 - \frac{1}{8}\,\mathrm{tr}\, R^2(\,\mathrm{tr}\, F_1^2 + \,\mathrm{tr}\, F_2^2) \\ &+ \frac{1}{4}(\,\mathrm{tr}\, F_1^2)^2 + \frac{1}{4}(\,\mathrm{tr}\, F_2^2)^2 - \frac{1}{4}\,\mathrm{tr}\, F_1^2\,\mathrm{tr}\, F_2^2\Bigg). \end{aligned} \quad (4.54)$$

4.3.2. *Anomaly inflow and anomaly cancellation*

To begin with, there is a slight subtlety concerning the coefficients of the Chern–Simons and Green–Schwarz terms in the upstairs formalism. To see this, we start in the downstairs formalism where S_{CS} and S_{GS} are given by integrals over an honest manifold with boundary which is M_{10} times the interval $I = S^1/\mathbf{Z}_2$. Then clearly the coefficients must be those given in the preceding subsections,

$$S_{CS} = -\frac{1}{12\kappa^2}\int_{M_{10}\times I} C\wedge G\wedge G\,, \qquad S_{GS} = -\frac{1}{(4\pi\kappa^2)^{1/3}}\int_{M_{10}\times I} G\wedge X_7\,. \quad (4.55)$$

Here κ is the eleven-dimensional κ as before. This can be rewritten in the upstairs formalism by replacing $\int_I \ldots = \frac{1}{2}\int_{S^1}\ldots$ and appropriately identifying the fields so that the integrand is $\mathbf{Z}_2$-even. This introduces an extra $\frac{1}{2}$ in the coefficients. It is nevertheless customary to absorb this $\frac{1}{2}$ in a redefinition of κ as

$$\kappa_{\mathrm{U}}^2 = 2\kappa^2 \equiv 2\kappa_{\mathrm{D}}^2\,. \quad (4.56)$$

[i] $I_{4,A}$ is exactly what was called $\tilde I_{4,i}$ in [28].

Then one has

$$S_{CS} = -\frac{1}{12\kappa_{\mathrm{U}}^2}\int_{M_{10}\times S^1} C\wedge G\wedge G\,,$$
$$S_{GS} = -\frac{1}{2^{2/3}(4\pi\kappa_{\mathrm{U}}^2)^{1/3}}\int_{M_{10}\times S^1} G\wedge X_7\,, \tag{4.57}$$

and the Chern–Simons term looks conventionally normalized. However, due to the different dependence on κ, the Green–Schwarz term, when written in the upstairs formalism, has an extra factor of $2^{-2/3}$. This will be important later on.

The factorized form (4.51) of the anomaly on each ten-plane is a necessary condition to allow for local cancellation through inflow. Clearly, the $I_{4,A}X_8$-term has the right form to be cancelled through inflow from the Green–Schwarz term, provided G satisfies a modified Bianchi identity $\mathrm{d}G \sim \sum_{A=1,2}\delta_A\wedge I_{4,A}$, where δ_A is a one-form Dirac distribution such that $\int_{M_{10}\times S^1}\xi_{(10)}\wedge\delta_A = \int_{M_{10}^A}\xi_{(10)}$ for any 10-form $\xi_{(10)}$. This is equivalent to prescribing a boundary value for G on the boundary planes in the down-stairs approach. Such a modified Bianchi identity is indeed necessary to maintain supersymmetry in the coupled 11-dimensional supergravity/10-dimensional super-Yang–Mills system [5]. In principle, this allows us to deduce the coefficient $-\zeta$ on the right-hand side of the Bianchi identity in the upstairs approach. It is given by $-(4\pi)^2\frac{\kappa_{\mathrm{U}}^2}{\lambda^2}$ where λ is the (unknown) Yang–Mills coupling constant.

Hence, we start with a Bianchi identity [5]

$$\mathrm{d}G = -\zeta\sum_{A=1,2}\delta_A\wedge I_{4,A}\,. \tag{4.58}$$

The variation of the Green–Schwarz term then is (recall $\delta X_7 = \mathrm{d}X_6^1$)

$$\begin{aligned}\delta S_{GS} &= -\,\frac{1}{2^{2/3}(4\pi\kappa_{\mathrm{U}}^2)^{1/3}}\int_{M_{10}\times S^1} G\wedge \mathrm{d}X_6^1\\ &= -\,\frac{\zeta}{2^{2/3}(4\pi\kappa_{\mathrm{U}}^2)^{1/3}}\sum_A\int_{M_{10}^A} I_{4,A}\wedge X_6^1\,.\end{aligned} \tag{4.59}$$

Provided

$$\zeta = 2^{2/3}(4\pi\kappa_{\mathrm{U}}^2)^{1/3}\,, \tag{4.60}$$

δS_{GS} corresponds to an invariant polynomial

$$\hat{I}_{12}^{GS} = -\sum_A I_{4,A}\wedge X_8(R_A)\,. \tag{4.61}$$

As promised, this cancels the part of the anomaly (4.51) involving X_8. Moreover, this cancellation is local, i.e. cancellation occurs on each plane separately. We see that anomaly cancellation fixes the value of ζ to be (4.60), thereby determining the value of the 10-dimensional Yang–Mills coupling λ in terms of the 11-dimensional gravitational coupling κ. Although this latter aspect has drawn some attention, one has to realize that the more interesting relation between λ and the 10-dimensional κ_{10} involves the (unknown) radius r_0 of the circle, similarly to the relation between the type IIA string coupling constant and κ.

To study anomaly inflow from the Chern–Simons term we have to solve the Bianchi identity for G (as we did for the 5-brane). This involves several subtleties, discussed at length in [28]. One important point was to respect periodicity in the circle coordinate $x^{10} \in [-\pi r_0, \pi r_0]$ which led to the introduction of two periodic $\mathbf{Z}_2$-odd "step" functions $\epsilon_A(x^{10})$ such that $\epsilon_1(x^{10}) = \mathrm{sgn}(x^{10}) - \frac{x^{10}}{\pi r_0}$ and $\epsilon_2(x^{10}) = \epsilon_1(x^{10} \pm \pi r_0)$. They satisfy

$$\frac{1}{2}\,\mathrm{d}\epsilon_A = \delta_A - \frac{\mathrm{d}x^{10}}{2\pi r_0}\ . \tag{4.62}$$

Regularizing ϵ_A (and hence δ_A) properly gives

$$\delta_A \epsilon_B \epsilon_C \simeq \frac{1}{3}\,\delta_A\ \delta_{BA}\delta_{CA}\,, \tag{4.63}$$

where δ_{BA} and δ_{CA} denote the Kronecker symbol. When solving the Bianchi identity (4.58) one can (locally) trade terms $\frac{1}{2}\epsilon_A I_{4,A}$ for terms $-\left(\delta_A - \frac{\mathrm{d}x^{10}}{2\pi r_0}\right)\omega_{3,A}$, where

$$\mathrm{d}\omega_{3,A} = I_{4,A}\,, \qquad \delta\omega_{3,A} = \mathrm{d}\omega^1_{2,A}\,, \tag{4.64}$$

since their difference is a total derivative ($\omega_{3,A}$ is given in terms of the Chern–Simons forms on M^A_{10} and has no $\mathrm{d}x^{10}$ component). This introduces an arbitrary real parameter b into the solution,

$$\begin{aligned} G &= \mathrm{d}C - b\,\frac{\zeta}{2}\sum_A\left(\epsilon_A I_{4,A} + \omega_{3,A}\wedge\frac{\mathrm{d}x^{10}}{\pi r_0}\right) + (1-b)\,\zeta\sum_A \delta_A\wedge\omega_{3,A} \\ &= \mathrm{d}\Big(C - b\,\frac{\zeta}{2}\sum_A \epsilon_A\omega_{3,A}\Big) + \zeta\sum_A \delta_A\wedge\omega_{3,A} \\ &\equiv \mathrm{d}\,\widetilde{C} + \zeta\sum_A \delta_A\wedge\omega_{3,A}\,. \end{aligned} \tag{4.65}$$

Since G appears in the kinetic term $\sim \int G\wedge{}^*G$, as well as in the energy-momentum tensor, it must be gauge and local Lorentz invariant, $\delta G = 0$.

This is achieved if [28]

$$\delta C = b\,\zeta \sum_{A=1,2} \omega^1_{2,A} \wedge \frac{\mathrm{d}x^{10}}{2\pi r_0} + (1-b)\,\zeta \sum_A \delta_A \wedge \omega^1_{2,A}$$
$$\Leftrightarrow \quad \delta\widetilde{C} = \mathrm{d}\Big(-b\,\frac{\zeta}{2} \sum_A \epsilon_A \omega^1_{2,A} \Big) + \zeta \sum_A \delta_A \wedge \omega^1_{2,A}\,. \tag{4.66}$$

In [28] several arguments were given in favor of one particular value of b, namely $b = 1$, since only then G is globally well-defined. Furthermore, the higher Fourier modes of $C_{\bar\mu\bar\nu 10}$ are gauge invariant only for this value of b, which is a necessary condition for a safe truncation to the perturbative heterotic string. Last, but not least, it is only for $b = 1$ that G has no terms involving δ_A which would lead to divergent pieces in the kinetic term $\int G \wedge {}^*G$. Nevertheless, we will keep this parameter b for the time being and show in the end that anomaly cancellation also requires $b = 1$.

Note that, although $G \neq \mathrm{d}C$, we still have $G = \mathrm{d}\widetilde{C}$ as long as we stay away from the fixed planes. This motivates us to introduce a modified Chern–Simons term similar to what was done in Section 5 for the 5-brane or in [4] when discussing M-theory on singular G_2-manifolds. We take

$$\widetilde{S}_{CS} = -\frac{1}{12\kappa_{\mathrm{U}}^2} \int_{M_{10}\times S^1} \widetilde{C} \wedge G \wedge G\,, \tag{4.67}$$

which away from the fixed planes is just $\sim \int \widetilde{C} \wedge \mathrm{d}\widetilde{C} \wedge \mathrm{d}\widetilde{C}$. Then

$$\begin{aligned} \delta\widetilde{S}_{CS} &= -\frac{1}{12\kappa_{\mathrm{U}}^2} \int_{M_{10}\times S^1} \delta\widetilde{C} \wedge G \wedge G \\ &= -\frac{1}{12\kappa_{\mathrm{U}}^2} \int_{M_{10}\times S^1} \Bigg[\mathrm{d}\Big(-b\,\frac{\zeta}{2} \sum_A \epsilon_A \omega^1_{2,A} \Big) \wedge 2\,\mathrm{d}\widetilde{C} \wedge \zeta \sum_C \delta_C \wedge \omega_{3,C} \\ &\qquad\qquad + \zeta \sum_A \delta_A \wedge \omega^1_{2,A} \wedge \mathrm{d}\widetilde{C} \wedge \mathrm{d}\widetilde{C} \Bigg]\,. \end{aligned} \tag{4.68}$$

Note that we can freely integrate by parts (we assume that M_{10} has no boundary). Furthermore, since both δ_A and $\mathrm{d}C = \mathrm{d}\tilde{B} \wedge \mathrm{d}x^{10}$ always contain a $\mathrm{d}x^{10}$, on the r.h.s of Eq. (4.68) one can replace $\mathrm{d}\widetilde{C} \to -b\,\frac{\zeta}{2} \sum_B \epsilon_B I_{4,B}$, so that

$$\delta\widetilde{S}_{CS} = -\frac{1}{12\kappa_{\mathrm{U}}^2}\, b^2 \left(\frac{\zeta^3}{4}\right) \int_{M_{10}\times S^1} \sum_{A,B,C} \big(2\epsilon_A\epsilon_B\delta_C + \delta_A\epsilon_B\epsilon_C\big)\omega^1_{2,A} \wedge I_{4,B} \wedge I_{4,C}\,. \tag{4.69}$$

The modified Chern–Simons term contributes three terms $\epsilon\,\epsilon\,\delta$. This factor of 3 was absent in [28] where inflow from the unmodified Chern–Simons term was computed. Also the result of [28] was obtained only after using $\int_{S^1} \mathrm{d}x^{10}\epsilon_A\epsilon_B = \pi r_0(\delta_{AB} - \frac{1}{3})$ which somewhat obscured the local character of anomaly cancellation. Now, however, due to the explicit δ_A one-forms, the inflow from $\widetilde{S}_{CS}$ is localized on the 10-planes M_{10}^A. Using (4.63) we find

$$\delta\widetilde{S}_{CS} = -\frac{\zeta^3}{48\kappa_{\mathrm{U}}^2}\, b^2 \sum_{A=1,2} \int_{M_{10}^A} \omega_{2,A}^1 \wedge I_{4,A} \wedge I_{4,A}\,. \tag{4.70}$$

Upon inserting the value of ζ, equation (4.60), we see that this corresponds to an invariant polynomial

$$\hat{I}_{12}^{\widetilde{CS}} = -b^2\,\frac{\pi}{3} \sum_{A=1,2} I_{4,A}^3\,. \tag{4.71}$$

This cancels the remaining piece of the anomaly (4.51) precisely if

$$b^2 = 1\,. \tag{4.72}$$

As already mentioned there are many other arguments in favor of $b = 1$, but now we can conclude that also anomaly cancellation on $S^1/\mathbf{Z}_2$ *requires* $b = 1$, as argued in [28]. [j]

Thus we have shown that all the anomalies are cancelled *locally* through inflow from the Green–Schwarz [k] and (modified) Chern–Simons terms with exactly the same coefficients as already selected from cancellation of the 5-brane anomalies.

4.3.3. *Small radius limit and the heterotic anomaly cancelling term*

Finally, it is easy to show that in the small radius limit ($r_0 \to 0$) the sum $S_{GS} + \widetilde{S}_{CS}$ exactly reproduces the heterotic Green–Schwarz term. In this limit $X_8(R)$ and $X_7(R)$ are independent of x^{10} and have no $\mathrm{d}x^{10}$ components. From $C = \tilde{B} \wedge \mathrm{d}x^{10}$ and δC given in (4.66) we identify the correctly

[j] In [28] inflow from the unmodified Chern–Simons term was computed. This is three times smaller than (4.70). Also the factor $2^{2/3}$ in ζ was missing, so that the overall inflow δS_{CS} appeared 12 times smaller. This discrepancy remained unnoticed since the anomaly cancellation condition was expressed as $\frac{(4\pi)^5\kappa^4 b^2}{12\lambda^6} = 1$. It is only after relating $\frac{\lambda^2}{\kappa^2}$ to the coefficient of the Green–Schwarz term that one can use $\frac{(4\pi)^5\kappa^4}{\lambda^6} = 1$ and then $\frac{b^2}{12} = 1$ clearly is in conflict with $b = 1$.

[k] It is interesting to note that $\widetilde{S}_{GS} = -\frac{1}{2^{2/3}(4\pi\kappa_{\mathrm{U}}^2)^{1/3}} \int_{M_{10}\times S^1} \widetilde{C} \wedge X_8$ would have led to the same result.

normalized heterotic B-field as the zero mode of $\tilde{B}$ times $\frac{(4\pi)^2}{\zeta}2\pi r_0$,

$$B = \frac{(4\pi)^2}{\zeta}\int_{S^1}\tilde{B}\wedge \mathrm{d}x^{10}\,, \qquad \delta B = (4\pi)^2\sum_A \omega^1_{2,A} = \omega^1_{2,YM} - \omega^1_{2,L} \tag{4.73}$$

where $\omega^1_{2,YM}$ and $\omega^1_{2,L}$ are related to $\mathrm{tr}\,F_1^2 + \mathrm{tr}\,F_2^2$ and $\mathrm{tr}\,R^2$ via descent. Next, using (4.65) and (4.60), the Green–Schwarz term (4.57) gives in the small radius limit

$$\begin{aligned} S_{GS} &\to \frac{1}{(4\pi)^2}\int_{M_{10}}(\mathrm{d}B - \omega_{3,YM} + \omega_{3,L})\wedge X_7 \\ &= -\frac{1}{(4\pi)^2}\int_{M_{10}} B\wedge X_8 - \frac{1}{(4\pi)^2}\int_{M_{10}}(\omega_{3,YM} - \omega_{3,L})\wedge X_7\,. \end{aligned} \tag{4.74}$$

The second term is an irrelevant local counterterm; its gauge and local Lorentz variation corresponds to a vanishing I_{12}. Such terms can always be added and subtracted. The modified Chern–Simons term (4.67) gives (using (4.65) with $b = 1$, (4.60), (4.73) and integrating by parts on M_{10})

$$\begin{aligned} \widetilde{S}_{CS} \to -\sum_{A,B}\ &\int_{M_{10}}\left(\frac{\pi}{(4\pi)^2}B\wedge I_{4,A}\wedge I_{4,B} - \frac{2\pi}{3}\omega_{3,A}\wedge I_{4,B}\wedge\sum_C \omega_{3,C}\right) \\ &\times\int_{S^1}\epsilon_A\,\epsilon_B\,\frac{\mathrm{d}x^{10}}{2\pi\, r_0}\ . \end{aligned} \tag{4.75}$$

Using the relation

$$\int_{S^1}\epsilon_A\ \epsilon_B\ \frac{\mathrm{d}x^{10}}{2\pi\, r_0} = \frac{1}{2}\left(\delta_{AB} - \frac{1}{3}\right) \tag{4.76}$$

we get

$$\begin{aligned} \widetilde{S}_{CS} \to &-\frac{1}{(4\pi)^2}\int_{M_{10}} B\wedge\frac{\pi}{3}\left(I^2_{4,1} + I^2_{4,2} - I_{4,1}I_{4,2}\right) \\ &-\frac{2\pi}{9}\int_{M_{10}}(\omega_{3,1}+\omega_{3,2})\left(\omega_{3,1}I_{4,1} + \omega_{3,2}I_{4,2} - \frac{1}{2}\omega_{3,1}I_{4,2} - \frac{1}{2}\omega_{3,2}I_{4,1}\right)\,. \end{aligned} \tag{4.77}$$

Again, the second term is an irrelevant counterterm. Summing (4.74) and (4.77) we arrive at (cf. (4.53))

$$S_{GS} + \widetilde{S}_{CS} \to S_{het} = -\frac{1}{(4\pi)^2}\int_{M_{10}} B\wedge\hat{X}_8(R,F_1,F_2) + \text{ local counterterms}\,, \tag{4.78}$$

where $\hat{X}_8(R, F_1, F_2)$ is the standard heterotic 8-form given in (4.54). Equation (4.78) is the correctly normalized heterotic anomaly-cancelling term.[1]

5. Concluding remarks: Brane World Cosmologies, etc

We have studied anomaly cancellation by inflow from the bulk in two very different settings: the low-dimensional example of the Quantum Hall Effect and the high-dimensional examples of M-theory. There are certainly many other examples one could cite and study. One particularly interesting case are brane-world cosmologies. Here one has a 4-dimensional Minkowski manifold that is a "brane" embedded in a higher-dimensional manifold. Usually it is considered that the standard-model fields only live on the brane and only gravity propagates in the bulk. More sophisticated versions based on supergravity will also have certain gauge fields in the bulk and one can then study in the same way inflow of gauge and gauge-gravitational anomalies into the brane. This is somewhat reminiscent of what happens in the AdS/CFT correspondence where the five-dimensional AdS_5 supergravity has $SU(4)$ gauge fields and its action precisely involves a Chern–Simons term. On the boundary of AdS_5 lives the CFT, namely the $N = 4$ super Yang–Mills theory with a *global* R-symmetry $SU(4)$ which is anomalous. In this case, however, the non-invariance of the 5-dimensional Chern–Simons term does not provide an anomaly cancelling inflow, but explains the global $SU(4)$ anomaly of the CFT (see e.g. [29]). The mathematics is the same, but its interpretation is different. In brane world scenarios, on the other hand, anomaly cancellation may be a valuable constraint.

Acknowledgments

This contribution is based largely on work done together with Steffen Metzger [4].

[1] In order to facilitate comparison with [12] we note that $\hat{X}_8 = \frac{1}{(2\pi)^3 4!} X_8^{GSW}$, and S_{het} as given in (4.78) exactly equals minus the expression given in [12]. The missing minus sign in [12] is due to a sign error related to the subtle issues of orientation, and is corrected e.g. when using the anomaly polynomials as given in [14].

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