FROM SUPER-YANG–MILLS THEORY TO QCD: PLANAR EQUIVALENCE AND ITS IMPLICATIONS

A. ARMONI

Department of Physics, Theory Division
CERN, CH-1211 Geneva 23, Switzerland

M. SHIFMAN

William I. Fine Theoretical Physics Institute,
University of Minnesota,
Minneapolis, MN 55455, USA

G. VENEZIANO

Department of Physics, Theory Division
CERN, CH-1211 Geneva 23, Switzerland

We review and extend our recent work on the planar (large N) equivalence between gauge theories with varying degrees of supersymmetry. The main emphasis is on the planar equivalence between $\mathcal{N}=1$ gluodynamics (super-Yang–Mills theory) and a non-supersymmetric "orientifold field theory". We outline an "orientifold" large N expansion, analyze its possible phenomenological consequences in one-flavor massless QCD, and make a first attempt at extending the correspondence to three massless flavors. An analytic calculation of the quark condensate in one-flavor QCD starting from the gluino condensate in $\mathcal{N}=1$ gluodynamics is thoroughly discussed. We also comment on a planar equivalence involving $\mathcal{N}=1$ supersymmetry, on "chiral rings" in non-supersymmetric theories, and on the origin of planar equivalence from an underlying, non-tachyonic type-0 string theory. Finally, possible further directions of investigation, such as the gauge/gravity correspondence in large-N orientifold field theory, are briefly discussed.

Table of Contents

Dictionary and some notation

In the bulk of this review the gauge group is assumed to be SU(N), with the exception of Sects. 4 and 5. Different theories that we consider include:

Orientifold theory A (one two-index Dirac fermion in the antisymmetric representation);

Orientifold theory S (one two-index Dirac fermion in the symmetric representation);

General version: **Orientifold theory** (one of the two above);

Orienti/A$_n$ theory (n antisymmetric Dirac fermions);

Orienti/S$_n$ (n symmetric Dirac fermions);

Adjoint$_n$ theory (n adjoint Majorana fermions);

N^2 - QCD ($N_f = N_c = N$ Dirac flavors);

Orienti/nf theory (one two-index Dirac fermion in the antisymmetric representation plus n fundamental Dirac quarks);
Version: **Orienti/2f theory**;

kf-QCD (QCD with k Dirac fundamental flavors);
Version: **1f-QCD** (QCD with one Dirac fundamental flavor);

Common sector (for a definition see the discussion preceding Sect. 4.1).

⋆ ⋆ ⋆

Note that, from the string theory perspective, only some of the theories listed above deserve the name "orientifold theories" — not all appear as a result of *bona fide* orientifoldization. Keeping in mind, however, that our prime emphasis is on field theory, we will take the liberty of naming them in a way that is convenient for our purposes, albeit not quite justified from the string theory standpoint.

⋆ ⋆ ⋆

We normalize the Grassmann integration as

$$\int \theta^2 \, d^2\theta = 2\,. \tag{1}$$

The rest of the notation for superfield/superspace variables is essentially the same as in the classical textbook of Bagger and Wess [1], except that our metric is $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ while in Bagger and Wess's book it is $g^{\mathrm{BW}}_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$.

If we have two Weyl (right-handed) spinors ξ^α and η_β, transforming in the representations R and $\bar{R}$ of the gauge group, respectively, then the Dirac spinor Ψ can be formed as

$$\Psi = \begin{pmatrix} \xi \\ \bar{\eta} \end{pmatrix} . \tag{2}$$

The Dirac spinor Ψ has four components, while ξ^α and η_β have two components each.

⋆ ⋆ ⋆

The gluon field-strength tensor is denoted by $G^a_{\mu\nu}$. We use the abbreviation G^2 for

$$G^2 \equiv G^a_{\mu\nu}\, G^{\mu\nu\ a} , \tag{3}$$

while

$$G\tilde{G} \equiv G^a_{\mu\nu}\, \tilde{G}^{\mu\nu\ a} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} G^a_{\mu\nu} G^a_{\rho\sigma} . \tag{4}$$

⋆ ⋆ ⋆

Group-theory coefficients

As was mentioned, the gauge group is assumed to be SU(N). For a given representation R of SU(N), the definitions of the Casimir operators to be used below are

$$\mathrm{Tr}\,(T^a T^b)_R = T(R)\delta^{ab} , \qquad (T^a T^a)_R = C(R)\, I , \tag{5}$$

where I is the unit matrix in this representation. It is quite obvious that

$$C(R) = T(R)\,\frac{\dim(G)}{\dim(R)} , \tag{6}$$

where $\dim(G)$ is the dimension of the group (= the dimension of the adjoint representation). Note that $T(R)$ is also known as (one half of) the Dynkin index, or the dual Coxeter number. For the adjoint representation, $T(R)$ is denoted by $T(G)$. Moreover, $T(\mathrm{SU}(N)) = N$.

Renormalization-group conventions

The β function and anomalous dimensions are defined as

$$\mu\,\frac{\partial\alpha}{\partial\mu} \equiv \beta(\alpha) = -\frac{\beta_0}{2\pi}\alpha^2 - \frac{\beta_1}{4\pi^2}\alpha^3 + ... \tag{7}$$

and

$$\gamma = -d\ln Z(\mu)/d\ln\mu\,. \tag{8}$$

In supersymmetric theories

$$\beta(\alpha) = -\frac{\alpha^2}{2\pi}\left[3\,T(G) - \sum_i T(R_i)(1-\gamma_i)\right]\left(1 - \frac{T(G)\,\alpha}{2\pi}\right)^{-1}, \tag{9}$$

where the sum runs over all matter supermultiplets. The anomalous dimension of the i-th matter superfield is

$$\gamma_i = -2C(R_i)\,\frac{\alpha}{2\pi} + \dots\,. \tag{10}$$

Sometimes, when one-loop anomalous dimensions are discussed, the coefficient in front of $-\alpha/(2\pi)$ in (8) is also referred to as an "anomalous dimension."

1. Introduction

Gauge field theories at strong coupling are obviously of great importance in particle physics. Needless to say, exact results in such theories have a special weight. Supersymmetry proved to be an extremely powerful tool, allowing one to solve some otherwise intractable problems (for a review see e.g. [2]). During the last twenty years or so, a significant number of results were obtained in this direction, starting from the gluino condensate and exact β functions, to, perhaps, the most famous example — the Seiberg–Witten solution [3] of $\mathcal{N}=2$ supersymmetric theory, which (upon a small perturbation that breaks $\mathcal{N}=2$ to $\mathcal{N}=1$) explicitly exhibits the dual Meißner mechanism of color confinement conjectured by Mandelstam [4] and 't Hooft [5] in the 1970's and 80's. This review is devoted to a recent addition to the supersymmetry-based kit, which goes under the name of *planar equivalence*. Some *non*-supersymmetric gauge theories — quite close relatives of QCD — were shown [6–8] to be equivalent to supersymmetric gluodynamics in a bosonic sub-sector, provided the number of colors $N \to \infty$. Needless to say that, if true, this statement has far-reaching consequences. A large number of supersymmetry-based predictions, such as spectral degeneracies, low-energy theorems, and so on, hold (to leading order in $1/N$) in strongly coupled non-supersymmetric gauge theories!

We will start this review by summarizing the basic features of the "main" parent theory — supersymmetric gluodynamics, also known as supersymmetric Yang–Mills (SYM) theory. We review the genesis of the idea of planar equivalence and its physical foundation, both at the perturbative and non-perturbative levels. After a brief introduction on perturbative planar equivalence, we proceed to the *orientifold field theory*, whose bosonic sector was argued [6] to be fully (i.e. perturbatively *and* non-perturbatively) equivalent to SYM theory at $N \to \infty$. At $N = 3$ the daughter theory is also equivalent to *one-flavor* QCD. We explore the detailed consequences of these equivalences.

Among other aspects, our review will demonstrate cross-fertilization between field theory and string theory. The idea of planar equivalence, born in the depths of string theory, was reformulated and adapted to field theory, where it found a life of its own and was developed to a point where practical applications are looming.

Instead of summarizing here the main results and topics to be covered — normal practice in an introductory section — we refer the reader to the Table of Contents and to Sect. 15.

2. Genesis of the idea [a]

Kachru and Silverstein studied [9] various orbifolds of R^6 within the framework of the AdS/CFT correspondence. Starting from $\mathcal{N}=4$, they obtained distinct four-dimensional (daughter) gauge field theories with matter, with varying degree of supersymmetry, $\mathcal{N}=2,1,0$, all with vanishing β functions. In the latter case, $\mathcal{N}=0$, Kachru and Silverstein predicted the first coefficient of the Gell-Mann–Low function, β_0, to vanish in the large-N (planar) limit in the daughter theories, while for $\mathcal{N}=2,1$ they predicted $\beta_0=0$ even for finite N. Shortly after their study, this analysis was expanded in [10], where generic orbifold projections of $\mathcal{N}=4$ were considered, and those preserving conformal invariance of the resulting — less supersymmetric — gauge theories (at two loops) were identified.

A decisive step was made [11] by Bershadsky *et al.* These authors abandoned the AdS/CFT limit of large 't Hooft coupling. They considered the string perturbative expansion in the presence of D-branes embedded in orbifolded space-time in the limit $\alpha' \to 0$. In this limit the string perturbative expansion coincides with the 't Hooft $1/N$ expansion [12]. The genus-zero string graphs can be identified with the leading (planar) term in the 't Hooft expansion, the genus-one is the next-to-leading correction, and so on. In [11] one can find a proof that in the large-N limit the β functions of all orbifold theories considered previously [9, 10] vanish to any finite order in the gauge coupling constant. Moreover, a remarkable theorem was derived in [11]. The authors showed that, not only the β functions, but a variety of amplitudes which can be considered in the orbifold theories coincide in the large-N limit (upon an appropriate rescaling of the gauge coupling) with the corresponding amplitudes of the parent $\mathcal{N}=4$ theory, order by order in the gauge coupling.

In a week or so, Bershadsky and Johansen abandoned the string theory set-up altogether. They proved [13] the above theorem in the framework of field theory *per se*.

In a parallel development, Kakushadze suggested [14] considering orientifold rather than orbifold daughters of $\mathcal{N}=4$ theory. His construction was close to that of Ref. [11], except that instead of the string expansion in the presence of D-branes he added orientifold planes. The main emphasis was on the search of $\mathcal{N}=2,1$ daughters with gauge groups other than SU(N) and matter fields other than bifundamentals.

The first attempt to apply the idea of orbifoldization to non-conformal

[a] This section can be omitted in a first reading. Subsequent sections explain much of its material.

field theories was carried out by Schmaltz who suggested [15] a version of Seiberg duality between a pair of non-supersymmetric large-N orbifold field theories. Later on, the Z_2 version of Schmaltz' suggestion was realized in a brane configuration of type-0 string theory [16].

After a few years of relative oblivion, interest in the issue of planar equivalence was revived by Strassler [17]. In the inspiring paper entitled "On methods for extracting exact non-perturbative results in non-supersymmetric gauge theories" he shifted the emphasis away from the search of the conformal daughters, towards engineering QCD-like daughters. Similarly to Schmaltz, Strassler noted that one does not need to limit oneself to an $\mathcal{N}=4$ parent; $\mathcal{N}=1$ gluodynamics, which is much closer to actual QCD, gives rise to orbifold planar-equivalent daughters too. Although this equivalence was proved only at the perturbative level, Strassler formulated a "non-perturbative orbifold conjecture" (NPO). According to this conjecture, for "those vacua which appear in both theories (i.e. parent and orbifold daughter) the Green's functions which exist in both theories are identical in the limit $N \to \infty$ after an appropriate rescaling of the coupling constant".

Unfortunately, it turned out [18,19] that the NPO conjecture could not be valid. The orbifold daughter theories "remember" that they have fewer vacua than the parent one, which results in a mismatch [18] in low-energy theorems. In string theory language, the killing factor is the presence of tachyons in the twisted sector. This is clearly seen in the light of the calculation presented in Ref. [19].

The appeal of the idea was so strong, however, that the searches continued. They culminated in the discovery of the orientifold field theory [6] whose planar equivalence to SYM theory, both perturbative and non-perturbative, rests on a solid footing. The orientifold daughter has no twisted sector. The SU(N) orientifold daughter and its parent, SU(N) $\mathcal{N}=1$ gluodynamics, have the same numbers of vacua.

3. The basic parent theory

Let us describe a prototypical parent theory of this review. It is a generic supersymmetric gauge theory of the form

$$\mathcal{L} = \left\{ \frac{1}{4g^2} \int \mathrm{d}^2\theta \, \mathrm{Tr}\, W^2 + \mathrm{h.c.} \right\} + \mathcal{L}_{\mathrm{matter}} \,, \tag{3.1}$$

where the spinorial superfield W_α in the Wess-Zumino gauge is

$$W_\alpha = \frac{1}{8}\, \bar{D}^2 \left(e^{-V} D_\alpha e^V \right) = i \left(\lambda_\alpha + i\theta_\alpha D - \theta^\beta \, G_{\alpha\beta} - i\theta^2 \mathcal{D}_{\alpha\dot\alpha} \bar\lambda^{\dot\alpha} \right) \,, \tag{3.2}$$

and $\mathcal{L}_{\rm matter}$ is the matter part of the Lagrangian,

$$\mathcal{L}_{\rm matter} = \sum_{\rm matter} \left\{ \frac{1}{4} \int d^2\theta d^2\bar{\theta}\, \bar{Q} e^V Q + \left(\frac{1}{2} \int d^2\theta\, \mathcal{W}(Q) + {\rm h.c.} \right) \right\}, \tag{3.3}$$

where Q is a generic notation for the matter chiral superfields and $\mathcal{W}(Q)$ is their superpotential. As is seen from Eq. (3.2), W_α is nothing but the superspace field strength.

The gauge group is assumed to be $\mathrm{SU}(N)$. The matter fields Q can belong to various representations R of the gauge group. The definition of various group-theoretical quantities is given at the very beginning of this review. For the representations used in this paper the relevant Casimir operators are collected in Table 1.

Table 1. The Casimir coefficients for relevant representations of $\mathrm{SU}(N)$.

Casimirs \ Reps	Fundamental	Adjoint	2-index A	2-index S
$T(R)$	$\frac{1}{2}$	N	$\frac{N-2}{2}$	$\frac{N+2}{2}$
$C(R)$	$\frac{N^2-1}{2N}$	N	$\frac{(N-2)(N+1)}{N}$	$\frac{(N+2)(N-1)}{N}$

Most of this review will deal with $\mathcal{N}=1$ SYM theory, also known as supersymmetric gluodynamics — the theory of gluons and gluinos. The Lagrangian of the theory is obtained from Eq. (3.1) by omitting the matter part. In component form

$$\mathcal{L} = -\frac{1}{4g^2} G^a_{\mu\nu} G^a_{\mu\nu} + \frac{i}{g^2} \lambda^{a\alpha} \mathcal{D}_{\alpha\dot{\beta}} \bar{\lambda}^{\dot{\alpha}\dot{\beta}}, \tag{3.4}$$

where λ is the Weyl (Majorana) fermion in the adjoint representation, and spinorial notation is used in the fermion sector. One can add, if one wishes, a θ term,

$$\mathcal{L}_\theta = \frac{\theta}{32\pi^2}\, G^a_{\mu\nu} \widetilde{G}^a_{\mu\nu}\,. \tag{3.5}$$

For the time being we will set $\theta = 0$.

Later on we will let $N \to \infty$, following 't Hooft, i.e. keeping [b]

$$\lambda \equiv \frac{g^2\,N}{8\pi^2} \equiv \frac{N\alpha}{2\pi} = \text{const.} \tag{3.6}$$

The distinction between QCD and SUSY gluodynamics lies in the fermion sector — the QCD quarks are Dirac fermions in the fundamental representation of the gauge group.

The theory (3.4) is supersymmetric. The conserved spin-3/2 current J^μ_β has the form (in spinorial notation)

$$J_{\beta\alpha\dot\alpha} \equiv (\sigma_\mu)_{\alpha\dot\alpha} J^\mu_\beta = \frac{2i}{g^2}\, G^a_{\alpha\beta}\bar\lambda^a_{\dot\alpha}\,. \tag{3.7}$$

Moreover, the theory is believed to be confining, with a mass gap. Some new semi-theoretical–semi-empirical arguments substantiating this point will be provided later, see Sect. 6.3.

The spectrum of supersymmetric gluodynamics consists of composite (color-singlet) hadrons which enter in degenerate supermultiplets. Note that there is no conserved fermion-number current in the theory at hand. The only U(1) symmetry of the classical Lagrangian (3.4),

$$\lambda \to e^{i\alpha}\lambda\,, \quad \bar\lambda \to e^{-i\alpha}\bar\lambda\,,$$

is broken at the quantum level by the chiral anomaly. A discrete Z_{2N} subgroup, $\lambda \to e^{\pi i k/N}\lambda$, is non-anomalous. It is known to be dynamically broken down to Z_2. The order parameter, the gluino condensate [c] $\langle\lambda\lambda\rangle$, can take N values,

$$\langle \lambda^a_\alpha \lambda^{a\,,\alpha}\rangle = -6N\Lambda^3 \exp\left(\frac{2\pi i k}{N}\right)\,, \quad k = 0, 1, ..., N-1\,, \tag{3.8}$$

labeling the N distinct vacua of the theory (3.4), see Fig. 1. Here $\langle ...\rangle$ means averaging over the given vacuum state, and Λ is a dynamical scale, defined

[b] Here and below we follow the standard convention $g^2 = 4\pi\alpha$.

[c] The gluino condensate in supersymmetric gluodynamics was first conjectured, on the basis of the value of his index, by E. Witten [20]. It was confirmed in an effective Lagrangian approach by G. Veneziano and S. Yankielowicz [21], and exactly calculated (by using holomorphy and analytic continuations in mass parameters [22]) by M. A. Shifman and A. I. Vainshtein [23]. The exact value of the coefficient $6N$ in Eq. (3.8) can be extracted from several sources. All numerical factors are carefully collected for SU(2) in the review paper [24]. A weak-coupling calculation for SU(N) with arbitrary N was carried out in [25]. Note, however, that an unconventional definition of the scale parameter Λ is used in Ref. [25]. One can pass to the conventional definition of Λ either by normalizing the result to the SU(2) case [24] or by analyzing the context of Ref. [25]. Both methods give one and the same result.

in the standard manner (i.e. in accordance with Ref. [26]) in terms of the ultraviolet parameters,

$$\Lambda^3 = \frac{2}{3}\, M_{\rm UV}^3\, \frac{8\pi^2}{N g_0^2}\, \exp\left(-\frac{8\pi^2}{N g_0^2}\right) = \frac{2}{3}\, M_{\rm UV}^3\, \frac{1}{\lambda_0}\, \exp\left(-\frac{1}{\lambda_0}\right), \tag{3.9}$$

where $M_{\rm UV}$ is the ultraviolet (UV) regulator mass, while g_0^2 and λ_0 are the bare coupling constants.

Note that since Λ is expressible in terms of the 't Hooft coupling, it is explicitly N-independent. Equation (3.9) is exact [27] in supersymmetric gluodynamics. If $\theta \neq 0$, the exponent in Eq. (3.8) is replaced by

$$\exp\left(\frac{2\pi i k}{N} + \frac{i\,\theta}{N}\right).$$

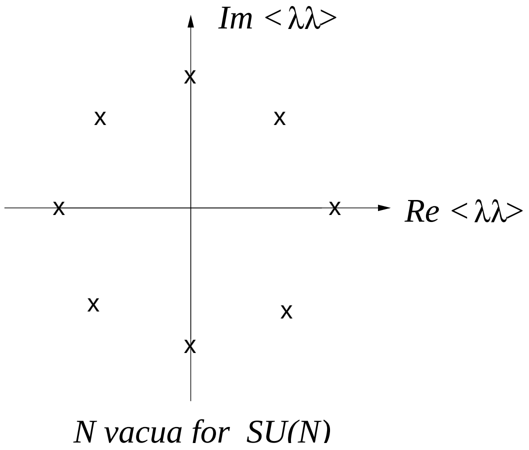


Figure 1. The gluino condensate $\langle\lambda\lambda\rangle$ is the order parameter labeling distinct vacua in supersymmetric gluodynamics. For the SU(N) gauge group there are N discrete degenerate vacua.

All hadronic states are arranged in supermultiplets. The simplest is the so-called chiral supermultiplet, which includes two (massive) spin-zero mesons (with opposite parities), and a Majorana fermion with a Majorana mass (alternatively, one can treat it as a Weyl fermion). The interpolating operators producing the corresponding hadrons from the vacuum are G^2, $G\tilde{G}$ and $G\lambda$. The vector supermultiplet consists of a spin-1 massive vector particle, a 0^+ scalar and a Dirac fermion. All the particles from one supermultiplet have degenerate masses. Two-point functions are degenerate too (modulo obvious kinematical spin factors). For instance,

$$\langle G^2(x)\,,\, G^2(0)\rangle = \langle G\tilde{G}(x)\,,\, G\tilde{G}(0)\rangle = \langle G\lambda(x)\,,\, G\lambda(0)\rangle\,. \tag{3.10}$$

Unlike what happens in conventional QCD, both the meson and fermion masses in SUSY gluodynamics are expected to scale as N^0.

4. Orbifoldization and perturbative equivalence

The technique of orbifoldization was developed in the context of string/brane theory, and it will be discussed in the string part of this review (Sect. 14). Starting from supersymmetric gluodynamics with the gauge group SU(kN), where k is an integer, one can perform a Z_k orbifoldization.

The only thing we need to know at the moment, is that in orbifolding the parent theory (3.4) one obtains a self-consistent daughter field theory by judiciously discarding a number of fields in the Lagrangian (3.4). This is illustrated in Fig. 2 corresponding to $k = 2$. The square presents the color contents of the fields — gluon and gluino — of the daughter theory. The gauge fields, which are *retained* in the daughter theory, belong to two blocks on the main diagonal. Thus, the daughter theory has the gauge symmetry SU(N)×SU(N). (Needless to say, there is no distinction at large N between SU(N) and U(N).) This specific theory has a realization in type-0 string theory [16]. To emphasize the fact that the gauge group is a direct product it is convenient to use "tilded" indices for one SU(N), e.g. $(A_\mu)^{\tilde{i}}_{\tilde{j}}$, and "untilded" for another, e.g. $(G_{\mu\nu})^i_j$, where $i, j, = 1, 2, ..., N$ while $\tilde{i}, \tilde{j} = N + 1, ..., 2N$.

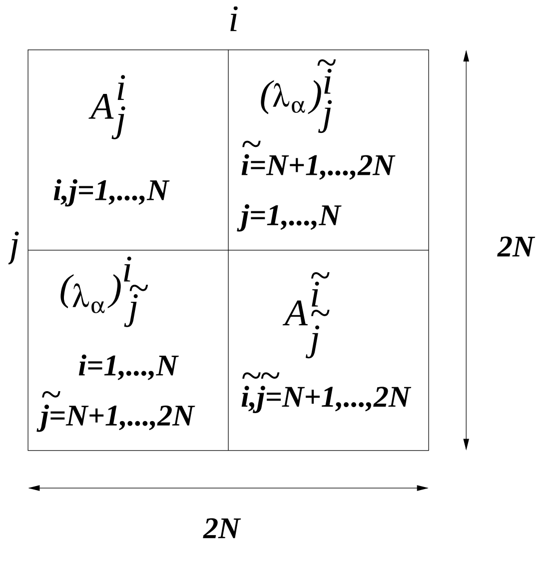


Figure 2. Color decomposition of fields in the Z_2-orbifold daughter.

The fermion fields to be retained in the daughter theory belong to two off-diagonal blocks in Fig. 2. Thus, each fermion carries one tilded and one untilded color index. In other words, they are fundamentals with respect to the first SU(N) and antifundamentals with respect to the second SU(N). Such fields are called *bifundamentals*. Thus, in the daughter theory we deal with two Weyl bifundamentals, $(\lambda_\alpha)^i_{\tilde{j}}$ (we will call it χ) and $(\lambda_\alpha)^{\tilde{i}}_j$ (we will call it η). It is quite clear that they can be combined to form the four-component (Dirac) bifundamental field

$$\eta,\ \bar{\chi} \to \Psi^{\tilde{i}}_j\,.$$

The Lagrangian of the daughter theory is

$$\mathcal{L} = -\frac{1}{4g^2} G^i_j G^j_i - \frac{1}{4g^2} G^{\tilde{i}}_{\tilde{j}}\, G^{\tilde{j}}_{\tilde{i}} + \frac{\theta}{32\pi^2}\left(G^i_j \widetilde{G}^j_i + G^{\tilde{i}}_{\tilde{j}} \widetilde{G}^{\tilde{j}}_{\tilde{i}}\right)$$
$$+\, \overline{\Psi^{\tilde{i}}_j}\; i\, \not{D}\; \Psi^{\tilde{i}}_j\,, \tag{4.1}$$

where

$$D_\mu = \partial_\mu \; - i\, A^a_\mu T^a \; - i\, \tilde{A}^a_\mu \tilde{T}^a\,, \tag{4.2}$$

and the θ term is included. Note that the perturbative planar equivalence requires the gauge couplings of both SU(N)'s to be *the same*. Below they will be denoted by g^2_D. We will then assume that the vacuum angles are the same too, as is indicated in Eq. (4.1).

The perturbative planar equivalence that holds between supersymmetric gluodynamics and its orbifold daughter (4.1) requires a rescaling of the gauge couplings in passing from the parent to the daughter theory, namely

$$g^2_D = 2 g^2_P\,. \tag{4.3}$$

In the general case of Z_k orbifoldization the relation is

$$g^2_D = k g^2_P\,. \tag{4.4}$$

The orbifold daughter is *not* supersymmetric. The color-singlet supercurrent (3.7) cannot be formed now since the color indices of the gauge fields and the fermion fields cannot be contracted. In this way the orbifold daughter seems to be closer to genuine QCD than SUSY gluodynamics. In fact, for $N = 3$ this is nothing but three-color/three-flavor "QCD" with a gauged flavor group. (Note, that flavor SU(2) is gauged anyway in the electroweak theory, albeit that the corresponding gauge coupling is small.)

The absence of supersymmetry is not the only feature of (4.1) that distinguishes this theory from SUSY gluodynamics. The daughter theory possesses a global U(1). The conserved fermion number current is $J_\mu = \overline{\Psi}\gamma_\mu\Psi$. Applying this vector current to the vacuum, one produces a vector meson that has no counterpartners in the parent theory. One says that the operator J_μ has no projection onto the parent theory. One can construct other operators with similar properties. There are many hadronic states in the daughter theory that have no analogues in the parent one. For instance, for odd N one expects composite states with the quark content $\Psi\Psi...\Psi\ \varepsilon\ \tilde{\varepsilon}$ and masses scaling as N. For even N, the daughter theory has no color-singlet fermions at all.

The inverse is also true. For instance, the operator of the supercurrent produces a spin-3/2 hadron from the vacuum state of SUSY gluodynamics, which has no counterpart in the daughter theory. Therefore, in confronting these theories it makes sense to compare only those sectors that have projections onto one another. Below we will refer to such sectors as *common.*

Under the Z_k orbifoldization the gauge symmetry of the orbifold daughter is $\mathrm{SU}(N)^k$, and all fermion fields in the daughter theory are bifundamentals. Note that the daughter theory is chiral for $k > 2$, i.e. no mass term can be introduced for the "quark" fields. Such daughter theories can be only remotely related to QCD, if at all.

4.1. *Perturbation theory*

As was mentioned in Sect. 2, it was discovered in 1998 [13] that the Green functions for operators which exist in both theories have identical planar graphs [d] provided the rescaling (4.3) is implemented. In other words, perturbatively these Green's functions in the parent (supersymmetric) and daughter (non-supersymmetric) theories coincide order by order in the large-N limit. Motivation for the work [13] came from the string side. A detailed discussion of orbifoldization in various gauge theories can be found in Ref. [15].

Rather than presenting a general proof (it can be found in the original publications, see e.g. [13]) we will illustrate this statement (see Fig. 3) by a simple example of the correlation function of two axial currents,

$$(2/g^2)(\bar{\lambda}_{\dot{\alpha}})^i_j(\lambda_\alpha)^j_i\,.$$

[d] This statement assumes that the daughter theory under consideration inherits the vacuum from the parent theory, an important reservation given that there are more vacua in the parent theory than in the daughter one. Perturbation theory does not distinguish between distinct vacua as long as they are "untwisted", i.e. Z_2-symmetric.

For computational purposes, it is convenient to keep the fermion fields in the daughter theory in the same form $(\lambda_\alpha)^i_j$, just constraining the color indices. In the parent theory i, j run from 1 to $2N$. In the daughter theory either $i \in [1, N]$, $j \in [N+1, 2N]$ or *vice versa*, $i \in [N+1, 2N]$, $j \in [1, N]$. In the first case we deal with $\lambda^i_{\tilde{j}}$ in the second $\lambda^{\tilde{i}}_j$.

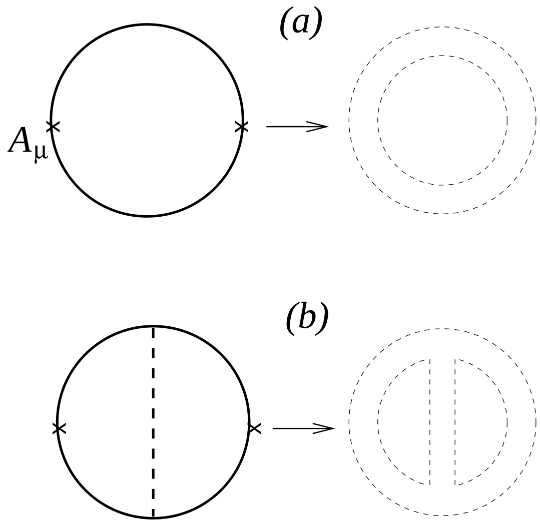


Figure 3. Counting N factors in the correlation function of the axial currents. The 't Hooft diagrams are shown in dashed lines.

The easiest way to check the match is by analyzing relevant graphs in the 't Hooft representation — each gluon or fermion line carrying two indices is represented by a double line, see Fig. 3. In this way we readily see that the weight of the graph 3a is $4N^2$ (parent), $2N^2$ (daughter), while that of 3b is $8N^3 g_P^2$ (parent), $2N^3 g_D^2$ (daughter). To get the latter estimate one notes that if the first index of λ — the one forming the outside loop — is untilded, the second index (corresponding to two inside loops in Fig. 3b) is tilded, and *vice versa*. The parent-to-daughter ratio is 2 for both graphs 3a and 3b, provided $g_D^2 = 2\, g_P^2$. Proceeding along these lines it is quite easy to see that the parent-to-daughter ratio stays the same for any planar graph. This proves the perturbative planar equivalence in the case at hand. The proof can be readily extended to any correlation function that exists in both supersymmetric gluodynamics and its orbifold daughter. In the common sector these two theories are perturbatively equivalent at $N = \infty$.

4.2. *Consequences of perturbative planar equivalence*

As we have just seen, a relationship can be established between two theories in the large-N limit — one is supersymmetric, the other is not. Although this relationship is admittedly perturbative, a natural question to ask is whether one can benefit from it and, if so, how.

The *exact* β function of SU($2N$) SUSY gluodynamics, established [27] almost twenty years ago, is

$$\beta(\alpha_P) \equiv \frac{d\alpha_P}{d\ln\mu} = -\frac{1}{2\pi}\,\frac{6N\alpha_P^2}{1-(N\alpha_P)/\pi}\,. \tag{4.5}$$

Because of planar equivalence, it implies that the full β function in the (non-supersymmetric) daughter theory is also known. Namely, in the 't Hooft limit,

$$\beta(\alpha_D) = -\frac{1}{2\pi}\,\frac{3N\alpha_D^2}{1-(N\alpha_D)/(2\pi)}\,, \tag{4.6}$$

where the relation (4.3) between the coupling constants in the parent and orbifold theories is taken into account. Curiously, the β function (4.6) is exactly the same as in SU(N) SUSY gluodynamics.

5. Non-perturbative orbifold conjecture

It is non-perturbative dynamics which is our prime concern in strongly coupled gauge theories. Non-perturbative effects are at the heart of key phenomena such as confinement, chiral symmetry breaking, and so on. Strassler conjectured [17] that the planar Feynman graph equivalence discussed in Sect. 4 extends to non-perturbative phenomena, namely Green's functions for the color-singlet operators that appear in both theories are identical at $N \to \infty$, including non-perturbative effects. This suggestion is referred to as the *non-perturbative orbifold (NPO) conjecture*.

Unfortunately, Strassler's suggestion, in its original form, is not valid [18], as we will see shortly. It played an important role, however, in paving the way for further developments. Analyzing why orbifold daughters fail to retain planar equivalence at non-perturbative level, one can guess how one can engineer daughters free from these drawbacks and thus have a better chance to be non-perturbatively planar-equivalent to their supersymmetric parents (Sect. 6).

5.1. *The vacuum structure and low-energy theorems*

In order to go over to a non-perturbative analysis, let us first compare the number of vacua in the SU(kN) supersymmetric gluodynamics to that of its Z_k daughter.

The vacua in the parent theory are counted by Witten's index [20]. They are labeled by the gluino condensate, see Eq. (3.8), which for SU(kN) takes kN distinct values, so that we have kN degenerate vacua. For $k = 2$, the simplest case with which we will deal below, there are $2N$ supersymmetric vacua in the parent theory.

At the same time, the number of vacua in the orbifold theory is twice smaller. This is easy to see from the discrete chiral invariance of the daughter theory. Generally speaking, the theory (4.1) is *not* invariant under the chiral rotation

$$\chi \to \chi\, e^{i\alpha}\,, \qquad \eta \to \eta e^{i\alpha}\,, \tag{5.1}$$

since under this rotation the vacuum angle θ is shifted, $\theta \to \theta + 2\,N\,\alpha$. However, if

$$\alpha = k\pi/N\,, \qquad k \text{ integer}\,, \tag{5.2}$$

then $\delta\theta = 2\pi\, k$, and such a discrete rotation *is* an invariance of the theory. The integer parameter k in Eq. (5.2) clearly runs from 1 to $2N$. However, only the phase of the bilinear operator $\chi\eta$ distinguishes the different vacua. In other words, the condensate $\langle\chi\eta\rangle$ breaks (spontaneously) Z_{2N}, the discrete symmetry surviving the axial anomaly in the theory (4.1), down to Z_2. We conclude that the daughter theory has N distinct vacua.

The phase of the bifermion operator *per se* is unobservable. One makes it observable by introducing a small mass term (which we will need anyway in order to regularize fermion determinants, see Sect. 6),

$$\mathcal{L}_m = \begin{cases} -\frac{m}{2g_P^2}\,\lambda^a\lambda^a + \text{h.c.} \\ -\frac{m}{g_D^2}\,\chi\eta + \text{h.c.} \end{cases}. \tag{5.3}$$

The vacua are no longer degenerate; the vacuum energy densities split,

$$\mathcal{E}_{\text{vac}} = \begin{cases} \frac{m}{2g_P^2}\,\langle \text{vac}_k|\lambda^a\lambda^a|\text{vac}_k\rangle_0\, e^{i\theta/(2N)} + \text{h.c.} \\ \frac{m}{g_D^2}\,\langle \text{vac}_k|\chi\eta|\text{vac}_k\rangle_0\, e^{i\theta/N} + \text{h.c.} \end{cases} + O(m^2)\,, \tag{5.4}$$

where $|\text{vac}_k\rangle$ denotes the k-th vacuum, while the subscript 0 marks the value of the condensate at $\theta = 0$. Finally, besides the vacuum energy densities, we will consider topological susceptibilities defined as

$$\mathcal{T} = -\frac{\partial^2 \mathcal{E}_{\rm vac}}{\partial \theta^2}\bigg|_{\theta=0} = i\int d^4x \left\langle \frac{1}{32\pi^2}\, G\widetilde{G}(x),\ \frac{1}{32\pi^2}\, G\widetilde{G}(0)\right\rangle_{\rm conn}, \qquad (5.5)$$

where the definition of $G\widetilde{G}$ in the parent and orbifold theories is given in (3.5) and (4.1) respectively.

Now, we can assemble all the above elements to prove that the NPO conjecture does not work. Indeed, assume it does. Then $\mathcal{E}^D_{\rm vac} = \mathcal{E}^P_{\rm vac}$. Here we choose a pair of "corresponding" vacua. Remember, the parent theory has twice as many vacua. We take the one that can be projected onto the daughter theory. Equation (5.4) implies that in the given vacuum $\mathcal{T}_P = \frac{1}{4}\,\mathcal{T}_D$.

On the other hand, the topological susceptibilities in the given vacuum are measurable, for instance, on lattices. The statement that $\mathcal{T}_P = \frac{1}{4}\,\mathcal{T}_D$ is in disagreement with the definition (5.5) by a factor of 2 [18]. Thus, the non-perturbative sector of the orbifold theory remembers the mismatch between the number of vacua in the parent and daughter theories.

5.2. *What makes orbifold daughters unsuitable?*

The underlying reason for the failure of the NPO conjecture is most clearly seen in the string-theory language; it is the presence of *tachyons* in the twisted sector. We defer the corresponding discussion to Sect. 14. Here we will present some field-theoretical arguments illustrating the devastating consequences of the presence of the twisted sector.

First, let us explain what the twisted sector is in the field-theoretical language. To this end we will recast Fig. 2, summarizing the content of the Z_2 orbifold daughter, in a slightly different way, see Table 2.

Table 2. The Z_2 orbifold theory.

	$\mathrm{U}_e(N)\times$	$\mathrm{U}_m(N)$
A^e_e	Adj	1
A^m_m	1	Adj
λ^e_m	$\square$	$\overline{\square}$
λ^m_e	$\overline{\square}$	$\square$

We replaced the untilded indices in Fig. 2 by generic sub/superscripts e (meaning electric) while the tilded indices by m (meaning magnetic). As we will see in Sect. 14, the Z_2 orbifold theory has a realization in type 0A theory [16]. It lives on a brane configuration of type 0A, which consists of "electric" and "magnetic" D-branes. This is the origin of the sub/superscripts e and

m in Table 2. The Z_2 orbifold theory is obviously Z_2-symmetric under the exchange of all indices $e \to m$ and *vice versa*.

Let us divide all color-singlet operators of the Z_2 daughter in two classes: (i) those that are invariant (even) under the above Z_2 symmetry, and (ii) those that are non-invariant (odd). The operators from the second class are called twisted. For instance, $G_e^e G_e^e - G_m^m G_m^m$ is a twisted operator, whereas $G_e^e G_e^e + G_m^m G_m^m$ is untwisted.

The perturbative relation between the orbifold theory and its supersymmetric parent concerns only the untwisted sector [11,13]. The parent theory does not carry information about the twisted sector of the daughter theory. The NPO conjecture assumes that the vacuum of the daughter theory is Z_2-invariant. However, this need not be the case.

A possible sign of the Z_2 instability can be obtained just in perturbation theory. Indeed, let us assume that at some ultraviolet (UV) scale, where perturbation theory is applicable, the gauge couplings of the two SU(N) factors in the orbifold theory are slightly different. We will denote them by $8\pi^2/(Ng_e^2) = 1/\lambda_e$ and $8\pi^2/(Ng_m^2) = 1/\lambda_m$. It is not difficult to find the renormalization group flow of $\delta\lambda$ towards the infrared (IR) domain. As long as $\delta\lambda \ll \lambda_{e,m}$ we have

$$\frac{d(\delta\lambda)}{d\,\ln\mu} = -3\,\lambda^2\;(\delta\lambda) + \text{higher orders}\,. \tag{5.6}$$

Neglecting the weak logarithmic μ dependence of λ, we obtain

$$\delta\lambda(\mu) = \delta\lambda(\mu_0)\left(\frac{\mu_0}{\mu}\right)^{3\lambda^2}\,. \tag{5.7}$$

Even if $\delta\lambda$ is small in the UV, it grows exponentially towards the IR domain. A similar analysis of the conformal orbifold daughter of $\mathcal{N}=4$ SUSY theory was carried out in Ref. [28], and a similar conclusion about such IR instability was reached.

The advantage of the orientifold theory over the orbifold theory is the absence of the twisted sector. Moreover, the gauge groups are the same in the parent and daughter theories, and so are the patterns of the spontaneous breaking of the global symmetry and the numbers of vacua. This is also seen from the string-theory standpoint. The orbifold field theory originates from the type 0A string theory, which is obtained by a Z_2 orbifold of type IIA. The result is a bosonic string theory with a tachyon in the twisted sector. In contrast, the orientifold field theory originates from a configuration with an orientifold that removes the twisted sector (the result is very similar to the bosonic part of the type I string).

Another way to search for instabilities in the twisted sector was suggested in [18]. The logic is as follows. Since the perturbative planar equivalence between the SUSY parent and orbifold daughter does not depend on the geometry of space-time, one can compactify one or more dimensions, making the compactified dimension small, so that the theory in question becomes weakly coupled. Then one can compare the parent and the orbifold theories at weak coupling.

Tong considered [19] an $R^3 \times S$ compactification. On $R^3 \times S$, the third spatial component of the gluon field becomes a scalar field with a flat (vanishing) potential at the classical level, $(A_3)^i_j \to \phi^i_j$. (Alternatively, one can speak of the Polyakov line in the x_3 direction.) In the parent theory the flatness is perturbatively maintained to all orders owing to supersymmetry. Non-perturbatively, the flatness is lifted by $R^3 \times S$ instantons (monopoles), which generate a superpotential. As it turns out, in supersymmetric vacua the non-Abelian gauge symmetry is broken by $\langle \phi^i_j \rangle = v_i \delta^i_j$, down to $\mathrm{U}(1)^{2N}$, so that the theory is in the Coulomb phase [25].

In the daughter theory the situation is drastically different; a potential *is generated at one loop*, making the point of the broken gauge symmetry unstable. Shifting away from this point, for a trial, one finds oneself in the Z_2-non-invariant (twisted) sector, for which no planar equivalence exists, and which proves to be energetically favored in the orbifold theory on $R^3 \times S$. In this true vacuum the energy density is negative rather than zero, and the full gauge symmetry is restored. The daughter theory is in the confining phase. Obviously, there can be no equivalence.

6. Orientifold field theory and $\mathcal{N}=1$ gluodynamics

Having concluded that, regretfully, the planar equivalence of the orbifold daughters does not extend to the non-perturbative level, we move on to another class of theories, called *orientifolds*, which have lately given rise to great expectations. In this section we will argue that in the $N \to \infty$ limit there is a sector in the orientifold theory exactly identical to $\mathcal{N}=1$ SYM theory, and, therefore, exact results on the IR behavior of this theory can be obtained. This sector is referred to as the *common sector*.

The parent theory is $\mathcal{N}=1$ SUSY gluodynamics with gauge group $\mathrm{SU}(N)$. The daughter theory has the same gauge group and the same gauge coupling. The gluino field λ^i_j is replaced by two Weyl spinors $\eta_{[ij]}$ and $\xi^{[ij]}$, with two antisymmetrized indices. We can combine the Weyl spinors into one Dirac spinor, either $\Psi_{[ij]}$ or $\Psi^{[ij]}$. Note that the number of fermion degrees of freedom in $\Psi_{[ij]}$ is $N^2 - N$, as in the parent theory in the large-N limit. We call this daughter theory *orientifold A*.

There is another version of the orientifold daughter — *orientifold S*. Instead of the antisymmetrization of the two-index spinors, we can perform symmetrization, so that $\lambda^i_j \to (\eta_{\{ij\}}\,,\ \xi^{\{ij\}})$. The number of degrees of freedom in $\Psi_{\{ij\}}$ is N^2+N. The field contents of the orientifold theories is shown in Table 3. We will mostly focus on the antisymmetric daughter since it is of more physical interest; see Sect. 9.

Table 3. The field content of the orientifold theories. Here, η and ξ are two Weyl fermions, while A_μ stands for the gauge bosons. In the left (right) parts of the table the fermions are in the two-index symmetric (antisymmetric) representation of the gauge group SU(N). Moreover, $\mathrm{U}_V(1)$ is the conserved global symmetry while the $\mathrm{U}_A(1)$ symmetry is lost at the quantum level, because of the chiral anomaly. A discrete subgroup remains unbroken.

	SU(N)	$\mathrm{U}_V(1)$	$\mathrm{U}_A(1)$
$\eta_{\{ij\}}$	$\square\square$	1	1
$\xi^{\{ij\}}$	$\overline{\square\square}$	-1	1
A_μ	Adj	0	0

	SU(N)	$\mathrm{U}_V(1)$	$\mathrm{U}_A(1)$
$\eta_{[ij]}$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	1	1
$\xi^{[ij]}$	$\overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$	-1	1
A_μ	Adj	0	0

The hadronic (color-singlet) sectors of the parent and daughter theories are different. In the parent theory composite fermions with mass scaling as N^0 exist, and, moreover, they are degenerate with their bosonic SUSY counterparts. In the daughter theory any interpolating color-singlet current with the fermion quantum numbers (if it exists at all) contains a number of constituents growing with N. Hence, at $N=\infty$ the spectrum contains only bosons.

Classically the parent theory has a single global symmetry — an R symmetry corresponding to the chiral rotations of the gluino field. Instantons break this symmetry down to Z_{2N}. The daughter theory has, on top, a conserved anomaly-free current

$$\bar\eta_{\dot\alpha}\eta_\alpha - \bar\xi_{\dot\alpha}\xi_\alpha\,. \tag{6.1}$$

In terms of the Dirac spinor this is the vector current $\bar\Psi\gamma_\mu\Psi$. If the corresponding charge is denoted by Q, then necessarily $Q=0$ in the color-singlet bosonic sector. Thus, the only residual global symmetry in both theories is Z_{2N} spontaneously broken down to Z_2 by the respective bi-fermion condensates. This explains the existence of N vacua [e] in both cases.

[e] At finite N the parent theory has N vacua, while the orientifold daughters have $N-2$ and $N+2$

September 11, 2004 11:55 WSPC/Trim Size: 9.75in x 6.5in for Proceedings armoni

We will compare the bosonic sectors of the parent and daughter theories. Note that the part of the daughter theory's bosonic sector probed by operators of the type (6.1), which have no analogs in the parent theory, is inaccessible. Such a sector of the theory does not belong to the common sector.

6.1. *Perturbative equivalence*

Let us start from perturbative considerations. The Feynman rules of the planar theory are shown in Fig. 4.

The difference between the orientifold theory and $\mathcal{N}=1$ gluodynamics is that the arrows on the fermionic lines point in the same direction, since the fermion is in the antisymmetric representation, in contrast to the supersymmetric theory where the gaugino is in the adjoint representation and the arrows point in opposite directions. This difference between the two theories does not affect planar graphs, provided that each gaugino line is replaced by the sum of $\eta_{[..]}$ and $\xi^{[..]}$.

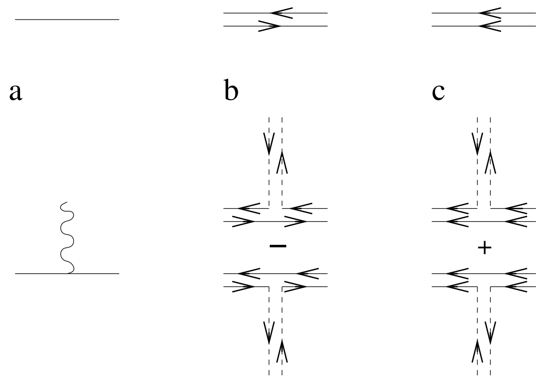


Figure 4. (a) The fermion propagator and the fermion-fermion-gluon vertex. (b) $\mathcal{N}=1$ SYM theory. (c) Orientifold daughter.

There is a one-to-one correspondence between the planar graphs of the two theories. Diagrammatically this works as follows (see, for example, Fig. 5). Consider any planar diagram of the daughter theory: by definition of planarity, it can be drawn on a sphere. The fermionic propagators form closed, non-intersecting loops that divide the sphere into regions. Each time we cross a fermionic line the orientation of color-index loops (each one

in the antisymmetric and symmetric versions, respectively.

producing a factor N) changes from clock to counter-clockwise, and *vice-versa*, as is graphically demonstrated in Fig. 5c. Thus, the fermionic loops allow one to attribute to each of the above regions a binary label (say ± 1), according to whether the color loops go clock or counter-clockwise in the given region. Imagine now that one cuts out all the regions with a -1 label and glues them again on the sphere after having flipped them upside down. We will get a planar diagram of the SYM theory in which all color loops go, by convention, clockwise. The number associated with both diagrams will be the same since the diagrams inside each region always contain an even number of powers of g, so that the relative minus signs of Fig. 4 do not matter.

In fact, in the above argument, we cut corners, so that the careful reader might get somewhat puzzled. For instance, in the parent theory gluinos are Weyl (Majorana) fermions, while fermions in the daughter theory are Dirac fermions. Therefore, we hasten to add a few explanatory remarks, which will, hopefully, leave the careful reader fully satisfied.

First, let us replace the Weyl (Majorana) gluino of $\mathcal{N}=1$ gluodynamics by a Dirac spinor Ψ^i_j. Each fermion loop in the original theory is then obtained from the Dirac loop by multiplying the latter by 1/2. Let us keep this factor 1/2 in mind.

On the daughter theory side, instead of considering the antisymmetric spinor $\Psi_{[ij]}$ *or* the symmetric one, $\Psi_{\{ij\}}$, we will consider a Dirac spinor in the *reducible* two-index representation Ψ_{ij}, without imposing any conditions on $i,\ j$. Thus, this reducible two-index representation is a sum of $\square\square + \begin{smallmatrix}\square\\\square\end{smallmatrix}$. It is rather obvious that at $N\to\infty$ any loop of $\Psi_{[ij]}$ yields the same result as the very same loop with $\Psi_{\{ij\}}$, which implies, in turn, that in order to get the fermion loop in, say, an antisymmetric orientifold daughter, one can take the Dirac fermion loop in the above reducible representation, and multiply it by 1/2.

Given the same factor 1/2 on the side of the parent and daughter theories, what remains to be done is to prove that the Dirac fermion loops for Ψ^i_j and Ψ_{ij} are identical at $N\to\infty$. We will therefore focus on the color factors.

Let the generator of SU(N) in the fundamental representation be T^a, while that in the antifundamental one will be $\bar{T}^a$,

$$T^a = T^a_{\square}, \qquad \bar{T}^a = T^a_{\bar{\square}}. \tag{6.2}$$

Then the generator in the adjoint representation is

$$T^a_{\rm Adj} \sim T^a_{\square\times\bar{\square}} = T^a_{\square}\otimes 1 + 1\otimes T^a_{\bar{\square}} \equiv T^a\otimes 1 + 1\otimes \bar{T}^a\,, \tag{6.3}$$

where we used the large-N limit to neglect the singlet. Moreover, in the daughter theory the generator of the reducible $\square \otimes \square$ representation can be written as

$$T^a_{\text{two-index}} = T^a_{\square} \otimes 1 + 1 \otimes T^a_{\square} \equiv T^a \otimes 1 + 1 \otimes T^a$$
$$\text{or} \quad \bar{T}^a \otimes 1 + 1 \otimes \bar{T}^a \,. \tag{6.4}$$

One more thing which we will need to know is the fact that

$$\bar{T} = -\tilde{T} = -T^* \,, \tag{6.5}$$

where the tilde denotes the transposed matrix.

Let us examine the color structure of a generic planar diagram for a gauge-invariant quantity. For example, Fig. 5a exhibits a four-loop planar graph for the vacuum energy. The color decomposition (6.3) and (6.4) is equivalent to using the 't Hooft double-line notation, see Figs. 5b,c. In the parent theory each fermion–gluon vertex contains T^a_{Adj}; in passing to the daughter theory we replace $T^a_{\text{Adj}} \to T^a_{\text{two-index}}$.

Upon substitution of Eqs. (6.3) and (6.4) the graph at hand splits into two (disconnected!) parts; the inner one (inside the dotted ellipse in Fig. 5c), and the outer one (outside the dotted ellipse in Fig. 5c). These two parts do not "talk to each other," because of planarity (large-N limit). The outer parts are the same in Figs. 5b,c. They are proportional to the trace of the product of two T's in the two cases

$$\mathrm{Tr}\,(\bar{T}^a \bar{T}^a) \,.$$

This is the first factor. The second one comes from the inner part of Figs. 5b,c. In the parent theory the inner factor is built out of 6 T's — one in each fermion–gluon vertex, and three T's in the three-gluon vertex $\mathrm{Tr}\,([A_\mu A_\nu]\,\partial_\mu A_\nu)$, where $A_\mu \equiv A^a_\mu T^a$. In the daughter theory the inner factor is obtained from that in the parent one by replacing all T's by $\bar{T}$'s. According to Eq. (6.5), $\bar{T} = -\tilde{T}$ (remember that a tilde denotes the transposed matrix). This fact implies that the only difference between the inner blocks in Figs. 5b,c is the reversal of the direction of the color flow on each of the 't Hooft lines. Since the inner part is a color-singlet by itself, the above reversal has no impact on the color factor — they are identical in the parent and daughter theories.

It is, perhaps, instructive to illustrate how this works using a more conventional notation. For the inner part of the graph in Fig. 5b we have the color factor $\mathrm{Tr}\,(T^a T^b T^c)\, f^{abc}$, while in the daughter theory we have

$\mathrm{Tr}\,(\bar{T}^a\bar{T}^b\bar{T}^c)\,f^{abc}$. Using the fact that

$$[T^aT^b] = if^{abc}T^c \quad \text{and} \quad [\bar{T}^a\bar{T}^b] = if^{abc}\bar{T}^c\,,$$

we immediately come to the conclusion that the above two expressions coincide.

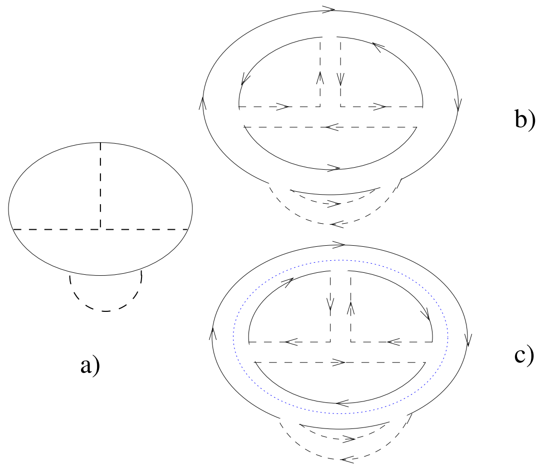


Figure 5. (a) A typical planar contribution to the vacuum energy. (b and c) The same in the 't Hooft notation for (b) $\mathcal{N}=1$ SYM theory; (c) orientifold daughter.

Thus, all perturbative results that we are aware of in $\mathcal{N}=1$ SYM theory apply in the orientifold model as well. For example, the β function of the orientifold field theory is

$$\beta = -\frac{1}{2\pi}\,\frac{3N\alpha^2}{1-(N\alpha)/(2\pi)}\left\{1+O\left(\frac{1}{N}\right)\right\}. \tag{6.6}$$

In the large-N limit it coincides with the $\mathcal{N}=1$ SYM theory result [27]. Note that the corrections are $1/N$ rather than $1/N^2$. For instance, the exact first coefficient of the β function is $-3N-4/3$ versus $-3N$ in the parent theory.

6.2. *Non-perturbative equivalence proof*

Now we will argue that the perturbative argument can be elevated to the non-perturbative level in the case at hand. A heuristic argument in favor of the non-perturbative equivalence is that the coincidence of all planar graphs of the two theories implies that the relevant Casimir operators of the two representations are equivalent in the large-N limit. The partition functions of the two theories depend on the Casimir operators and, therefore, must coincide as well.

A more formal line of reasoning is as follows. It is essential that the fermion fields enter bilinearly in the action, and that for any *given* gauge-field configuration in the parent theory there is *exactly the same* configuration in the daughter one. Our idea is to integrate out fermion fields for any *fixed* gluon-field configuration, which yields respective determinants, and then compare them.

Consider the partition function of $\mathcal{N}=1$ SYM theory,

$$\mathcal{Z}_0 = \int \mathcal{D}A\, \mathcal{D}\lambda\, \exp\left(iS[A,\lambda,J]\right)\,, \tag{6.7}$$

where J is any source coupled to color-singlet gluon operators. (Appropriate color-singlet fermion bilinears can be considered too.)

For any given gluon field, upon integrating out the gaugino field, we obtain

$$\mathcal{Z}_0 = \int \mathcal{D}A\, \exp\left(iS[A,J]\right) \det\left(i\, \partial\!\!\!/ + A\!\!\!/^a T^a_{\rm Adj}\right)\,, \tag{6.8}$$

where $T^a_{\rm Adj}$ is a generator in the adjoint representation.

If one integrates out the fermion fields in the non-supersymmetric orientifold theory, at fixed A, one arrives at a similar expression, but with the generators of the antisymmetric (or symmetric) representation instead of the adjoint, $T^a_{\rm Adj} \to T^a_{\rm asymm}$ or $T^a_{\rm symm}$.

To compare the fermion determinants in the parent and daughter theories (assuming that the gauge field configuration $A^a_\mu(x)$, $a = 1,...,N^2$, is the same and fixed), we must cast both fermion operators in similar forms. To this end we will extend both theories. In the parent one we introduce a second adjoint Weyl fermion ξ^i_j and combine two Weyl fermions into one Dirac adjoint fermion Ψ^i_j. The determinant in this extended theory is the square of the original one.

In the daughter theory, instead of $\Psi^{[ij]}$, we will work with the reducible representation, combining both symmetric and antisymmetric into the Dirac field Ψ^{ij} (no (anti)symmetrization over the color indices). Then the number of fermion Dirac fields, N^2, is the same as in the extended parent. Again, as in the previous case, the determinant in the extended daughter theory is the square of the original one. To avoid zero modes, which might make determinants ill-defined, we must impose an IR regularization. To this end we will introduce small fermion mass terms, which will render determinants well-defined. As usual, it is assumed that the vanishing mass limit is smooth, which must be the case in the confining regime.

Since the theory, being vector-like, is anomaly free, the determinant in Eq. (6.8) is a gauge-invariant object and, thus, can be expanded in Wilson-loops operators

$$\mathcal{W}_{\mathcal{C}}[A_{\rm Adj}] = {\rm Tr}\, P\, \exp\left(i \int_{\mathcal{C}} A^a_\mu T^a_{\rm Adj}\; dx^\mu \right) . \tag{6.9}$$

For other representations, the Wilson-loop operator is defined in a similar manner. Thus, one can write

$$D \equiv \det\left(i\, \not\partial + \not A^a T^a_{\rm Adj} - m \right) = \sum_{\mathcal{C}} \alpha_{\mathcal{C}}\, \mathcal{W}_{\mathcal{C}}[A_{\rm Adj}] \,. \tag{6.10}$$

Equation (6.3) then implies that

$$D = \sum_{\mathcal{C}} \alpha_{\mathcal{C}}\, {\rm Tr}\, P\, \exp\left(i \int_{\mathcal{C}} A^a_\mu \left(T^a \otimes 1 + 1 \otimes \bar T^a \right) dx^\mu \right) . \tag{6.11}$$

Moreover, since the commutator is such that

$$[(T^a \otimes 1),\; (1 \otimes \bar T^a)] = 0 \,,$$

the determinant (6.11) can be rewritten as

$$D = \sum_{\mathcal{C}} \alpha_{\mathcal{C}}\, {\rm Tr}\, P\, \exp\left(i \int_{\mathcal{C}} A^a_\mu T^a\, dx^\mu \right)\, {\rm Tr}\, P\, \exp\left(i \int_{\mathcal{C}} A^a_\mu \bar T^a\, dx^\mu \right) . \tag{6.12}$$

As a result, the partition function takes the form

$$\mathcal{Z}_0 = \sum_{\mathcal{C}} \alpha_{\mathcal{C}}\, \langle\, \mathcal{W}_{\mathcal{C}}[A_\square]\; \mathcal{W}^*_{\mathcal{C}}[A_\square]\, \rangle \,. \tag{6.13}$$

One of the two most crucial points of the proof is the applicability of factorization in the large-N limit,

$$\mathcal{Z}_0 = \sum_{\mathcal{C}} \alpha_{\mathcal{C}}\, \langle\, \mathcal{W}_{\mathcal{C}}(A_\square)\, \rangle\, \langle\, \mathcal{W}^*_{\mathcal{C}}(A_\square)\, \rangle = \sum_{\mathcal{C}} \alpha_{\mathcal{C}}\, \langle\, \mathcal{W}_{\mathcal{C}}(A_\square)\, \rangle^2 \,. \tag{6.14}$$

In the second equality in Eq. (6.14) we used the second most crucial point, the reality of the Wilson loop,

$$\langle\, \mathcal{W}_{\mathcal{C}}\, \rangle = \langle\, \mathcal{W}^*_{\mathcal{C}}\, \rangle \,. \tag{6.15}$$

The partition function (6.14) is *exactly* the same as that obtained in the (extended) orientifold theory upon exploiting Eq. (6.4), factorization,

$$\mathcal{Z}_{\rm orientifold} = \sum_{\mathcal{C}} \alpha_{\mathcal{C}}\, \langle\, \mathcal{W}_{\mathcal{C}}(A_\square)\, \rangle\, \langle\, \mathcal{W}_{\mathcal{C}}(A_\square)\, \rangle \,, \tag{6.16}$$

and a third ingredient; independence of the expansion coefficients $\alpha_{\mathcal{C}}$ of the fermion representation.

At this point we can take the square root of the determinants of the two extended theories. Owing to the non-vanishing mass, the determinants do not vanish and no sign ambiguities arise. In the parent theory we recover the (softly broken) super-Yang–Mills determinant, while in the daughter theory, given the planar equivalence of the symmetric and antisymmetric representations, we recover either one of them. We finally take the (supposedly) smooth massless limit and, thus, prove our central result.

Let us ask, *"What is the critical difference between the orbifold and orientifold theories, which makes the elevation of the perturbative planar equivalence possible to the non-perturbative level in the latter case, while this does not work in the former?"* The answer is quite clear. In the orientifold theories one can choose the very same gauge field configurations as in $\mathcal{N}=1$ gluodynamics, one by one, and compare the fermion determinants in the given gauge field background. This is due to the fact that the gauge groups are the same in the parent and daughter theories. In the orbifold theories the gauge group is different from that of the parent ($\mathrm{SU}(N)^k$ vs. $\mathrm{SU}(kN)$). The presence of the twisted sector makes such a strategy impossible.

6.3. *Special case: one-flavor QCD*

Let us focus our attention on the orientifold A daughter. Why does it seem to be of more practical importance than orientifold S? Let us assume that $1/N$ corrections are manageable. We will discuss them in more detail in Sect. 9. Here, anticipating this discussion, it will be sufficient to assume that nothing dramatic happens on the way from $N=\infty$ down to $N=3$, no phase transition, for instance. This is a pretty natural assumption implying that the dynamical regime of the orientifold A theory does not qualitatively change in the descent from $N=\infty$ down to $N=3$.[f]

At $N=3$ the two-index antisymmetric Dirac field is *identical* to the Dirac field in the fundamental representation. This is nothing but a standard *quark* field. Thus, we arrive at a correspondence between $\mathcal{N}=1$ gluodynamics and one-flavor QCD! Although one-flavor QCD is still not *the* theory of our world, it is a very close relative into which QCD theorists and lattice practitioners have a rich insight. One-flavor QCD is known to be a confining theory with a mass gap. Therefore, we can reverse the direction of our consideration and

[f] If for some reasons beyond our grasp there were a phase transition in N *en route*, this would be a remarkable discovery by itself.

state that $\mathcal{N}=1$ gluodynamics enjoys the same features — confinement and a mass gap. Certainly, this was a general belief in the past. The status of this statement can now be elevated; we can turn a folklore statement into a motivated argument.

6.4. *Summary of the main results*

To conclude this key section we briefly summarize its main results. We have shown that the partition functions of the orientifold field theory and $\mathcal{N}=1$ SYM theory coincide at large N. All color-singlet correlation functions, which involve gluons as sources, are the same in the parent and daughter theories. This statement is also valid for the fermion-bilinear sources that can be projected onto each other in both theories. As a consequence, the glueball spectrum in the daughter theory is identical to that of SUSY gluodynamics. In particular, there is a degeneracy between $0^{\pm}$ glueballs. This degeneracy of the bosonic spectrum extends much further: all mesons originating from one and the same supermultiplet in the parent theory are inherited by the daughter theory, and their mass degeneracy is part of this inheritance. There is an *infinite number* of such "unexpected" degeneracies in the orientifold theories at $N=\infty$. At the same time, unlike $\mathcal{N}=1$ gluodynamics, the orientifold daughter theories have no color-singlet fermions at $N=\infty$. Finally, at $N=3$, the orientifold A theory is identical to one-flavor massless QCD.

7. Non-supersymmetric parent–daughter pairs

A simple proliferation of the fermion fields in the form of "flavors" leads to non-supersymmetric parent–daughter pairs. This was mentioned in passing in Ref. [18] in the context of Z_2 orbifolds, while in the orientifold field theories the issue of non-supersymmetric planar-equivalent pairs was raised in Ref. [6] and analyzed in [29]. For instance, gauge theories with the same number of Dirac fermions either in the antisymmetric two-index or symmetric two-index representations are planar-equivalent.

Here we would like to discuss in more detail parent–daughter pairs which are obtained from $\mathcal{N}=1$ gluodynamics and its orientifold A daughter by introducing fermion replicas. Thus, as previously, the parent and daughter theories share one and the same gauge group, SU(N), and one and the same gauge coupling.[g] The two respective fermion sectors are:

[g] As previously, at large N we disregard the distinction between SU(N) and U(N).

(i) k species of the Weyl fermions in the adjoint, to be denoted as $\left(\lambda^A\right)^i_j$;
(ii) k species of the Dirac fermions $\left(\Psi^A\right)_{[ij]}$ or $\left(\Psi^A\right)^{[ij]}$.

Here i, j are (anti)fundamental indices running $i, j = 1, 2, ..., N$, while A is the flavor index running $A = 1, 2, .., k$. Note that $k \leq 5$. Otherwise we loose asymptotic freedom. Each Dirac fermion is equivalent to two Weyl fermions,

$$\Psi_{[ij]} \to \{\eta_{[ij]}\,, \ \xi^{[ij]}\}\,.$$

Of course, since both theories are non-supersymmetric, the predictive power is significantly reduced with respect to the case of the SUSY parent. Still, one can benefit from the comparison of both theories, in particular, the Nambu–Goldstone boson sectors. Of some interest is also the comparison of the respective Skyrme models. We will say a few words on that at the end of the section.

Let us start with the case (ii), k species of $\Psi_{[ij]}$. Since the fermion fields are Dirac and belong to the complex representation of the gauge group, the theory has the same (non-anomalous) chiral symmetry as QCD with k flavors, namely, $\mathrm{SU}(k)_L \times \mathrm{SU}(k)_R$. Various arguments tell us [30–32] that the pattern of the chiral symmetry breaking is the same as in QCD too, namely

$$\mathrm{SU}(k)_L \times \mathrm{SU}(k)_R \to \mathrm{SU}(k)_V\,. \tag{7.1}$$

The only distinction is that in QCD the constant f scales as $\sqrt{N}\,\Lambda$ while in our case its scaling law is $N\,\Lambda$. Moreover, the coefficient n in front of the Wess–Zumino–Novikov–Witten term Γ (see e.g. Ref. [33]) equals N in QCD and $(1/2)N(N-1)$ in the case at hand (see below).

All axial (non-anomalous) currents are spontaneously broken, giving rise to $k^2 - 1$ Nambu–Goldstone "mesons." Some of them — those coupled to the axial currents that can be elevated from the daughter theory (ii) to the parent theory (i) — persist in the parent theory (i), where the fermion fields belong to the real representation. In these channels the couplings of the "parent" Nambu–Goldstone bosons to the corresponding axial currents are the same as in the daughter theory. This is because of the planar equivalence of two theories in the common sector.

A crucial point is that not all axial currents can be elevated from the daughter to the parent theory. Remember, say, that the currents with the structure $\bar\xi\xi - \bar\eta\eta$ cannot be elevated from (ii) to (i). Therefore, in comparing these two theories, one must identify such currents and exclude them from the parent theory.

It is not difficult to count the number of the axial currents that are elevated from (ii) to (i); there are $(1/2)k(k-1)$ off-diagonal currents of the type $\bar{\xi}^A\xi^B + \bar{\eta}^A\eta^B$ $(A \neq B)$ plus $(k-1)$ diagonal axial currents of the type $\sum c_A(\bar{\xi}^A\xi^A + \bar{\eta}^A\eta^A)$, with $\sum c_A = 0$ and $C_1 = 1$. Altogether, we get

$$\frac{k(k+1)}{2} - 1$$

Nambu–Goldstone bosons. This corresponds to the following pattern of chiral symmetry breaking in the parent theory:

$$\mathrm{SU}(k) \to \mathrm{SO}(k)\,, \tag{7.2}$$

with the Nambu-Goldstone bosons in the symmetric and traceless two-index representation of $\mathrm{SO}(k)$.

The pattern of the chiral symmetry breaking for the quarks belonging to real representations of the gauge group indicated in Eq. (7.2) was advocated many times in the literature [30–32, 34], but no complete proof was ever given.

We conclude this section by a brief comment on the topological properties of the chiral Lagrangian, corresponding to the above patterns of the chiral symmetry breaking, and how they match the underlying gauge field theory expectations. The theory (i) is confining and supports flux tubes — fundamental color charges cannot be screened. On the other hand, we do not expect stable baryons with mass growing with N. Color-singlet states composed of gluons and $\left(\lambda^A\right)^i_j$ form baryons with $M \sim N^0$.

At the same time, the theory (ii) does *not* have baryons with $M \sim N^0$. Here the baryon masses grow with N. The theory is expected to be confining too, but two flux tubes (each attached to a color source in the fundamental representation) can be screened [h] by $\left(\Psi^A\right)_{[ij]}$.

As was suggested in Refs. [33, 34], at large N one can try to identify baryons with the Skyrmions supported by the corresponding chiral Lagrangians. Since

$$\pi_3\left\{\mathrm{SU}(k)/\mathrm{O}(k)\right\} = Z_4 \;\text{ at }\; k=3, \quad \pi_3\left\{\mathrm{SU}(k)/\mathrm{O}(k)\right\} = Z_2 \;\text{ at }\; k \geq 4,$$
$$\pi_3\left\{\mathrm{SU}(k)\right\} = Z \;\text{ at all }\; k\,, \tag{7.3}$$

[h] Although two fundamental strings put together can break via creation of a pair $\Psi_{[ij]} + \overline{\Psi}_{[ij]}$, the breaking is suppressed by $1/N$ factors and does not occur at $N = \infty$. For a recent review see [35].

both theories, (i) and (ii), yield Skyrmions with $M_{\rm Skyrme} \sim N^2$, albeit the theory (ii) has a richer spectrum. The above scaling law, $M_{\rm Skyrme} \sim N^2$, is due to the fact that f scales as N in the theories under consideration.

The Skyrmion statistics is determined by the (quantized) factor in front of the Wess–Zumino–Novikov–Witten term,

$$(-1)^{N(N-1)/2}\,. \tag{7.4}$$

It has half-integer spin provided that $N(N-1)/2$ is odd, i.e. $N = 4p + 2$ or $N = 4p + 3$ where p is an integer. In both cases one can construct, in the microscopic theory, interpolating baryon currents with an odd number of constituents scaling as N. Why then does the Skyrmion mass scale as N^2?

A possible explanation is as follows. For quarks in the fundamental representation of SU(N) the color wave function is antisymmetric, which allows all of them to be in an S wave in coordinate space. With antisymmetric two-index spinor fields, the color wave function is symmetric, which would require them to occupy orbits with angular momentum up to $\sim N$. Then the scaling law $M_{\rm Skyrme} \sim N^2$ seems natural.

Since $\pi_2\,\{\mathrm{SU}(k)\} = 0$ the chiral sector of the theory (ii) does not support flux tubes. Though disappointing, such a situation was anticipated by Witten [34] who noted that the topology of the full space of the large-N theory need not coincide with the topology of its Nambu–Goldstone sector.

In the light of this remark we can understand the complete failure of the Skyrmion description of theory (i). In particular, since $\pi_2\,\{\mathrm{SU}(k)/\mathrm{O}(k)\} = Z_2$, at $k \geq 3$ we have flux tubes in the chiral theory, while we do not expect them in the microscopic theory. Moreover, stable Skyrmions of the chiral sector should become unstable in the full theory.

It remains to be seen whether meaningful predictions regarding QCD baryons can be obtained from the large-N limit of the orientifold theory with two or three flavors.

8. Orientifold large-N expansion

8.1. *General features ($N_f > 1$ fixed, N large)*

We will now abandon for a while the topic of planar equivalence, and look at the SU(N) orientifold theories with N_f flavors discussed in Sect. 7 from a more general perspective. We will focus on the antisymmetric orientifold theories, assuming that N_f does not scale with N at large N, say, $N_f = 1$, 2 or 3. We will return to the issue of planar equivalence later.

As was noted in Sect. 6.3, if $N = 3$ (i.e. if the gauge group is SU(3)) the two-index antisymmetric quark is identical to the standard quark in the fundamental representation. Therefore, it is quite obvious that extrapolation to large N, with the subsequent $1/N$ expansion, can have distinct starting points: (i) quarks in the *fundamental* representation; (ii) quarks in the *two-index antisymmetric* representation; (iii) a combination thereof. The first option gives rise to the standard 't Hooft $1/N$ expansion [12, 36], while the second and third lead to a new expansion [7,8], to which we will refer as the *orientifold large-N expansion*.[i]

The 't Hooft expansion is the simplest and the oldest $1/N$ expansion to have been designed specifically for QCD. The 't Hooft limit assumes N to be large, while keeping the 't Hooft coupling (3.6) fixed. If the number of quark flavors is fixed too (i.e. does not scale with N) then each quark loop is suppressed by $1/N$. Only quenched planar diagrams (i.e. those with no quark loops) survive in leading order in $1/N$. Non-planar diagrams with "handles" are suppressed by $1/N^2$ per handle. Each extra quark-loop insertion in any planar graph costs (N_f/N). Thus, the corrections to the leading approximation run in powers of (N_f/N) and $1/N^2$.

The 't Hooft expansion enjoyed a significant success in phenomenology. It provided a qualitative explanation for the well-known regularities of the hadronic world, first and foremost the Zweig rule, a relative smallness of the meson widths, the rarity of the four-quark mesons and so on. We hasten to add, though, that, except in two dimensions ($D = 2$), nobody succeeded in getting *quantitative* results for QCD, even to leading order in the $1/N$ expansion.

Although the standard large-N ideology definitely captures some basic regularities, it gives rise to certain puzzles, as far as subtle details are concerned. Indeed, in the 't Hooft expansion, the width of $q\bar{q}$ mesons scales as $1/N$, while that of glueballs scales as $1/N^2$. In other words, the latter are expected to be narrower than quarkonia, which is hardly the case in reality. No glueballs have been reliably identified so far, in spite of decades of searching.

Moreover, the Zweig rule is not universally valid. It is known to be badly

[i] It is curious that Corrigan and Ramond suggested [37] replacing the 't Hooft model by a model with one two-index antisymmetric quark $\Psi_{[ij]}$ and two fundamental ones $q^i_{1,2}$, as early as 1979. Their motivation originated from some awkwardness in the treatment of baryons in the 't Hooft model, where all baryons, being composed of N quarks, have masses scaling as N and thus disappear from the spectrum at $N \to \infty$. If the fermion sector contains $\Psi_{[ij]}$ and $q^i_{1,2}$, then, even at large N, there are three-quark baryons of the type $\Psi_{[ij]}\, q^i_1\, q^j_2$, which apparently realize a smoother extrapolation from $N = 3$ to large N than the one in 't Hooft's model.

violated for scalar and pseudoscalar mesons. Another example of this type, thoroughly discussed in Ref. [38], refers to the quark dependence of the vacuum energy. In the 't Hooft limit the vacuum energy density is obviously independent of the quark mass, since all quark loops die out. At the same time, an estimate based on QCD low-energy theorems tells us [38] that changing the strange-quark mass from ~ 150 MeV to zero would roughly double the value of the vacuum energy density.

It is clear that the 't Hooft expansion underestimates the role of quarks. This was noted long ago, and a remedy was suggested [39], a *topological expansion*. The topological expansion (TE) assumes that the number of flavors N_f scales as N in the large-N limit, so that the ratio N_f/N is kept fixed.

The graphs that survive in the leading order of TE are *all* planar diagrams, including those with the quark loops. This is easily seen by slightly modifying [39] the 't Hooft double-line notation — adding a flavor line to the single color line for quarks. In the leading (planar) diagrams the quark loops are "empty" inside, since gluons do not attach to the flavor line. Needless to say, obtaining analytic results in TE is even harder than in the 't Hooft case.

The orientifold large-N expansion opens the way for a novel and potentially rich large-N phenomenology in which the quark loops (i.e. dynamical quarks) do play a non-negligible role. An additional bonus is that in the orientifold large-N expansion, one-flavor QCD gets connected to supersymmetric gluodynamics, potentially paving the way to a wealth of predictions.

In order to demonstrate the difference between the standard large-N expansion and the orientifold large N expansion we exhibit a planar contribution to the vacuum energy in two ways in Fig. 6. Moreover, we will illustrate the usefulness of the orientifold large-N expansion at the qualitative, semi-quantitative and quantitative levels by a few examples given below.

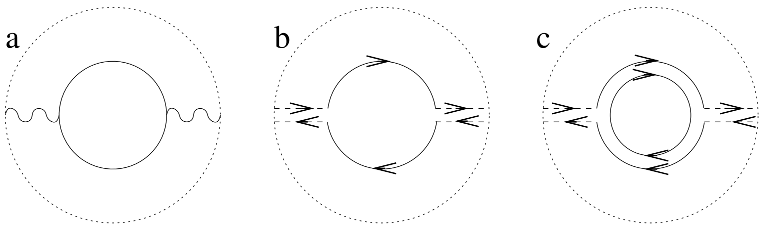


Figure 6. (a.) A typical contribution to the vacuum energy. (b.) The planar contribution in 't Hooft large-N expansion. (c.) The orientifold large-N expansion. The dotted circle represents a sphere so that every line hitting the dotted circle gets connected "on the other side."

Probably the most notable distinctions from the 't Hooft expansion are as follows: (i) the decay widths of *both* glueballs and quarkonia scale with N in a similar manner, as $1/N^2$; this can easily be deduced by analyzing appropriate diagrams with the quark loops of the type displayed in Fig. 6b; (ii) unquenching quarks in the vacuum produces an effect that is not suppressed by $1/N$; in particular, the vacuum energy density does depend on the quark masses in the leading order in $1/N$.

8.2. *Qualitative results for one-flavor QCD from the orientifold expansion*

In this subsection we list some predictions for one-flavor QCD, keeping in mind that they are expected to be valid up to corrections of the order of $1/N = 1/3$ (barring large numerical coefficients).

(i) Confinement with a mass gap is a common feature of one-flavor QCD and SUSY gluodynamics.

(ii) Degeneracy in the color-singlet bosonic spectrum. Even/odd parity mesons (typically mixtures of fermionic and gluonic color-singlet states) are expected to be degenerate. In particular,

$$\frac{m^2_{\eta'}}{m^2_\sigma} = 1 + O(1/N)\,, \qquad \text{one-flavor QCD}\,, \tag{8.1}$$

where η' and σ stand for the lightest 0^- and 0^+ mesons, respectively. This follows from the exact degeneracy in $\mathcal{N}=1$ SYM theory. Note that the σ meson is stable in this theory, as there are no light pions for it to decay into.

The prediction (8.1) should be taken with care: a rather large numerical coefficient in front of $1/N$ is not at all ruled out, since the η' mass is anomaly-driven (the Witten–Veneziano (WV) formula [40, 41]), whereas the σ mass is more "dynamical".

In more general terms the mass degeneracy is inherited by all those "daughter" mesons that fall into one and the same supermultiplet in the parent theory. The accuracy of the spectral degeneracy is expected to improve at higher levels of the Regge trajectories, as $1/N$ corrections that induce splittings are expected to fall off (see e.g. [39]).

(iii) Bifermion condensate. $\mathcal{N} = 1$ SYM theory has a bifermion condensate. Similarly we predict a condensate in one-flavor QCD. A detailed calculation is discussed in Sect. 9.

(iv) One can try to get an idea of the size of $1/N$ corrections from perturbative arguments. In one-flavor QCD the first coefficient of the β function is $b = 31/3$, while in adjoint QCD with $N_f = 1$ it becomes $b = 27/3$. One can go beyond one loop too. As was mentioned, in the very same approximation the β function of the one-flavor QCD coincides with the exact NSVZ β function,

$$\beta = -\frac{1}{2\pi}\,\frac{9\alpha^2}{1-3\alpha/2\pi}\,. \tag{8.2}$$

Thus, for the (relative) value of the two-loop β-function coefficient, we predict $+3\alpha/2\pi$, to be compared with the exact value in one-flavor QCD,

$$+\frac{134}{31}\,\frac{\alpha}{2\pi} \approx 4.32\,\frac{\alpha}{2\pi}\,.$$

We see that the orientifold large-N expansion somewhat overemphasizes the quark-loop contributions, and, thus, makes the theory less asymptotically free than in reality — the opposite of what happens in 't Hooft's expansion. Parametrically, the error is $1/N$ rather than $1/N^2$. This is because there are N^2-1 gluons and N^2-N fermions in the orientifold field theory.

If one-flavor QCD is related, through the planar equivalence, to SYM theory, while the 't Hooft $1/N$ expansion connects it with pure Yang–Mills theory, it is tempting to combine both to ask a seemingly heretical question: "Are there any traces of supersymmetry in *pure Yang–Mills theory*?" We try to answer this question in Appendix A, which presents some evidence for a "residual SUSY" in pure Yang–Mills theory.

8.3. *An estimate of the conformal window in QCD from the orientifold large-N expansion*

One can try to use the orientifold large-N expansion for an estimate of the position of the *lower* edge of the conformal window. The logic behind this analysis is as follows. Since this expansion overestimates the role of the quark loops, in leading order in $1/N$ one can expect to obtain a lower boundary on the onset of the conformal regime, which, hopefully, will be close to the actual lower edge of the conformal window. The 't Hooft large-N limit [36] is not suitable for this purpose at all, since in this limit the quark effects disappear altogether.

We start from a brief summary of the issue of the conformal window in QCD. The standard definition of the coefficients of the β function from the

Particle Data Group (PDG) is[j]

$$\mu\frac{\partial\alpha}{\partial\mu}\equiv\beta(\alpha)=-\frac{\beta_0}{2\pi}\alpha^2-\frac{\beta_1}{4\pi^2}\alpha^3+... \tag{8.3}$$

(see Ref. [26]). The first two coefficients of the Gell-Mann–Low function in QCD have the form [26]

$$\beta_0=11-\frac{2}{3}N_f\,,\ \ \beta_1=51-\frac{19}{3}N_f\,, \tag{8.4}$$

where N_f is the number of quark flavors, all taken to be massless. At small α_s the β function is negative (asymptotic freedom!) since the $O(\alpha_s^2)$ term dominates. With the scale μ decreasing the running gauge coupling constant grows, and eventually the second term becomes important. The second term takes over the first one at $\alpha_s/\pi\sim 1$, when all terms in the α_s expansion are equally important, i.e. in the strong coupling regime. We have no firm knowledge regarding what happens at $\alpha_s/\pi\sim 1$.

Assume, however, that for some reason the first coefficient β_0 is abnormally small, and this smallness does not propagate to higher orders. Then the second term catches up with the first one when $\alpha_s/\pi\ll 1$: we are at weak coupling, and higher-order terms are inessential. Inspection of Eq. (8.4) shows that this happens when N_f is close to 33/2, say 16 or 15 (N_f has to be less than 33/2 to ensure asymptotic freedom). For these values of N_f the second coefficient β_1 turns out to be negative!

This means that the β function develops a zero at weak coupling, at

$$\frac{\alpha_*}{2\pi}=\frac{\beta_0}{-\beta_1}\ll 1\,. \tag{8.5}$$

For instance, if $N_f=15$ or 16 the critical values are at

$$\lambda_*\equiv\frac{N\alpha_*}{2\pi}=\frac{3}{44}\,,\quad N_f=15;\qquad \lambda_*=\frac{3}{151}\,,\quad N_f=16, \tag{8.6}$$

where in the expression above $N=3$. At such values of α, the Gell-Mann–Low function vanishes, i.e. $\beta(\alpha_*)=0$. This zero is nothing but the infrared fixed point of the theory. At large distances $\alpha_s\to\alpha_s^*$, and $\beta(\alpha_s^*)=0$, implying that the trace of the energy-momentum tensor vanishes. Then the theory is in the conformal regime. There are no localized particle-like states in the spectrum of this theory; rather we deal with massless unconfined interacting "quarks and gluons." All correlation functions at large distances

[j] With respect to the PDG, we dropped a factor 2 in the middle equality, $2\beta(\alpha)_{\mathrm{PDG}}\to\beta(\alpha)$, which is neither conventional nor convenient.

exhibit a power-like behavior. In particular, the potential between two heavy static quarks at large distances R will behave as $\sim \alpha_s^*/R$, a pure Coulomb behavior. The situation is not drastically different from conventional QED. As long as α_s^* is small, the interaction of the massless quarks and gluons in the theory is weak at all distances, short and large, and is amenable to the standard perturbative treatment (renormalization group, etc.). Chiral symmetry is not broken spontaneously, quark condensates do not develop, and QCD becomes a fully calculable theory. The fact that, at N_f close to 16, QCD becomes conformal and weakly coupled in the infrared limit has been known for about 30 years [42, 43]. If the conformal regime takes place at $N_f = 16$ and 15, how far can we descend in N_f without ruining the conformal nature of the theory in the infrared? Certainly, when N_f is not close to 16, the gauge coupling α_s^* is not small. The theory is *strongly coupled*, like conventional QCD, and, simultaneously, it is conformal in the infrared. Presumably, chiral symmetry is unbroken, $\langle \bar{q}q \rangle = 0$. The vacuum structure is totally different, and so are all properties of the theory.

The range of N_f where this phenomenon occurs is called the *conformal window*. The right edge of the conformal window is at $N_{f<} = 16$. According to our conjecture, the orientifold large-N expansion allows one to get an idea of the left edge of the conformal window, $N_{f>}$, which happens to lie not too far from actual QCD (i.e. not too far from $N_f = 3$)!

For antisymmetric quark "flavors"

$$\beta_0 = \frac{11}{3}N - \frac{2}{3}NN_f + \frac{4}{3}N_f\,, \tag{8.7}$$

$$\beta_1 = \frac{17}{3}N^2 - \frac{5}{3}N(N-2)N_f - \frac{(N-2)^2(N+1)}{N}N_f\,. \tag{8.8}$$

We will consider the limit $N \to \infty$. At $N_f = 6$, asymptotic freedom is lost. At $N_f = 5$ we have a conformal infrared fixed point with

$$\lambda_* \equiv \frac{N\alpha_*}{2\pi} = \frac{1}{23}\,. \tag{8.9}$$

The above assertion seems to be solid, since λ_* is small enough, cf. Eq. (8.5). At $N_f < 5$ the position of the would-be infrared fixed point shifts to rather large values of λ_* (namely, $\lambda_* = 1/5$), and is thus unreliable.

Interpolating down to the actual value of the number of quark colors, $N = 3$, in the most naive manner, we would conclude that the left edge of the conformal window is around $N_f = 5$.

How does this compare with other theoretical and/or numerical data? First, in the instanton liquid model it was found [44] that the chiral condensate disappears at $N_f = 5$, $\langle \bar{q}q \rangle = 0$. Although nothing is said about the onset of the conformal regime in this work, the vanishing of $\langle \bar{q}q \rangle$ may be a signal. It is believed that color confinement in QCD implies chiral symmetry breaking [45]. More evidence in favor of $N_f = 5$ arises from the condition $\gamma(N_f) = 1$ [46].

More definite is the conclusion of lattice investigations [47]. It is claimed that at $N_f = 7$ not only does the quark condensate disappear, but the decay law of correlation functions at large distances changes from exponential to power-like, i.e. the theory switches from the confining regime to the conformal one.

Of course, at $N_f = 3$ we know from experiment that QCD is in the confining rather than conformal regime. From the above argument, it looks likely that going up from $N_f = 3$ to $N_f = 5$ or so — quite a modest variation in the number of massless quarks — one drastically changes the vacuum structure of the theory, with the corresponding abrupt change of the picture of the hadron world. Let us emphasize again that this phenomenon would be totally missed by 't Hooft's expansion.

An interesting question is what happens slightly below the left edge of the conformal window, i.e. say at $N_{f>} - 1$. If N_f were a continuous parameter, and the phase transition in N_f were of the second order, slightly below $N_{f>}$ the string tension σ would be parametrically small in its natural scale given by Λ^2. In reality N_f changes discretely. Still it may well happen that at $N_f = N_{f>} - 1$ the ratio σ/Λ^2 is numerically small. This would mean that the string and its excitations, whose scale is set by σ, are abnormally light. Under the circumstances one might hope to build an effective low-energy approach analogous to the chiral Lagrangians of actual QCD. In the limit $\sigma/\Lambda^2 \to 0$ the string becomes, in a sense, classical; quantum corrections are unimportant. We do not rule out that the string representation of QCD — the Holy Grail of two generations of theorists — is easier to construct in this limit. One could even dream of an expansion in $N_{f>} - 3$. At the moment, this is pure speculation, though.

8.4. *Orientifold field theories in two and three dimensions*

The large-N equivalence of gauge theories with adjoint matter and anti-symmetric (or symmetric) matter is not restricted to four dimensions. The equivalence holds in any number of dimensions.

Let us consider first the two-dimensional case. We assert that planar two-dimensional Yang–Mills theory with fermions in the adjoint representation (this theory is *not* supersymmetric) is equivalent to planar two-dimensional

QCD with fermions in the antisymmetric representation.

When the fermions are massless, the equivalence of the two theories can be obtained (besides our derivation, see Sect. 6.2) from the universality theorem of Kutasov and Schwimmer [48]. The theorem states that physics of massless two-dimensional QCD depends on the total chiral anomaly, but not on the specific representation of the fermions. The anomaly in two dimensions is dictated by the vacuum polarization diagram, $\mathrm{Tr}\, T^a T^b = T(R)\delta^{ab}$. For the adjoint representation, $T = N$, whereas for the antisymmetric representation $T = N - 2$. At large N the two anomalies coincide and, hence, so do physics of the two theories.

The large-N equivalence extends further; for instance, it takes place for $m \neq 0$, where m is the fermion mass term. In this case we have two dimensionful parameters, m and g^2, and one dimensionless, N. The most interesting physical question one can ask is the rate of convergence of the $1/N$ expansion.

Of course, this question is of great interest in four dimensions too. It seems that it will be much easier and faster to answer it first in two (or, perhaps, three) dimensions. Indeed, two-dimensional Yang-Mills theory with fermions in the adjoint representation can be readily studied on lattices, by methods of discrete light-cone quantization (DLCQ) and, to some extent, analytically. In fact, it was studied quite extensively in this way in the 1990's, see Refs. [49]– [54]. Now these analyses can be repeated with two-index antisymmetric (or symmetric) fermions, with the aim of studying the rate of convergence of the orientifold expansion.

Also very promising is three-dimensional $\mathcal{N}=1$ Yang–Mills theory. In this case the parent theory (with adjoint fermions) is supersymmetric, provided one deals with three-dimensional Majorana fermion fields.[k] In three dimensions this theory admits a Chern–Simons term, which generates a mass for the gauge fields, and a counterpart for the fermions. A remarkable feature of this theory is the occurrence of a phase transition at a certain critical value of the Chern–Simons coefficient, and the possibility of an analytic study of this phase transition [55]. It is not ruled out that consequences of the large-N equivalence between the three-dimensional SYM theory and the orientifold theory could be used in analytical studies of confinement in a $T \neq 0$ regime of four-dimensional one-flavor QCD (T standing here for the temperature).

[k] Introducing adjoint Dirac fermion fields would render the theory at hand non-supersymmetric. Even so, studying the parent–daughter planar equivalence in this case is likely to be rewarding. This case is similar to the "flavor proliferation" treated in Sect. 7.

8.5. $1/N$ *corrections*

The statement that large-N Yang–Mills theories with adjoint and two-index fermions are equivalent both perturbatively and non-perturbatively will become a critical advance, only provided that we will be able to get an idea of $1/N$ corrections. Indeed, our final goal, after all, is understanding nature; and in nature $N = 3$. No matter how good toy models are for various explorations, they acquire a real value only as long as they properly reflect the pattern of actual QCD.

From this standpoint, the orientifold large-N expansion is far from being optimal, since, as was mentioned, corrections to the planar limit scale as $1/N$ rather than $1/N^2$; they can thus be large, leading to drastic deviations from the predictions obtained on the basis of planar equivalence. In fact, as we will see below (Sect. 9), in some problems one can encounter corrections of the type $1 - (2/N)$. Further progress will depend on whether we will be able to handle them.

We would like to make a remark concerning this aspect. Our first observation will be useful for lattice-QCD practitioners. On the lattices it does not make much difference whether one simulates fermions in the antisymmetric, symmetric or reducible two-index representation (i.e. no (anti)symmetrization at all). Let us focus on the latter case. The number of flavors need not be equal to 1; it can be, say, 2 or 4. The important thing is that these numbers match. By, say, considering two Majorana adjoint fermions in the parent theory one requires one Dirac Ψ^{ij} (no (anti)symmetrization) in the daughter theory. This is the sum of two irreducible representations A and S.

Assume that we have certain information regarding the mass of a meson with the quantum numbers 0^- in the parent theory. Then, on the lattice, one can measure the mass of its counterpart in the orientifold theory described above. Our assertion is that in such a study the deviation from the adjoint theory will be $O(1/N^2)$ rather than $O(1/N)$!

Another idea for minimizing the $1/N$ corrections is to add extra fermions in the fundamental representation to the fermion in the antisymmetric representation. At least perturbatively, it is obvious that an adjoint fermion (the gluino) is better approximated at finite N by an antisymmetric fermion and two additional fundamental fermions. For example, the one-loop β-function coefficient of the latter theory is $3N$, exactly as the one-loop β function of $\mathcal{N}=1$ SUSY Yang–Mills theory. The similarity of the two theories at finite N and the potential phenomenological applications will be further discussed in Sect. 10.

In addition, some insight about $1/N$ corrections can be obtained from string theory, where this problem translates into the understanding of string loop corrections.

Work on other methods of minimizing (or suppressing) $1/N$ corrections in the transition from the parent theories to orientifold daughters is in progress.

9. Calculating the quark condensate in one-flavor QCD from supersymmetric gluodynamics [l]

The gluino condensate in SU(N) SYM theory is presented in Eq. (3.8). Let us reproduce it here again, for convenience,

$$\langle \lambda^a_\alpha \lambda^{a\,,\alpha} \rangle = -6N\Lambda^3\,, \tag{9.1}$$

where we set $\theta = 0$, $k = 0$, and the parameter Λ in Eq. (9.1) refers to the Pauli-Villars scheme. As is explained in detail in Appendix B, $\Lambda_{\rm Pauli-Villars}$ identically coincides in supersymmetric theories with the scale parameter defined in the dimensional-reduction scheme, DRED $+\overline{\rm MS}$ procedure. In perturbative calculations one usually uses dimensional regularization rather than reduction (supplemented by $\overline{\rm MS}$). The difference between these two Λ's is insignificant — it amounts to a modest factor $\exp(1/12)$, see Appendix B — and we will disregard it hereafter.

It is not difficult to show [m] that the correspondence between the bifermion operators is as follows:

$$\langle \lambda^a_\alpha \lambda^{a\,,\alpha} \rangle \leftrightarrow \langle \bar\Psi\,\Psi \rangle\,. \tag{9.2}$$

The left-hand side is in the parent theory, the right-hand side is in the orientifold theory A, and they project onto each other with the unit coefficient.

[l] Estimates presented in this section are arranged in a somewhat different manner from in the original work [8]. The final result is the same.

[m] One can establish the mapping of $\langle \lambda^{a\,,\alpha} \lambda^a_\alpha \rangle$ onto $\langle \bar\Psi_{[ij]} \Psi^{[ij]} \rangle$ as follows. The simplest way is the comparison of the corresponding mass terms. The Dirac fermion Ψ of the orientifold theory can be replaced by two Weyl spinors, $\xi_{[ij]}$ and $\eta^{[ij]}$, so that the fermion mass term becomes:

$$m\bar\Psi\Psi = m\xi\eta + \text{h.c.}\,,$$

while in softly broken SYM the mass term has the form

$$\frac{m}{2}\,\lambda\lambda + \text{h.c.}\,.$$

Thus,

$$\frac{1}{2}\langle\lambda\lambda\rangle \leftrightarrow \langle\xi\eta\rangle\,, \quad \text{or} \quad \langle\lambda\lambda\rangle \leftrightarrow \langle\bar\Psi\Psi\rangle \quad \text{at} \quad \theta = 0\,.$$

The same identification is obtained from a comparison of the two-point functions in the scalar and/or pseudoscalar channels in both theories.

It is worth emphasizing that Eq. (9.2) assumes that $\langle \lambda^a_\alpha \lambda^{a,\alpha} \rangle$ is real and negative in the vacuum under consideration (which amounts to a particular choice of vacuum). It also assumes that the gluino kinetic term is normalized non-canonically, as in Eq. (9.4).

Thus, the planar equivalence gives us a prediction for the quark condensate in the one-flavor orientifold theory at $N = \infty$, for free. Our purpose here is to go further, and to estimate the quark condensate at $N = 3$, i.e. in one-flavor QCD. To get an idea of the value of $1/N$ corrections, as a first step, one can examine the first two coefficients in the β function and the anomalous dimension of the matter field in three distinct gauge theories, presented in Table 4. We confront pure Yang-Mills theory (YM), QCD with one Dirac fermion in the fundamental representation (1f-QCD), orientifold theory A with one flavor (Orientifold A), and, finally, SYM theory — all with the gauge group SU(N).

Table 4. Comparison of the anomalous dimensions and the first two coefficients of the β function. The notation is explained in the text.

Theory / Coeff.	YM	1f-QCD	Orienti A	SYM
β_0	$\frac{11}{3}\,N$	$\frac{11}{3}\,N - \frac{2}{3}$	$3N + \frac{4}{3}$	$3N$
β_1	$\frac{17}{3}\,N^2$	$\frac{17}{3}N^2 - \frac{13}{6}N + \frac{1}{2N}$	$3N^2 + \frac{19}{3}N - \frac{4}{N}$	$3N^2$
γ	$\star$	$\frac{3(N^2-1)}{2N}$	$\frac{3(N-2)(N+1)}{N}$	$3N$

We use the standard definition of the coefficients of the β function from PDG, see Eq. (8.3). The coefficients can be found in [56], where formulae up to three loops are given. The anomalous dimension γ of the fermion bilinear operators $\bar{\Psi}\Psi$ is normalized in such a way that

$$\left(\bar{\Psi}\Psi\right)_Q = \kappa^{\gamma/\beta_0}\,\left(\bar{\Psi}\Psi\right)_\mu\,, \qquad \kappa \equiv \frac{\alpha(\mu)}{\alpha(Q)}\,, \tag{9.3}$$

and μ and Q denote the normalization points. For our present purposes we can limit ourselves to the two-loop β functions and the one-loop anomalous dimensions. We can easily check that the various coefficients of the orientifold theory A go smoothly from those of YM ($N = 2$) through those of

1f-QCD ($N = 3$) to those of SYM theory at $N \to \infty$. Note, however, that some corrections are as large as $\sim 2/N$. This is an alarming signal.

The gluino condensate $\langle \lambda^a_\alpha \lambda^{a,\alpha} \rangle$ is renormalization-group-invariant (RGI) at any N, and so is Λ. This is not the case for the quark condensate in the non-SUSY daughter (i.e. the orientifold theory) at finite N. Unlike SUSY theories, where it is customary to normalize the gluino kinetic term as

$$\frac{1}{g^2}\,\bar{\lambda}\, i\, \not{D} \lambda\,, \tag{9.4}$$

the standard normalization of the fermion kinetic term in non-SUSY theories is canonical,

$$\bar{\Psi}\, i\, \not{D} \Psi\,. \tag{9.5}$$

Hence, in fact, the correspondence between operators is

$$\langle \lambda^a_\alpha \lambda^{a,\alpha} \rangle \leftrightarrow \langle g^2 \bar{\Psi}\,\Psi \rangle\,. \tag{9.6}$$

In the canonic normalization (9.5), the RGI combination is

$$\left(g^2\right)^{\gamma/\beta_0}\; \bar{\Psi}\Psi \equiv \left(g^2\right)^{1-\delta(N)}\; \bar{\Psi}\Psi\,, \tag{9.7}$$

where

$$\frac{\gamma}{\beta_0} = \frac{\left(1-\frac{2}{N}\right)\left(1+\frac{1}{N}\right)}{1+\frac{4}{9N}} \tag{9.8}$$

and

$$\delta(N) \equiv 1 - \frac{\gamma}{\beta_0} = O(1/N)\,, \qquad \delta(3) = \frac{19}{31}\,. \tag{9.9}$$

Combining Eqs. (9.1), (9.6) and (9.7) we conclude that

$$\left[g^2(\mu)\right]^{\delta(N)} \left\langle \left(g^2\right)^{1-\delta(N)}\; \bar{\Psi}\Psi \right\rangle = -6(N-2)\,\Lambda^3_{\overline{\rm MS}}\,\mathcal{K}(\mu, N)\,, \tag{9.10}$$

where μ is some *fixed* normalization point; the correction factor

$$\mathcal{K}(\mu, N = \infty) = 1\,, \tag{9.11}$$

simultaneously with $\delta(N = \infty) = 0$. At finite N the correction factor $\mathcal{K}(\mu, N) - 1 = O(1/N)$ and $\mathcal{K}$ depends on μ in the same way as $\left[g^2(\mu)\right]^{\delta(N)}$. The combination $\left(g^2\right)^{1-\delta(N)}\; \bar{\Psi}\Psi$ on the left-hand side is singled out because of its RG invariance. Equation (9.10) is our master formula.

The factor $N-2$ on the right-hand side of Eq. (9.10), a descendant of N in Eq. (9.1), makes $\left\langle \left(g^2\right)^{1-\delta(N)}\; \bar{\Psi}\Psi \right\rangle$ vanish at $N = 2$. This requirement is obvious, given that at $N = 2$ the antisymmetric fermion loses color. In fact, we replaced N in Eq. (9.1) by $N-2$ by hand, assembling all other $1/N$ corrections in $\mathcal{K}$, in the hope that all other $1/N$ corrections collected in $\mathcal{K}$ are not so large. There is no obvious reason for them to be large. Moreover, we can try to further minimize them by a judicious choice of μ.

It is intuitively clear that $1/N$ corrections in $\mathcal{K}$ will be minimal, provided that μ presents a scale "appropriate to the process", which, in the case at hand, is the formation of the quark condensate. Thus, μ must be chosen as low as possible, but still in the interval where the notion of $g^2(\mu)$ makes sense. Our educated guess is $\lambda(\mu) = (1/2)$ corresponding (at $N = 3$) to

$$\left[g^2(\mu)\right]^{\delta(3)} \approx 4.9\,. \tag{9.12}$$

As a result, we arrive at the conclusion that in one-flavor QCD

$$\left\langle \left(g^2\right)^{1-\delta(3)}\ \bar{\Psi}\Psi \right\rangle = -1.2\,\Lambda^3_{\overline{\rm MS}}\,\mathcal{K}(N)\,. \tag{9.13}$$

Empiric determinations of the quark condensate with which we will confront our theoretical prediction are usually quoted for the normalization point 2 GeV. To convert the RGI combination on the left-hand side of Eq. (9.13) to the quark condensate at 2 GeV, we must divide by $[g^2(2\text{ GeV})]^{1-\delta(3)} \approx 1.4$. Moreover, as has already been mentioned, we expect non-planar corrections in $\mathcal{K}$ to be in the ballpark $\pm 1/N$. If so, three values for $\mathcal{K}$,

$$\mathcal{K} = \{2/3,\ 1,\ 4/3\}\,, \tag{9.14}$$

give a representative set. Assembling all these factors together we end up with the prediction that for one-flavor QCD

$$\left\langle\, \bar{\Psi}_{[ij]}\Psi^{[ij]} \right\rangle_{2\text{ GeV}} = -\,(0.6\text{ to }1.1)\ \Lambda^3_{\overline{\rm MS}}\,. \tag{9.15}$$

Next, our task is to compare it with empiric determinations, which, unfortunately, are not very precise. The problem is that one-flavor QCD is different both from actual QCD, with three massless quarks, and from quenched QCD, in which lattice measurements have recently been carried out [57]. In quenched QCD there are no quark loops in the running of α_s; thus, it runs steeper than in one-flavor QCD. On the other hand, in three-flavor QCD the running of α_s is milder than in one-flavor QCD.

To estimate the input value of $\lambda_{\overline{\rm MS}}$ we resort to the following procedure. First, starting from $\alpha_s(M_\tau) = 0.31$ (which is close to the world average) we determine $\Lambda^{(3)}_{\overline{\rm MS}}$. Then, with this Λ used as an input, we evolve the coupling constant back to 2 GeV according to the *one-flavor* formula. In this way we obtain

$$\lambda(2\text{ GeV}) = 0.115\,. \tag{9.16}$$

A check exhibiting the scatter of the value of $\lambda(2\text{ GeV})$ is provided by lattice measurements. Using the results of Ref. [58] referring to *pure* Yang–Mills theory one can extract $\alpha_s(2\text{ GeV}) = 0.189$. Then, as previously, we find $\Lambda^{(0)}_{\overline{\rm MS}}$, and evolve back to 2 GeV according to the *one-flavor* formula. The result is

$$\lambda(2\text{ GeV}) = 0.097\,. \tag{9.17}$$

The estimate (9.17) is smaller than (9.16) by approximately one standard deviation σ. This is natural, since the lattice determinations of α_s lie on the low side, within one σ of the world average. In passing from Eq. (9.13) to Eq. (9.15) we used the average value $\lambda_{\overline{\rm MS}}(2\text{ GeV}) = 0.1$.

One can summarize the lattice (quenched) determinations of the quark condensate, and the chiral theory determinations extrapolated to one flavor, available in the literature, as

$$\left\langle\, \bar\Psi_{[ij]}\Psi^{[ij]} \right\rangle_{2\text{ GeV, ``empiric''}} = -\,(0.4\text{ to }0.9)\ \Lambda^3_{\overline{\rm MS}}\,. \tag{9.18}$$

We put empiric in quotation marks, given all the uncertainties discussed above.

Even keeping in mind all the uncertainties involved in our numerical estimates, both from the side of supersymmetry/planar equivalence $1/N$ corrections, and from the "empiric" side, a comparison of Eqs. (9.15) and (9.18) reveals an encouraging overlap. We claim, with satisfaction, that we are on the right track!

10. The orientifold Large-N expansion and three-flavor QCD

So far we have established an analytical relation between the orientifold theory (A or S) and $\mathcal{N}=1$ SYM for $N\to\infty$. This relation was used to make predictions for one-flavor QCD. It seems that our method is restricted to it, since the $\mathcal{N}=1$ theory contains one gluino. Could we use our new technique to calculate quantities in three-flavor QCD, while using $\mathcal{N}=1$ SYM as a parent theory?

To this end we can try to deform the orientifold A theory in such a way that the leading planar approximation will remain intact, while $1/N$ corrections hopefully will change in the direction of making the $N = 3$

deformed model closer to the parent theory. Amusingly enough, at $N = 3$, the daughter theory will be QCD with three massless flavors!

The model to be outlined in this section was in fact formulated in 1979 [37] (see footnote in Sect. 8.1); we refer to it as orienti/2f model. Its fermion sector consists of a single two-index antisymmetric Dirac field $\Psi_{[ij]}$ plus two fundamental Dirac fields q_1^i and q_2^i $(i, j = 1, 2, ..., N)$. At $N = 3$ it reduces to *three-flavor QCD*, in contradistinction with orientifold A theory whose $N = 3$ limit is one-flavor QCD. On the other hand, at $N \to \infty$ fundamental quark loops can be neglected. Each of the two determinants — one for q_1 and another for q_2 — becomes unity, up to $1/N$ corrections. As a result, the orienti/2f theory is planar-equivalent to supersymmetric gluodynamics, much in the same way as the orientifold theory. It is important to note that only those operators of the daughter theory that do not explicitly contain $q_{1,2}$ can be matched onto operators of the parent one. The latter can appear only in loops.

Why may one hope that by adding $q_{1,2}$ one suppresses $1/N$ corrections? A hint comes from perturbation theory. Indeed, in orienti/2f the first coefficient of the β function exactly coincides with that of SUSY gluodynamics, while the deviation of the second coefficient is $\sim 1/3$ of the deviation inherent to orientifold A (cf. Tables 4 and 6 and Appendix C). On the non-perturbative side, orientifold A has no color-singlet baryons with mass scaling as N^0, while orienti/2f does have such baryons (remember, in SUSY gluodynamics there are baryons of the type $G\lambda$ whose mass $\sim N^0$).

The orienti/2f theory has not been analyzed in earnest yet. This is an obvious task for the future. Here we limit ourselves to an introductory remark. The vacuum structure in the orienti/2f theory is quite different from that in SUSY gluodynamics. For generic values of N the orienti/2f theory possesses the global symmetries [n]

$$\{\mathrm{U}(1)_V \times \mathrm{U}(1)_A\} \times \{\mathrm{U}(1)_V \times \mathrm{U}(1)_A\} \times \{\mathrm{SU}(2)_R \times \mathrm{SU}(2)_L\}\,, \qquad (10.1)$$

where the first two factors correspond to Ψ while the last four to $q_{1,2}$. All vector symmetries are realized linearly. As for the axial symmetries, $\mathrm{SU}(2)_A$ is spontaneously broken by $\langle \bar{q}_1 q_1 + \bar{q}_2 q_2\rangle \neq 0$. The two $\mathrm{U}(1)_A$'s are separately anomalous, but one can construct an anomaly-free combination, which is spontaneously broken by $\langle \bar{\Psi}\Psi\rangle \neq 0$ as well as by $\langle \bar{q}_1 q_1 + \bar{q}_2 q_2\rangle \neq 0$. The

[n] The symmetry is enlarged to $\mathrm{U}(1)_V \times \mathrm{U}(1)_A \times \mathrm{SU}(3)_R \times \mathrm{SU}(3)_L$ for $N = 3$.

anomaly-free $\mathrm{U}(1)_A$ is generated by the current

$$A^\mu = \bar{\Psi}\gamma^\mu\gamma^5\Psi - \frac{N-2}{2}\left(\bar{q}_1\gamma^\mu\gamma^5 q_1 + \bar{q}_2\gamma^\mu\gamma^5 q_2\right). \tag{10.2}$$

The anomalous $\mathrm{U}(1)_A$ current is ambiguous, since we can always add to it a multiple of the conserved current. The above symmetries give rise to a vacuum manifold of the type $S_3 \times S_1$, corresponding to four Nambu-Goldstone bosons. The fact that the anomalous $\mathrm{U}(1)_A$ current has an anomaly-free discrete subgroup has no impact on the vacuum structure.

Thus, the parent theory has a mass gap while the daughter one does not. It is not difficult to see, however, that the massless modes in the daughter theory, in the common sector, manifest themselves only at a subleading in $1/N$ level. For instance, the Nambu–Goldstone bosons associated with the spontaneous breaking of $\mathrm{SU}(2)_A$ are pair-produced. The Nambu–Goldstone boson associated with the spontaneous breaking of the anomaly-free $\mathrm{U}(1)_A$ can be produced by an appropriate operator, but the residue will necessarily be suppressed by $1/N$.

The most visible effect of the $q_{1,2}$ quarks in loops is the change in the gauge coupling β function mentioned at the beginning of this section. If this is numerically the most important effect, one can readily arrive at an estimate of the quark condensate in the three-flavor theory. In fact, $\langle\bar{\Psi}\Psi\rangle$ would be predicted to be the same as in Eq. (9.10), up to modifications in δ due to the changes in β_0 and β_1. Good agreement is obtained once more with phenomenological estimates and with lattice data, although the latter are not very precise. In this case there is no need for extrapolation in the quark flavors.

11. Domain walls and the cosmological constant in the orientifold field theories

Domain walls are BPS objects in $\mathcal{N}=1$ supersymmetric gluodynamics [59]. As we know from Sect. 3, if the gauge group is $\mathrm{SU}(N)$, there are N distinct discrete vacua labeled by the order parameter, the gluino condensate,

$$\langle\lambda\lambda\rangle_k = -N\Lambda^3 \exp\left(i\frac{2\pi k}{N}\right), \qquad k = 0, 1, 2, ..., N-1 \tag{11.1}$$

(the vacuum angle is set to zero, and in this section we include the factor 6, cf. Eqs. (3.8) and (9.1), in the definition of Λ). The "elementary" domain wall $W_{\{k,k+1\}}$ interpolates between the k-th and $(k+1)$-th vacua. Moreover, at $N \to \infty$ two parallel domain walls $W_{\{k,k+1\}}$ and $W_{\{k+1,k+2\}}$ are also BPS — there is neither attraction nor repulsion between them [59].

It is known that the BPS domain walls in $\mathcal{N}=1$ gluodynamics present a close parallel to D branes in string theory [60]. In particular, a fundamental flux tube can end on the BPS domain wall, much as F1 end on D-branes in string theory [60] (see also [61]– [65]).

The orientifold theory has N discrete degenerate vacua.[o] Hence, one can expect domain walls. Indeed, the daughter theory inherits domain walls from its supersymmetric parent. Two parallel walls of the type $W_{\{k,k+1\}}$ and $W_{\{k+1,k+2\}}$ are in an indefinite equilibrium. In this sense they are "BPS", although the standard definition of "BPS-ness", through central charges and supercharges, is certainly not applicable in the non-supersymmetric theory. In this section we elaborate on physics of the "BPS" domain walls in the orientifold theory.

We will show that these walls carry charges similar to the Neveu-Schwarz–Neveu-Schwarz (NS-NS) and Ramond–Ramond (R-R) charges. In addition, we will argue that an open-closed string channel duality holds for the analogous field theory annulus amplitude. Moreover, by exploiting the similarity between string theory and field theory we will provide a reason why the vacuum energy density (the "cosmological constant") in the orientifold gauge theory vanishes at order N^2 despite the fact that the hadronic spectrum of the theory contains only bosons — for color-singlet states there is no Bose-Fermi degeneracy typical of supersymmetric theories.

We will assume that our gauge theory has a string theory dual in the spirit of Ref. [66] (yet to be found, though). It is presumably of the type 0B on a curved background, similarly to the orientifold field theory analogue of $\mathcal{N}=4$ SYM theory, which is type 0B on $\mathrm{AdS}_5 \times RP^5$ [67]. In this picture the closed strings correspond to the infrared (IR) degrees of freedom: the glueballs and "quarkonia". Indeed, the type-0 (closed) strings are purely bosonic, in agreement with our expectation from the confining orientifold field theory. Moreover, the bosonic IR spectrum of the gauge theory is even/odd parity-degenerate, in accordance with degeneracies between the NS-NS and R-R towers of the type-0 string.

11.1. *Parallel domain walls versus D branes*

As has been mentioned, gauge theory fundamental flux tubes can end on a BPS domain wall. Let us assume that, for $\mathcal{N}=1$ gluodynamics/orientifold theory, the domain walls have a realization in terms of Dp-branes ($p > 1$) of the corresponding type IIB/0B string theory. Their world volume is

[o] At $N\to\infty$. In fact, the number of vacua in orientifold A is $N-2$, while it is $N+2$ in orientifold S.

$012 + (p-2)$ directions transverse to the four-dimensional space-time 0123. Specific AdS/CFT realizations of domain walls in $\mathcal{N}=1$ theories are given in Refs. [63]– [68], mostly in terms of wrapped D5-branes. We deliberately do not specify which particular branes are used to model the BPS walls, since we do not perform actual AdS/CFT calculations.

D-branes carry the NS-NS charge, as well as the R-R charge [69]. Moreover, interactions induced by these charges exactly cancel, ensuring that the parallel D-branes neither attract nor repel each other. Let us see how this is realized in $\mathcal{N}=1$ SYM theory. We start from two parallel BPS walls $W_{\{k,k+1\}}$ and $W_{\{k+1,k+2\}}$. Each of them is BPS, with the tension [59]

$$T_{\{k,k+1\}} = T_{\{k+1,k+2\}} = \frac{N}{8\pi^2}\,|\langle\mathrm{tr}\lambda\lambda\rangle|\,2\,\sin\frac{\pi}{N}\,. \tag{11.2}$$

The tension of the configuration $W_{\{k,k+2\}}$ is

$$T_{\{k,k+2\}} = \frac{N}{8\pi^2}\,|\langle\mathrm{tr}\lambda\lambda\rangle|\,2\,\sin\frac{2\pi}{N}\,. \tag{11.3}$$

This means that at leading order in N (i.e. N^1) two parallel walls $W_{\{k,k+1\}}$ and $W_{\{k+1,k+2\}}$ do not interact. There is no interaction at the level N^0 either. An attraction emerges at the level N^{-1}. That the inter-wall interaction potential is $O(N^{-1})$ can be shown on general kinematic grounds. A relevant discussion can be found in Ref. [70], see Eq. (37); see also Ref. [71]. This and other aspects of the dynamics of the inter-wall separation are challenging questions, which are currently under investigation. Here we summarize some findings [29].

Our task is to understand the dynamics of this phenomenon from the field theory side. Assume that two walls under consideration are separated by a distance $Z \gg m^{-1}$, where m is the mass of the lightest composite meson. What is the origin of the force between these walls?

The interaction is due to the meson exchange in the bulk. Consider the lightest mesons, scalar and pseudoscalar. The scalar meson σ, the "dilaton", is coupled [72] to the trace of the energy-momentum tensor θ_μ^μ,

$$\frac{\sigma}{f}\,\theta^\mu_\mu = \frac{3N}{16\pi^2 f}\,\sigma\,\mathrm{tr}\,G^2\,. \tag{11.4}$$

It is very easy to show that the coupling constant f scales as

$$f \sim N\,\Lambda\,. \tag{11.5}$$

Integrating over the transverse direction, and using the fact that θ_μ^μ translates into mass, we find that the "dilaton"–wall coupling (per unit area) is

$$T\,\sigma/f\,, \tag{11.6}$$

where T is the wall tension, see Eq. (11.2). Since the wall tension scales as N, the σ–wall coupling scales as N^0.

The scalar meson is degenerate in mass with the pseudoscalar one, the "axion" or "η'" which, thus, must be included too. For brevity we will denote this field by η; it will have a realization in terms of the RR 0-form of type IIB/0B. The coupling of this "axion" to the wall is related to the change of the phase of the gluino condensate across the wall. Therefore, a natural estimate of the axion-wall coupling is [p]

$$\frac{\eta}{f}\int dz\,\frac{N}{8\pi^2}\,\mathrm{tr}\,G\tilde{G} \to \frac{\eta}{f}\int dz\,\frac{N}{8\pi^2}\,|\,\mathrm{tr}\,\lambda\lambda\,|\,\frac{\partial\alpha}{\partial z}\,, \tag{11.7}$$

where z is the coordinate transverse to the wall, and α is the phase of the order parameter. At large N the absolute value of the order parameter stays intact across the wall, while $\int dz\,(\partial\alpha/\partial z) = 2\pi/N$. Thus, the axion–wall coupling scales as N^0. Note that the sign of the coupling depends on whether we cross the wall from left to right or from right to left. This is why the exchange of the dilaton between two parallel BPS walls leads to the wall attraction, while that of the axion leads to repulsion.

Since the dilaton and axion couplings to the wall are proportional to N^0, we get no force between the walls at the level N^1 just for free. The underlying reason is that the BPS domain-wall tension scales not as N^2, as could be naively expected to be the case for solitons, but rather as N^1, the D-brane type of behavior.

We want to make a step further, however. We will show shortly that σ and η contributions (at the level N^0) are exactly equal in absolute value but are opposite in sign. Needless to say, this requires the degeneracy of their masses and their couplings to the walls, except for the relative sign.

Taking the right-hand side of Eq. (11.7) at its face value we get the axion–wall coupling in the form

$$T\,\eta/f + O(1/N)\,, \tag{11.8}$$

which is indeed the same as Eq. (11.6).

At finite N the wall thickness, which scales as [62] $(N\Lambda)^{-1}$, is finite too. On the wall world volume one has a (2 + 1)-dimensional supersymmetry,

[p] Equation (11.7) guarantees, automatically, that the axion–wall coupling is saturated inside the wall. Outside the wall, in the vacuum, $\alpha = \text{const.}$, while $\langle \mathrm{Tr}\,G^2\rangle = \langle \mathrm{Tr}\,G\tilde{G}\rangle = 0$.

which places scalars and pseudoscalars into distinct (non-degenerate) supermultiplets [73]. At $N = \infty$ the wall thickness vanishes, and it is natural that in this limit we deal with the lowest component of $(3+1)$-dimensional chiral superfield which has the form $\sigma + i\eta$.

Summarizing, the wall "R-R charge" is due to the axion-like nature of η. The "NS-NS charge" is due to the wall tension. Both charges are indeed equal and scale as $N^0\,\Lambda^2$. This guarantees that at order N^0 there is no force. The σ and η coupling to the wall split at order N^{-1}.

The very same arguments can be repeated *verbatim* in the orientifold theories. In the limit $N \to \infty$ the degeneracy of the σ and η masses holds, and so does the degeneracy of the wall "NS-NS and R-R charges". The reason why the couplings to the wall (associated with the trace of the energy-momentum tensor for the dilaton and the axial charge for the axion) are the same is that these charges and couplings are inherited from the parent supersymmetric theory at $N \to \infty$, see [6]. It will be useful, for what follows, to note that σ exchange alone (before cancellation) generates an interaction potential between the walls (per unit area)

$$\frac{V}{A} \sim N^0\,\Lambda^3\,e^{-mz}\,. \tag{11.9}$$

Let us now see how this picture is implemented in string theory. From the open string standpoint the result of a vanishing force is natural. This is the Casimir force between the walls. Since the UV degrees of freedom are Bose–Fermi-degenerate, the vacuum energy and, hence, the Casimir force is zero.

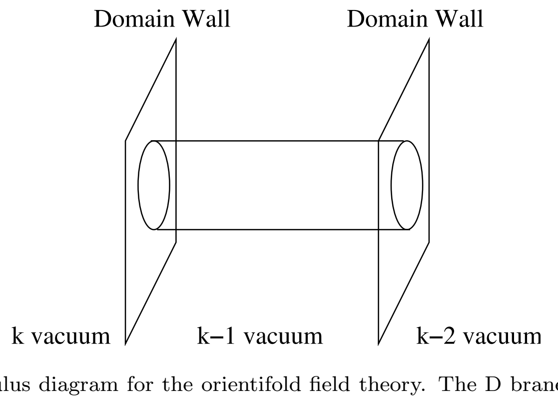


Figure 7. The annulus diagram for the orientifold field theory. The D branes are domain walls. Closed strings are bosonic glueballs and open strings are UV degrees of freedom.

In the gauge theory the closed strings are field-theoretic glueballs [74,75].

Let us consider first the large separation limit. At the lowest level we have a massive scalar and a pseudoscalar (we assume a mass gap). These two exactly degenerate states correspond to the dilaton and axion (the RR 0-form of type IIB/0B). They are expected to become massive when the theory is defined on a curved background [74]. The type IIB/0B action contains the couplings

$$e^{-\Phi}\,\mathrm{Tr}\,G^2 + C\,\mathrm{Tr}\,G\tilde{G}\,, \tag{11.10}$$

where Φ denotes the dilaton and C the R-R 0-form. In addition we have a coupling of the graviton and the R-R 4-form

$$\eta^{\mu\rho}h^{\nu\lambda}\,\mathrm{Tr}\,G_{\mu\nu}G_{\rho\lambda} + C^{\mu\nu\rho\lambda}\,\mathrm{Tr}\,G_{\mu\nu}G_{\rho\lambda}\,. \tag{11.11}$$

In the gauge theory the "graviton" (tensor meson) and the 4-form are heavier glueballs, since they carry higher spins than the dilaton and the 0-form. Similarly, the whole tower of degenerate bosonic hadrons of the orientifold field theory should correspond to the NS-NS and R-R fields of type II/0 string theory. This gives us a new picture of why the force between the domain walls vanishes in terms of the glueball exchanges; even-parity glueballs lead to an attractive force between the walls, whereas odd-parity glueballs lead to a repulsion. The sum of the two is exactly zero at the leading N^0 order. The $1/N$ force between the walls is related to a possible non-vanishing force between parallel D-branes in a curved space at $O(g_{\rm st}^2)$.

We can also exploit the above picture to estimate the potential between a given wall and an anti-wall. Since this configuration is not BPS a non-vanishing force is expected at leading N^0 order. At large separations, the force is controlled by an exchange of the lowest massive closed strings, the dilaton and the zero-form. Now their contributions add up and we get an attractive potential, as indicated in Eq. (11.9).

11.2. *Vanishing of the vacuum energy in $\mathcal{N}=1$ SYM and orientifold theories*

One of the surprising results is that the N^2 part of the vacuum energy density vanishes in the orientifold field theory. While this result makes sense from the UV point of view, where we do have Bose–Fermi degeneracy, it looks rather mysterious from the IR standpoint, since at the level of the composite color-singlet states we have only bosonic degrees of freedom and no Bose–Fermi degeneracy. Since at large N we have free bosons, it is legitimate to sum the bosonic contributions to the vacuum energy density $\mathcal{E}$ as follows:

$$\mathcal{E} = \sum_n \sum_{\vec{k}} \frac{1}{2}\sqrt{\vec{k}^2 + M_n^2}\ , \tag{11.12}$$

where M_n are the meson masses. An apparent paradox arises since the sum runs over positive contributions. How can positive contributions sum to zero?

In fact, there is no paradox and no contradiction with the statement of vanishing of $\mathcal{E}$ at $O(N^2)$. We will present below arguments that will, hopefully, make transparent the issue of the vanishing of the vacuum energy density (at the N^2 level) in the orientifold theory. Usually it is believed that one needs full supersymmetry to guarantee that $\mathcal{E} = 0$. It turns out that a milder requirement — the degeneracy between scalar and pseudoscalar glueballs/mesons — does the same job. Of course, in supersymmetric theories this degeneracy is automatic. The orientifold field theory is the first example where it takes place (to leading order in $1/N$) without full supersymmetry because it is inherited through the planar equivalence.

Two comments are in order here. First, the sum (11.12) is not well-defined since the expected Regge trajectories are not bounded from above and, therefore, regularization is needed. Second, in the above sum (11.12) the N dependence of each individual mode is N^0. The expected N^2 dependence of the vacuum energy is hidden in the sum over all hadronic modes and in the regularization.

Thus, even though (11.12) represents the vacuum energy density of the theory, it cannot be calculated from this equation. Below, we present an alternative definition of the vacuum energy density, with a UV regulator built in, which will illustrate the above surprising nullification.

Let us consider the contribution to $\mathcal{E}$ from the open string sector. At large N it is dominated by the annulus diagram where each boundary consists of N D-branes, and a summation over the various D-branes is assumed. The Möbius and Klein-bottle as well as higher-genus amplitudes are suppressed at large N. From this standpoint it is not surprising that the $\mathcal{E}$ vanishes, as we have N^2 bosons (NS open strings) and N^2 fermions (Ramond fermionic open strings).

The annulus diagram has another interpretation, however. It represents the force between the D-branes. The force is mediated by bosonic closed strings. In a SUSY setup (the type II string), D-branes are the BPS objects — hence, the zero force. As has been discussed above, the balance, at large separations, is achieved thanks to a cancellation between the dilaton, the graviton and the massless R-R forms.

It is interesting that the force between parallel self-dual D-branes also vanishes in type-0 string theory [67, 76],

$$\mathcal{A} = N^2(V_8 - S_8) \equiv 0\,. \tag{11.13}$$

This is due to the underlying supersymmetry on the world-sheet. The mechanism is exactly as in the type II case: the R-R modes cancel the contributions of the NS-NS modes. Note that since we are interested only in the planar gauge theory, we can restrict ourselves to $g_{st} = 0$ on the string theory side. Therefore, higher-genus amplitudes are irrelevant to our discussion. At this level the relevant type-0 amplitudes, as well as the bosonic spectrum, are identical to the type II ones, in a not too surprising similarity with the situation in the large-N dual gauge theories (the type-0 string becomes, in a sense, supersymmetric at tree level). In particular, the induced dilaton tadpole and cosmological constant are irrelevant and, thus, the background inherited from the supersymmetric theory remains intact.

The vanishing of the annulus diagram leads to an explanation of the mysterious vanishing of the cosmological constant in the orientifold field theory: if one views the hadrons (in the spirit of the AdS/CFT correspondence) as closed strings, the degenerate bosonic spectrum is the reason behind the vanishing result for both wall–wall interaction and $\mathcal{E}$.

We hasten to add that although the mechanism is similar, there is a difference between the two cases: the wall–wall interaction involves the force between "D2"-branes (wrapped D5-branes), whereas the vanishing cosmological constant involves the force between "D3"-branes. The two sorts of branes are not necessarily the same — everything depends on the specific realization. However, from the bulk point of view, the mechanism is identical. The only requirement is the degeneracy of the NS-NS and the R-R towers and their couplings to the branes.

The difference between string theory and field theory is that in string theory the force between D-branes, from the closed string standpoint, is related to the contribution to the cosmological constant from the open string sector. However, closed strings and open strings are independent degrees of freedom and the contribution of the closed strings to the cosmological constant should therefore be added. In the gauge theory string picture, the closed strings are simply hadrons made out of the constituent open strings — the gluons and quarks. Therefore, the value of the cosmological constant can be determined by either UV or IR degrees of freedom; it is the same quantity.

Let us now switch to the field theory language. At first, we will acquaint the reader with some general relations, relevant to $\mathcal{E}$, which are valid in any gauge theory with no mass scale other than the dynamically generated Λ. The vacuum energy density $\mathcal{E}$ is defined through the trace of the energy-momentum tensor,

$$\mathcal{E} = \frac{1}{4}\,\langle\theta^\mu_\mu\rangle = \frac{1}{4}\int DA\,D\Psi\;\theta^\mu_\mu\;\exp(iS)\,,$$
$$\theta^\mu_\mu = -\frac{3N}{32\pi^2}\,G^2\,, \tag{11.14}$$

where

$$G^2 \equiv G^a_{\mu\nu}\,G^{\mu\nu,a}\,, \qquad G\tilde{G} \equiv G^a_{\mu\nu}\,\tilde{G}^{\mu\nu,a}\,. \tag{11.15}$$

The second line in Eq. (11.14) is exact in $\mathcal{N}=1$ gluodynamics and is valid up to $1/N$ corrections in the orientifold theory.

Now, we use an old trick [38] to express the trace of the energy-momentum tensor in terms of a two-point function. The idea is to vary both sides of Eq. (11.14) with respect to $1/g^2$ and exploit the fact that the only dimensional parameter of the theory, Λ, exponentially depends on $1/g^2$. In this way one obtains [38]

$$\mathcal{E} = -i\int d^4x\,\left\langle \text{vac}\left|T\left\{\frac{1}{4}\,\theta^\mu_\mu(x)\,,\;\frac{1}{4}\,\theta^\nu_\nu(0)\right\}\right|\text{vac}\right\rangle_{\text{conn}}$$
$$\equiv -i\int d^4x\,\left\langle\frac{3N}{128\pi^2}\,G^2(x)\,,\;\frac{3N}{128\pi^2}\,G^2(0)\right\rangle_{\text{conn}}\,. \tag{11.16}$$

The above expression (11.16) is formal — it is not well defined in the UV. This is the same divergence that plagues any calculation of $\mathcal{E}$ (remember, Eq. (11.16) is general, not specific to supersymmetry).

In SUSY, we are prompted to a natural regularization. Indeed, let us consider the two-point function of the lowest components of two chiral superfields D^2W^2,

$$\langle\text{Tr}\,D^2W^2(x)\,,\;\text{Tr}\,D^2W^2(0)\rangle\,. \tag{11.17}$$

The supersymmetric Ward identity tells us that this two-point function vanishes identically. There are two remarkable facts encoded in Eq. (11.17). First, since $D^2W^2\;\propto\;\left(G^2+iG\tilde{G}\right)$, the vanishing of (11.17) is not due to the boson–fermion cancellation but, rather, to the cancellation between even/odd parity mesons (glueballs).

Second, Eq. (11.17) generalizes Eq. (11.16), so that the expression for the vacuum energy density takes the form

$$\mathcal{E} = -i\left(\frac{3N}{128\pi^2}\right)^2 \int d^4x \, \left(\langle G^2(x), G^2(0)\rangle - \langle G\tilde{G}(x), G\tilde{G}(0)\rangle\right), \tag{11.18}$$

where the connected correlators are understood on the right-hand side. To see that Eq. (11.18) is the same as (11.16), notice that

$$0 = -i\int d^4x \, \left\langle \frac{3N}{128\pi^2}\, G\tilde{G}(x)\,,\ \frac{3N}{128\pi^2}\, G\tilde{G}(0)\right\rangle\,. \tag{11.19}$$

The above zero is explained by the fact that $G\tilde{G}$ is proportional to the divergence of the axial current A_μ both in $\mathcal{N}=1$ gluodynamics and in the orientifold theory. Adding a formal zero we in fact achieved the ultraviolet regularization so that (11.18) obeys an unsubtracted dispersion relation.

It is absolutely clear that this regularization works perfectly both in SUSY theories and in the orientifold theories (at the N^2 level). Indeed, the part of the two-point function that involves G^2 is saturated by even-parity glueballs, while the part that involves $G\tilde{G}$ is saturated by odd-parity ones. We then get

$$\mathcal{E} = \sum_{\text{even parity}} \frac{\lambda_n^2}{M_n^2} - \sum_{\text{odd parity}} \frac{\lambda_n^2}{M_n^2}\,, \qquad \lambda_n^2 \sim N^2 \quad \text{for all } n\,, \tag{11.20}$$

where λ_n are the couplings to T_μ^μ and $\partial^\mu a_\mu$, respectively, and M_n are the glueball masses. Clearly, if the masses and the couplings of the glueballs are even/odd-parity degenerate, as is the case in $\mathcal{N}=1$ gluodynamics *and* in the large-N orientifold field theories, $\mathcal{E}$ vanishes.

In summary, in the UV calculation the Fermi–Bose degeneracy was responsible for the vanishing of the cosmological constant both in supersymmetric gluodynamics and in the orientifold theory (where the cancellation was at the N^2 level). In dealing with $\mathcal{E}$, a certain regularization procedure is needed. In SUSY it is implicit. In passing from the UV to the IR language, we make it explicit through Eq. (11.18).

Formula (11.20) is in remarkable agreement with our string theory picture. Only bosonic glueballs are involved, and the even/odd parity glueballs contribute with opposite signs.

12. A chiral ring for non-supersymmetric theories

In four-dimensional supersymmetric theories there is a notion of the *chiral ring*. An assembly of chiral operators (i.e. those annihilated by supersymmetries $\bar{Q}$) forms a ring (this is a mathematical term). The product of two chiral operators is also a chiral operator. Chiral operators are usually considered modulo operators of the form $\{\bar{Q}, ...\}$. The equivalence classes can be multiplied and form a ring. Chiral operators are the lowest components of chiral superfields.

A crucial property of the chiral operators is that their correlation functions are space-time independent [77], e.g.

$$\langle O_1(x) O_2(y) \rangle = \text{const.}, \tag{12.1}$$

where the constant on the right-hand side depends neither on x nor on y. Consider for example the correlation function

$$\langle T \left\{ \lambda^a_\alpha(x) \lambda^{a\alpha}(x) \, , \, \lambda^b_\beta(0) \lambda^{b\beta}(0) \right\} \rangle \, . \tag{12.2}$$

The proof of the constancy is quite straightforward [77] and is based on three elements: (i) the supercharge $\bar{Q}^{\dot\beta}$ acting on the vacuum state annihilates it; (ii) $\bar{Q}^{\dot\beta}$ anticommutes with $\lambda\lambda$; (iii) the derivative $\partial_{\alpha\dot\beta}(\lambda\lambda)$ is representable as the anticommutator of $\bar{Q}^{\dot\beta}$ and $\lambda^\beta G_{\beta\alpha}$. One differentiates Eq. (12.2), substitutes $\partial_{\alpha\dot\beta}(\lambda\lambda)$ by $\{\bar{Q}^{\dot\beta}, \lambda^\beta G_{\beta\alpha}\}$, and obtains zero.[q] Thus, supersymmetry requires the x derivative of (12.2) to vanish. It does not require the vanishing of the correlation function *per se*. A constant is allowed.

It is quite clear that the necessary and sufficient condition for the constancy of the correlation function (12.2) is the degeneracy of the glueball spectra in the scalar and pseudoscalar channels. If the spectra were not degenerate, the dispersion representation would require a non-trivial x dependence. Since the above degeneracy is inherited by the orientifold theories at $N \to \infty$, one expects that in this limit they do carry a remnant of this phenomenon. Our task is to identify this aspect.

Equations (12.1) imply that in SUSY theories the expectation values of the chiral operators factorize,

$$\langle O_1 O_2 ... O_n \rangle = \langle O_1 \rangle \langle O_2 \rangle ... \langle O_n \rangle . \tag{12.3}$$

This factorization is *exact*. Since in $\mathcal{N} = 1$ gluodynamics the chiral ring is spanned by the operator $S = \text{Tr}\, W^2_\alpha$, see [78] for a comprehensive discussion, for definiteness we can limit ourselves to the operators $\lambda^a_\alpha \lambda^{a\alpha}$. Their projection onto orientifold theory is $\bar\Psi_L \Psi_R \equiv \eta_{\{ij\}} \xi^{\{ij\}}$ or $\eta_{[ij]} \xi^{[ij]}$ in the notation of Sect. 6. The question to ask is what is the consequence of the

[q] The operator $\lambda\lambda$ is called Q-closed, while the operator $\partial_{\alpha\dot\beta}(\lambda\lambda)$ is Q-exact.

exact factorization,

$$\langle \mathrm{Tr}\,\lambda\lambda \mathrm{Tr}\,\lambda\lambda\rangle = \langle \mathrm{Tr}\,\lambda\lambda\rangle^2 \,, \tag{12.4}$$

in the parent theory for the daughter operators $\bar\Psi_L\Psi_R$.

As in any large-N theory, the nonfactorizable part of $\bar\Psi_L\Psi_R\bar\Psi_L\Psi_R$ is suppressed by $1/N$. More exactly, in the general case one can expect

$$\langle \bar\Psi_L\Psi_R\bar\Psi_L\Psi_R\rangle = \langle\bar\Psi_L\Psi_R\rangle^2 \left(1 + O(1/N^2)\right) \,, \tag{12.5}$$

where $O(1/N^2)$ corrections on the right-hand side is due to non-factorizable contributions; while the factorizable part is $\langle\bar\Psi_L\Psi_R\rangle^2 = O(N^4)$. The orientifold theory is special. The exact factorization (12.4) in conjunction with the planar equivalence implies that in the orientifold theories the non-factorizable part is *additionally suppressed*, namely,

$$\langle \bar\Psi_L\Psi_R\bar\Psi_L\Psi_R\rangle = \langle\bar\Psi_L\Psi_R\rangle^2 \left(1 + O(1/N^3)\right) \,. \tag{12.6}$$

The relative suppression $O(1/N^3)$ in Eq. (12.6) rather than $O(1/N^2)$ is the remnant of the chiral ring. This result replaces, in the orientifold theories, the statement of the existence of the chiral ring in SYM theories.

In more general terms one can write

$$\langle \bar\Psi_L\Psi_R(x_1)\,,\, \bar\Psi_L\Psi_R(x_2)\rangle = \langle\bar\Psi_L\Psi_R\rangle^2 \left[1 + O(N^{-3})\, f(x_1-x_2)\right] . \tag{12.7}$$

This is the meaning of a chiral ring in non-supersymmetric theories (orienti A/S). It would be interesting to explore the phenomenological consequences of our finding.

13. Parent theories related to $\mathcal{N}=2$ SUSY

13.1. *Planar equivalence between distinct supersymmetric theories and an estimate of the accuracy of the 1/N expansion*

While the original proof of the (non-perturbative) planar equivalence between a supersymmetric parent theory and its orientifold daughter [6] connects $\mathcal{N} = 1$ gluodynamics with an $\mathcal{N} = 0$ QCD-like theory, it can be generalized, practically *verbatim*, to include other parent–daughter pairs, in particular, supersymmetric pairs.[r]

Here we will consider the mass-deformed Seiberg–Witten theory as a parent, and its particular $\mathcal{N} = 1$ orientifold daughter. The presence of the

[r] *Orbifold* pairs with $\mathcal{N} = 1$ supersymmetric parent *and* daughter theories were analyzed in [79].

mass term is important in order to make relevant superdeterminants well-defined.

The Lagrangians of the parent/daughter theories are

$$\mathcal{L} = \left\{ \frac{1}{4g_0^2} \int \mathrm{d}^2\theta \, \mathrm{Tr}\, W^2 + \text{ h.c.} \right\} + \mathcal{L}_{\text{matter}} \, , \tag{13.1}$$

where the matter part for the parent theory is

$$\mathcal{L}_{\text{matter}} = \frac{1}{4} \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} \, \bar{\Phi} e^V \Phi + \left\{ \frac{m_0}{4} \int \mathrm{d}^2\theta \, \Phi^2 + \text{h.c.} \right\} \, , \tag{13.2}$$

while for the daughter theory it is

$$\mathcal{L}_{\text{matter}} = \frac{1}{4} \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} \, \left(\bar{\xi} e^V \xi + \bar{\eta} e^V \eta\right) + \left\{ \frac{m_0}{2} \int \mathrm{d}^2\theta \, \xi\eta + \text{h.c.} \right\} \, . \tag{13.3}$$

Here Φ is the chiral superfield in the adjoint representation, while ξ and η are two-index antisymmetric chiral superfields of the type $\xi^{[ij]}$ and $\eta_{[ij]}$. The gauge group is $\mathrm{SU}(N)$, the bare gauge coupling is denoted by g_0^2, while the bare-mass parameter is m_0. Remember that in our conventions $\int \theta^2 \, d^2\theta = 2$.

In the limit $m_0 \to 0$ the parent theory has $\mathcal{N} = 2$, while its orientifold daughter has $\mathcal{N} = 1$. We will keep m_0 fixed (i.e. N-independent); this is important for the proof of the non-perturbative planar equivalence. The ratio m/Λ can be arbitrary but *non-vanishing*.

The fact that, at $m = 0$ (or, more generally, $m \to 0$ as $N \to \infty$), the proof does not hold is seen from the fact that in this limit the parent theory develops a moduli space, which is much larger than that of the daughter theory. With m/Λ fixed both theories have N discrete vacua. Physically interesting is the case of non-vanishing but small m/Λ.

The advantage of having the supersymmetric orientifold daughter is that we can now calculate the gluon condensates exactly and independently in the two theories, and compare the degree of coincidence at large N. At $N = \infty$ they must coincide; the question we address is $1/N$ corrections.

We will start from the introduction of appropriate scale parameters. In general, the two-loop β-function equation leads to the convention [26]

$$\Lambda^{\beta_0} = M_{\text{UV}}^{\beta_0} \left(\frac{16\pi^2}{\beta_0 \, g_0^2} \right)^{\beta_1/\beta_0} \exp\left(-\frac{8\pi^2}{g_0^2} \right) \, . \tag{13.4}$$

In our specific case we thus have[s]

$$\Lambda_P = M_{\rm UV}\, \exp\left(-\frac{8\pi^2}{2N\, g_{P0}^2}\right)\,,$$

$$\Lambda_D = M_{\rm UV}\, \left[\frac{16\pi^2}{(2N+2)\, g_{D0}^2}\right]^{\frac{2}{N}\frac{N-1}{N+1}}\, \exp\left[-\frac{8\pi^2}{(2N+2)\, g_{D0}^2}\right]. \qquad (13.5)$$

In deriving the above expressions we used the data collected in Table 5.

Table 5. Coefficients β and γ pertinent to the discussion in Sect. 13.1.

Theory \ Data	β_0	β_1	γ
Parent	$2N$	0	$2N$
Daughter	$2N+2$	$\frac{8(N^2-1)}{N}$	$\frac{2(N-2)(N+1)}{N}$

Next, we would like to define renormalization-group invariant (RGI) masses in the two theories. Applying Eq. (9.3) and the data from Table 5 we get

$$\frac{m_P(\mu)}{Ng_P^2(\mu)} = {\rm RGI}\,, \qquad \frac{m_D(\mu)}{\left[Ng_D^2(\mu)\right]^{\frac{N-2}{N}}} = {\rm RGI}\,. \qquad (13.6)$$

It is clear, that physically it makes sense to compare the parent and daughter theories provided that they have one and the same dynamical scale, $\Lambda_P = \Lambda_D \equiv \Lambda$, and the same RGI mass,[t]

$$\frac{m_P}{Ng_P^2} = \frac{m_D}{(Ng_D^2)^{\frac{N-2}{N}}} \equiv m\,. \qquad (13.7)$$

[s] In fact, the expressions for Λ_P, the renormalization-group-invariant mass in the parent theory and – what is most important – for the gluino condensates in Eqs. (13.8) – (13.10), are exact. This is due to a holomorphic structure of the NSVZ β function and the gluino condensates in supersymmetric theories.

[t] In principle, there is a legitimate alternative; instead of identifying the RGI masses one could impose the condition of coincidence of the running masses at some relevant scale μ. The running masses are measurable. Distinction between these two options arises at the $1/N$ level and is insignificant in the limit of small m/Λ.

On general grounds, from generalized R parity, one can fix the dependence of the gluino condensate on the dynamical scale and the RGI mass [23]. Accordingly, the value of the gluino condensate in the parent theory is

$$\langle\lambda^a_\alpha\lambda^{a\,\alpha}\rangle_P = C_P \times \Lambda_P^2 \times \frac{m_P}{N g_P^2}\,, \tag{13.8}$$

and

$$\langle\lambda^a_\alpha\lambda^{a\,\alpha}\rangle_D = C_D \times \Lambda_D^{\frac{2N+2}{N}} \times \left[\frac{m_D}{(N g_D^2)^{(\frac{N-2}{N})}}\right]^{\frac{N-2}{N}} \tag{13.9}$$

in the daughter theory. Two constants C_P and C_D can be fixed by taking the limit $m_0 \to M_{\rm UV}$ in the two theories. Then the theories reduce to $\mathcal{N}=1$ gluodynamics, and the *exact* value of the condensate is known. One readily finds

$$C_P = -4N(8\pi^2), \quad C_D = -4N(8\pi^2)^{(1-2/N)^2}\left(1+\frac{1}{N}\right)^{\frac{4(N-1)}{N^2}}\,, \tag{13.10}$$

so that

$$\frac{\langle\lambda^a_\alpha\lambda^{a\,\alpha}\rangle_D}{\langle\lambda^a_\alpha\lambda^{a\,\alpha}\rangle_P} = \frac{C_D}{C_P}\left(\frac{\Lambda}{m}\right)^{2/N} = 1 + \frac{2}{N}\,\ln\frac{\Lambda}{64\pi^4 m} + O\left(\frac{1}{N^2}\right)\,. \tag{13.11}$$

The comparison of the gluino condensates is moderately interesting, since in both theories gluinos are in the adjoint representation, and orientifoldization has only an indirect impact on the daughter-theory gluino. Of more interest is the ratio of the matter condensates, since it is the matter field that is subject to orientifoldization. The matter condensates can be expressed in terms of those in Eq. (13.11) by virtue of the Konishi anomaly [80],

$$\bar{D}^2\,(\bar\Phi e^V\Phi) = 4\,m_0\Phi^2 + \frac{N}{2\pi^2}\,{\rm Tr}\,W^2\,,$$

$$\bar{D}^2\,(\xi e^V\xi + \bar\eta e^V\eta) = 4\,m_0\xi\eta + \frac{N-2}{2\pi^2}\,{\rm Tr}\,W^2\,, \tag{13.12}$$

which then implies

$$\frac{m_{D0}\,\langle\xi\eta\rangle}{m_{P0}\,\langle\Phi^2\rangle} = \frac{N-2}{N}\,\frac{\langle\lambda^a_\alpha\lambda^{a\,\alpha}\rangle_D}{\langle\lambda^a_\alpha\lambda^{a\,\alpha}\rangle_P} = \frac{N-2}{N}\,\frac{C_D}{C_P}\left(\frac{\Lambda}{m}\right)^{2/N} \to 1\,, \tag{13.13}$$

at $N\to\infty$. This formula, as well as Eq. (13.11), shows in an absolutely transparent form that, as the ratio m/Λ decreases, a critical value of N

needed for the onset of the planar equivalence increases logarithmically,

$$N_* \sim \ln \frac{\Lambda}{m} \,. \tag{13.14}$$

On the other hand, Eq. (13.13) exhibits the vanishing of the matter condensate in the daughter theory at $N = 2$. This is natural too, since at $N = 2$ the daughter-theory matter loses its color.

13.2. *Dijkgraaf–Vafa deformations*

So far we only considered the relation between mass deformed $\mathcal{N}=2$ SYM and a massive orientifold $\mathcal{N}=1$ theory. Recently there was a lot of interest in more general deformations of the $\mathcal{N}=2$ SYM theory, of the type

$$\mathcal{W}_{\rm tree} = \sum_k g_k \, {\rm Tr}\, \Phi^k. \tag{13.15}$$

Dijkgraaf and Vafa [81], following an earlier work by Cachazo, Intriligator and Vafa [82], showed that the effective superpotential of such a theory can be obtained from a matrix model. It was shown later that the same result could be derived using standard field theory techniques [78, 83] (generalized Konishi anomalies), hence confirming the Dijkgraaf–Vafa conjecture.

We would like to extend the analysis of Sect. 13.1 with a purely quadratic superpotential to a more general setup where we have an $\mathcal{N}=1$ SYM theory with matter in the antisymmetric representation and a generic tree-level superpotential

$$\mathcal{W}_{\rm tree} = \sum_k g_k \, {\rm Tr}\, (\xi\eta)^k \,. \tag{13.16}$$

We wish to compare it with $\mathcal{N}=1$ SYM theory with matter in the adjoint representation and a tree-level superpotential

$$\mathcal{W}_{\rm tree} = \sum_k g_k \, {\rm Tr}\, (\Phi)^{2k}. \tag{13.17}$$

The work is in progress, and it is premature to give a detailed report of relevant results. We would like to note only that at large N the effective superpotentials of the above two theories coincide. Therefore, the planar equivalence is instrumental here too, albeit only for holomorphic data.

13.3. *Non-holomorphic information*

In Sect. 13.2 we mentioned that softly broken $\mathcal{N}=2$ SYM and $\mathcal{N}=1$ SYM theories, with massive antisymmetric matter field, must effectively coincide at large N in the holomorphic sector.

The planar equivalence between SUSY gluodynamics and non-SUSY orientifold field theories extends further. Our results were not restricted to holomorphic data in the parent $\mathcal{N}=1$ theory. For example, we argued that the glueball spectra of the two theories coincide at large N.

A natural question is whether for $\mathcal{N}=2$ related parents and their daughters the correspondence is restricted to holomorphic data or can be extended, e.g. to D-terms. It is conceivable, for instance, that the prepotential of the softly broken $\mathcal{N}=2$ theory can tell us something about the D term of the orientifold $\mathcal{N}=1$ daughter theory, thus providing important information on its spectrum.

A "nearly holomorphic" result is the string tension formula of Douglas and Shenker for k-strings in softly broken $\mathcal{N}=2$ SYM theory [84]

$$\sigma_k \sim m\Lambda \sin \pi \frac{k}{N} \ . \tag{13.18}$$

An extended large-N correspondence would predict a similar sine formula for the k-string tension in $\mathcal{N}=1$ orientifold field theory, cf. [35].

14. A view from the string theory side

The orientifold field theory has a nice and natural description in type-0 string theory. We will describe this connection in detail. We start from a short review of orbifolds and orientifolds in type II string theory. We introduce the type-0 string and its various orientifolds. Then we explain how the orientifold field theory is realized via a type-0 brane configuration. We will use this realization to conjecture a supergravity description of the large-N theory. The lift to M-theory of the type 0A brane configuration is also briefly discussed.

14.1. *Orbifolds, orientifolds and quiver theories*

One of the most interesting aspects of string theory is that (in contrast to the point-particle theory) a consistent theory can be formulated on singular spaces, called "orbifolds". The textbooks by Polchinski [85] contain an excellent introduction to orbifolds.

Let us focus on closed string theories, such as type IIA/B. The Z_2 orbifold, obtained by the identification $X^m \to -X^m$, is the simplest example. Closed strings can propagate in the "bulk", but another consistent possibility is to consider also strings that are stuck at the fixed point $X^m = 0$. This is the twisted sector of the closed string theory.

One can introduce D-branes and place them on the orbifold singularity [86]. The massless open string spectrum on N D-branes placed on a Z_k singularity is identified with a $\mathrm{U}(N)^k$ gauge theory with chiral bifundamental matter. These field theories are also known in the literature as "quiver theories".

Following Maldacena [66], Kachru and Silverstein considered [9] D3-branes on an R^6/Z_k orbifold singularity. They conjectured that the $\mathrm{U}(N)^k$ quiver theory that lives on the branes is dual to type IIB string theory on $AdS_5 \times S^5/Z_k$. As we already mentioned in the introduction, some of these theories are non-supersymmetric and, hence, Kachru and Silverstein predicted a large class of large-N non-supersymmetric finite theories.

The AdS/CFT correspondence relates open strings to closed strings. Closed string modes are identified with operators in field theory. In general, untwisted ("bulk") closed strings are identified with the operators that are invariant under the orbifold action. For example, the dilaton is identified with the operator $G_1^2 + G_2^2 + ... + G_k^2$. Twisted modes are identified with non-invariant operators (therefore, they are called "twisted operators"). Examples of the correspondence in orbifold spaces are given in [87] and [88]. An important comment is that when the orbifold breaks supersymmetry, there are always tachyons in the twisted sector. From duality one then expects instabilities at strong coupling on the gauge theory side [28].

Similarly to orbifolds, one can consider orientifolds. The latter are obtained by gauging the transformation

$$\Omega : \, X^m(z,\bar{z}) \to -X^m(\bar{z},z) \, ,$$

a combination of world-sheet parity and a space-time reflection. One can think about orientifolds as mirrors that sit at the fixed point $X^m = 0$. The closed strings and their mirrors propagate in the bulk. The strings become unoriented. In contrast to orbifolds, there is *no* twisted sector in the case of orientifolds.

There are two options: one can place either (i) a D-brane on the orientifold fixed point, or (ii) a D-brane and its mirror with respect to the orientifold, see Fig. 8. Usually we have a $\mathrm{U}(N)$ gauge factor on a stack of N D-branes (N^2 combinations of massless open strings). If the D-branes sit on top of an orientifold, we will obtain either $\mathrm{SO}(N)$ or $\mathrm{Sp}(N)$ gauge groups. The reason is that only $(N^2 \pm N)/2$ combinations of open strings survive the projection. The second situation in which a closed string and its mirror are propagating between a D-brane and an orientifold plane is illustrated in Fig. 8. The same process can be described in terms of circulating open strings.

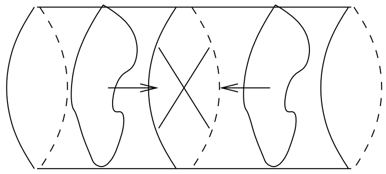


Figure 8. A D-brane, a closed string and their mirrors in the presence of an orientifold plane.

The generalization of the AdS/CFT correspondence to $\mathcal{N}=4$ theories with SO(N) or Sp(N) gauge groups is obtained by considering N D3-branes on top of an O3-plane. The resulting space is $\mathrm{AdS}_5 \times RP^5$. Since there are no twisted sectors for orientifolds, the AdS/CFT correspondence dictionary between the operators and fields for SO/Sp theories is obtained by selecting Z_2 invariant fields and their corresponding operators from the U(N) dictionary.

14.2. *Type-0 string theory*

Type 0A/B string theories are non-supersymmetric closed string theories with world-sheet supersymmetry. The world-sheet definition of these theories is exactly the same as that of the type II string theories. The difference between the theories is the non-chiral (diagonal) Gliozzi–Scherk–Olive (GSO) projection [89]. This projection omits the space-time fermions (the NS-R sector) but keeps the following bosonic sectors [76] in the spectrum:

$$\text{type 0A}: \ (NS-, NS-) \oplus (NS+, NS+) \oplus (R+, R-) \oplus (R-, R+)\,;$$

$$\text{type 0B}: \ (NS-, NS-) \oplus (NS+, NS+) \oplus (R+, R+) \oplus (R-, R-)\,.$$

in the spectrum.

In particular, there is a *tachyon* in the (NS-,NS-) sector. Because of the doubling (with respect to the type II) of the R-R sector, there will be two kinds of D-branes. These two kinds are often called “electric” and “magnetic” branes, or fractional branes. The type-0 string can also be thought of as an orbifold of the type-II string. In particular, type 0B is obtained by orbifolding type IIB by the space-time fermion number $(-1)^{F_s}$. Then, we can think about untwisted and twisted fields and, accordingly, about untwisted and twisted D-branes. In the following we will be interested in the brane configurations that consist of coincident pairs of N electric and N magnetic D-branes, or a set of N untwisted branes in the orbifold language. The untwisted branes are analogous to the type-II branes.

14.3. *Type-0 orientifolds*

The type-0B string in ten dimensions admits three kinds of orientifolds [90]. We will be interested mostly in Sagnotti's *non-tachyonic* orientifold [91,92]. The projection is $\Omega' = \Omega(-1)^{f_R}$. It acts on the oscillators as

$$\Omega' \ \alpha_n^\mu \ \Omega' = \tilde{\alpha}_n^\mu \,,$$

$$\Omega' \ \Psi_r^\mu \ \Omega' = \tilde{\Psi}_r^\mu \,,$$

$$\Omega' \ \tilde{\Psi}_r^\mu \ \Omega' = \Psi_r^\mu \,,$$

$$\Omega' \ \ |0\rangle_{\rm NS-NS} = - \ |0\rangle_{\rm NS-NS} \,.$$

In particular, the tachyon (the NS-NS vacuum) is not invariant under Ω' and, hence, it is projected out. In addition, Ω' projects out one set of the R-R fields. In order to cancel the R-R tadpole one has to introduce 32 D-branes. The massless open string spectrum on the branes forms the ten-dimensional U(32) gauge theory with an antisymmetric Dirac fermion. In this rather unusual orientifold, the projection acts only on the fermionic open strings and, therefore, there are N^2 gauge bosons as in the oriented theory and only $2 \times \frac{1}{2}(N^2 - N)$ massless fermions. In a sense, the closed string part of the resulting theory looks as the bosonic part of the type-I string. It is a tachyon-free bosonic string theory with an open string sector.

14.4. *Brane configurations*

The orientifold field theories (either orienti A or orienti S) can be embedded in a brane configuration of type-0A string theory. Let us briefly review the corresponding construction.

Let us first describe how $\mathcal{N}=1$ SYM theory is realized in type-IIA string theory [60, 93, 94] (see also [95]). One can consider a brane configuration that consists of N D4-branes suspended between orthogonal NS5-branes. The world-volume of the D4-branes is 01234, those of the NS5-branes are 012356 and 0123478. This brane configuration is depicted in Fig. 9.

Moreover, one can consider an analogous brane configuration in type-0A string theory [16], see Fig. 10. The type-IIA D4-branes have to be replaced by the untwisted D4-branes of type 0A (N electric plus N magnetic branes). The field theory on the brane is a gauge theory with a U(N)×U(N) gauge group and a bifundamental fermion (the Z_2 orbifold field theory discussed in detail in Sect. 4). Note that, since type-0A string theory contains a closed string tachyon whose mass is $-2/\alpha'$, it is not obvious at all that there is a decoupling limit. Moreover, the tachyon that lives in the twisted sector

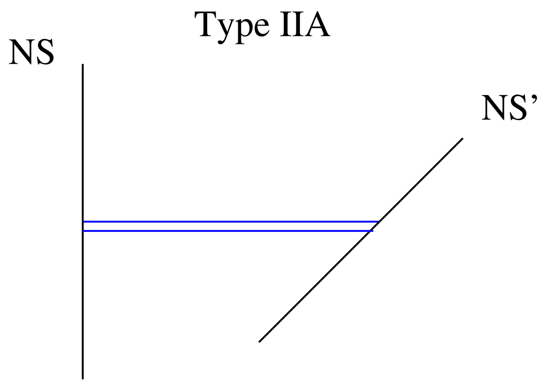


Figure 9. The type-IIA brane configuration. The field theory on the branes is $\mathcal{N}=1$ SYM theory.

couples to the twisted operator $\mathrm{Tr}\, G_{ee}^2 - \mathrm{Tr}\, G_{mm}^2$ on the brane. If we adopt an AdS/CFT philosophy, then the above-mentioned field theory should suffer from instabilities [28]. And it does! These problems were discussed in the field-theory part of this review, see Sect. 5.

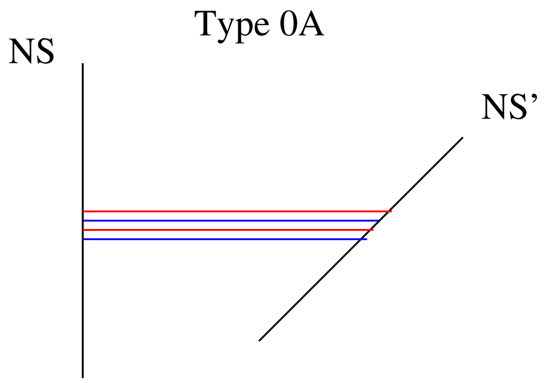


Figure 10. A type-0A brane configuration. The field theory on the branes is a gauge theory with gauge group $\mathrm{U}(N)\times\mathrm{U}(N)$ and a bifundamental Dirac fermion (the Z_2 orbifold field theory).

In order to have a better-behaved model, one can use the orientifold construction from the previous section. We can add an O'4-plane on top of the D4-branes. The resulting field theory is our orientifold field theory (A or S, depending on the orientifold charge).

This is the four-dimensional $\mathrm{U}(N)$ gauge theory with a Dirac fermion in the antisymmetric (symmetric) two-index representation. Note that the bulk tachyon does not couple to the field theory and, therefore, the decoupling limit is no longer problematic. In addition (and as a consequence), we do not expect any instabilities in field theory. This is the underlying reason why non-supersymmetric orientifold field theories are "better" than orbifold field theories. The brane configuration that realizes the orientifold field theory is depicted in Fig. 11.

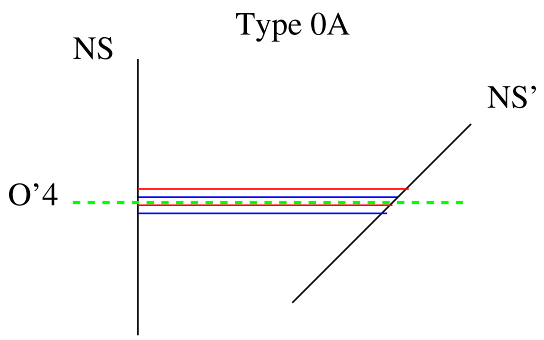


Figure 11. A type-0A brane configuration with an orientifold plane. The field theory on the branes is a U(N) gauge theory with a Dirac fermion in the antisymmetric (symmetric) two-index representation.

Since we discuss a non-supersymmetric brane configuration, a natural question is the stability issue. In general, non-BPS configurations are unstable. In the present case the situation is better, however. The untwisted D4-branes of type-0 string theory behave, at tree level ($g_{\rm st} = 0$), like the BPS branes of type-IIB string theory. The annulus diagram vanishes, implying that there are no forces between the branes.

From the analysis of the above brane configuration we learn that the orientifold field theory is better than the orbifold field theory due to the absence of the twisted sector in the former. The brane realization also teaches us that a useful way of thinking about the orientifold field theory is as if we had a U(N)×U(N) field theory, but with the two gauge factors identified. We exploited this way of thinking of the orientifold field theory in our proof [6] of planar equivalence between $\mathcal{N}=1$ SYM theory and orientifold theories.

14.5. *AdS/CFT*

In this section we would like to provide a dual-gravity description of the large-N orientifold field theory. The large-N gauge theories were conjectured to have a dual-string description. The prime example is the SU(N) $\mathcal{N}=4$ SYM theory, which is dual to the type-IIB string theory on $\mathrm{AdS}_5 \times S^5$.

Consider the orientifold field theory, which is the daughter of $\mathcal{N}=4$ SYM theory. The gauge group is SU(N) in both the parent and daughter theories. The orientifold daughter has six scalars in the adjoint representation of SU(N) and four Dirac fermions in the antisymmetric two-index representation.

This theory lives on a collection of self-dual (untwisted) D3-branes and an O'3-plane of type-0B string theory. It is natural, by considering the near-horizon geometry, to suggest [67] that its string dual is type-0B string theory on $\mathrm{AdS}_5 \times RP^5$. In the gravity approximation, the solution, as well as the dynamics, are almost identical to those of the parent supersymmetric theory. This is because the massless closed string modes of type 0B are the bosonic modes of type IIB. The effective gravity action of the non-tachyonic type 0B is also identical to the bosonic part of the type-IIB action. Hence, the dual string theory predicts that the large-N orientifold field theory will coincide with the supersymmetric theory in the bosonic sector. This is in agreement with field-theory expectations.

Let us consider now the field-theory analogue of $\mathcal{N} = 1$ SYM theory. In this case the problem is more involved, since there is no known supergravity dual to just $\mathcal{N} = 1$ SYM theory. There is, however, a supergravity dual to a theory that flows to $\mathcal{N} = 1$ SYM theory in the IR limit. This is the Maldacena-Nuñez solution [68]. We can consider an analogous solution in non-tachyonic type-0 string theory. Since the gravity action is identical to type II, it is guaranteed that such a solution exists also in the non-supersymmetric case. Thus, all predictions of supergravity can be copied in the large-N orientifold field theory, as long as only the bosonic sector is considered. In particular, all correlators will be the same and the bosonic glueball spectra should agree. Note that in type-0 string theory there are no fermionic closed strings, in agreement with the absence of fermionic glueballs in the orientifold field theory. Moreover, we will have N degenerate vacua and a gluino condensate. Again, all the above is in full agreement with the field-theory results.

Can we use the dual string theory description to calculate the $1/N$ corrections that make the supersymmetric parent theory and non-supersymmetric daughter different? This is a real challenge. On the string theory side, we expect that these effects will be due to a dilaton tadpole term that exists in the type-0 effective action but does not exist in the type-II action [67]. Understanding the finite-N theory could lead to predictions for one-flavor QCD from gravity! Were this feasible, it would represent a major breakthrough.

14.6. *The bifermion condensate from M-theory*

In this section we discuss the strong-coupling regime of the orientifold field theory at large-N starting from the relation of type-0A string theory to M-theory.

Similarly to type-IIA string theory, type-0A can be obtained from M-theory by a compactification of the eleventh dimension on a Scherk–Schwarz circle [96]. By using this conjecture we can lift the brane configuration displayed in Fig. 11 to M-theory, analogously to the lift of the analogous type-IIA brane configuration [60, 93, 94]. Note that the presence of the orientifold plane can be neglected in the large-N limit, since its R-R charge is negligible with respect to the R-R charge of the branes (N). Similarly to the type-IIA situation, we will obtain a smooth M5-brane and the resulting curve (the shape of the M5-brane) will be the same as the curve of $\mathcal{N}=1$ SYM theory [94],

$$S^N = 1. \tag{14.1}$$

The meaning of this curve is that there are N vacua with the bifermion condensate as the order parameter, in agreement with our field-theory results.

15. Conclusions

This review is devoted to a recent development, which will hopefully produce a strong impact on QCD and QCD-like theories: planar equivalence between distinct gauge theories at strong coupling, proved both at the perturbative and at the non-perturbative level. The "main" parent gauge theory is $\mathcal{N}=1$ gluodynamics. Although it is not completely solved, a rather extensive understanding of this theory emerged in the 1980s and '90s. The orientifold field theories, being planar-equivalent to $\mathcal{N}=1$ gluodynamics, inherit its most salient features: an infinite number of spectral degeneracies in the bosonic sector at $N=\infty$, discrete well-defined vacua labeled by an order parameter, the quark condensate, confinement with a mass gap, etc. The orientifold theory A is special. At $N=3$ it is not just another cousin of QCD; it identically coincides with one-flavor QCD. This gives a hope, for the first time ever, to obtain a direct relation between physical observables in SYM theories and those in one-flavor QCD. To this end working tools must be found, which will allow us to treat $1/N$ corrections (say, the leading post-planar-equivalence corrections). Finding such tools would amount to a major breakthrough in QCD. At the moment, progress in this direction is modest. Performing a rather crude estimate of $1/N$ corrections, we were able to evaluate the quark condensate in one-flavor QCD starting from the known exact expression for the gluino condensate in $\mathcal{N} = 1$ gluodynamics. Our result is in agreement with the existing empiric evaluations, although the uncertainties on both sides — theoretical and experimental — are rather large.

Our presentation sometimes follows and sometimes deviates from the historical path. The very notion of the planar equivalence between distinct gauge theories (with varying degrees of supersymmetry) was born in the depths of string theory. We discuss the genesis of the idea in the very beginning of our review, and then essentially leave string theory to Sect. 14. This section collects together, in concise form, string-theory interpretations of various results on the planar equivalence discussed in the bulk of the review. This is done on purpose. We hope that the above interpretations will turn out to be inspirational in solving the issue of $1/N$ corrections — today's number-one problem as we see it.

Meanwhile we initiate and carry out discussions of other issues, which are conceptually related to the statement of planar equivalence. These include "BPS" domain walls and "chiral rings" in orientifold theories (both turn out to be well-defined in the absence of SUSY!), the vanishing of the vacuum energy density (cosmological term) at $O(N^2)$, parent–daughter pairs with no supersymmetry (proliferation of flavors), and so on. Although each of the above issues can (and must) be elaborated in more detail in the future, we feel that outlining novel directions is as important now as the pursuit of the actual solutions. In this aspect our review is unconventional: besides summarizing results already published in the original literature, we pose and discuss new questions, e.g. in Sects. 10, 12, 13, and Appendix C.

Acknowledgments

We are grateful to O. Aharony, C. Angelantonj, J. Barbon, O. Bergman, I. Brunner, K. Chetyrkin, A. Czarnecki, P. H. Damgaard, P. de Forcrand, P. Di Vecchia, Y. Frishman, G. Gabadadze, L. Giusti, A. Gorsky, A. Hanany, A. Hoang, N. Itzhaki, A. Kataev, D. Kazakov, V. Khoze, I. Kogan, W. Lerche, A. Loewy, B. Lucini, M. Lüscher, J. Maldacena, Y. Oz, M. Peskin, J. Polchinski, R. Rabadan, E. Rabinovici, P. Ramond, A. Ritz, F. Sannino, A. Schwimmer, Y. Shadmi, A. Smilga, J. Sonnenschein, L. Susskind, M. Teper, D. Tong, and A. Vainshtein, for discussions. The initial stages of this work were carried out while M.S. was visiting CERN Theory Division, to which he is grateful for the kind hospitality. The work of M.S. was supported in part by DOE grant DE-FG02-94ER408.

Appendix A: SUSY relics in pure Yang–Mills theory

In Sect. 8 we suggested a new "orientifold" large-N expansion that connects one-flavor QCD to $\mathcal{N}=1$ gluodynamics. The 't Hooft $1/N$ expansion connects one-flavor QCD to pure SU(3) Yang–Mills theory. Therefore, the pure SU(3) gauge theory is also approximated, in a sense, by a supersymmetric theory.

Although it is reasonable to suspect that in this case the errors of the approximations made accumulate, it still makes sense to ask whether there are any relics of SUSY in the SU(3) Yang–Mills theory.

In fact, it has been known for a long time [38] that such relics do exist in pure gauge theory, although at that time they were not interpreted in terms of supersymmetry. Indeed, it was shown [38] that one can use an approximate holomorphic formula

$$\left\langle \mathrm{Tr}\, G^2 + i\, \mathrm{Tr}\, \tilde{G}G \right\rangle = M_{\mathrm{UV}}^4\, e^{-\tau/3}\,, \tag{A.1}$$

$$\tau = \frac{8\pi^2}{g^2} + i\,\theta\,. \tag{A.2}$$

(For this equation to be holomorphic in τ, renormalization-group-invariant and, in addition, to have the correct θ dependence, we must use $\beta_0 = 12$ rather than the actual value $\beta_0 = 11$ in SU(3) Yang–Mills theory. The decomposition $11 = 12 - 1$ has a deep physical meaning; -1 presents a unitary contribution in the one-loop β function, while 12 is a *bona fide* antiscreening, see e.g. [97]). The holomorphic dependence is a relic of supersymmetry. Equation (A.2) implies, in particular, that the topological susceptibility χ is given in terms of the expectation value of $\mathrm{Tr}\, G^2$ by

$$\begin{aligned} 12\chi &\equiv -i \int d^4x \left\langle \frac{\sqrt{3}}{16\pi^2}\, G^a \tilde{G}^a(x)\,,\ \frac{\sqrt{3}}{16\pi^2}\, G^a \tilde{G}^a(0) \right\rangle_{\mathrm{conn}} \\ &= \left\langle \frac{1}{8\pi^2}\, G^a G^a \right\rangle . \end{aligned} \tag{A.3}$$

The numerical value of the left-hand side is known, either from the WV formula [40,41] or from lattice measurements [98], to be $\approx 1.3 \times 10^{-2}$ GeV^4. The gluon condensate on the right-hand side is that of pure Yang–Mills theory and was estimated to be approximately twice larger (see Ref. [38], Sect. 14) than the gluon condensate in actual QCD,

$$\left\langle \frac{1}{4\pi^2} G^a G^a \right\rangle_{\mathrm{QCD}} \approx 0.012\, \mathrm{GeV}^4,$$

see Ref. [99]. If so, the numerical value of the right-hand side of (A.3) is $\approx 1.2 \times 10^{-2}$ GeV^4. Note that the phenomenological estimate of the gluon condensate and the factor 2 enhancement mentioned above are valid up to $\sim 30\%$.

Appendix B: Relation between $\Lambda_{\rm PV}$ and Λ's used in perturbative calculations

The standard regularization in performing loop calculations in perturbation theory in non-Abelian gauge theories is dimensional regularization, supplemented by $\overline{\rm MS}$ renormalization. Determinations of the scale parameter Λ in QCD that can be found in the literature refer to this scheme. At the same time, in calculating nonperturbative effects (e.g. instantons) one routinely uses the Pauli-Villars (PV) regularization scheme. This is because the instanton field is (anti)self-dual, and this notion cannot be continued to $4-\varepsilon$ dimensions. The PV regularization/renormalization scheme in this context was suggested by 't Hooft [100]; it was further advanced in supersymmetric instanton calculus in Ref. [101], see also the review paper [24].

This previously created no problem, since there were no analytical predictions for non-perturbative quantities in QCD in terms of Λ that could be compared with measured quantities. With the advent of the orientifold $1/N$ expansion, this becomes a problem. Indeed, in the parent SUSY theory non-perturbative quantities are known in terms of $\Lambda_{\rm PV}$. We then copy them in the large-N limit of the orientifold daughter theory, extrapolate down to $N=3$, and thus get a prediction in one-flavor QCD in terms of $\Lambda_{\rm PV}$. In order to calibrate the prediction one needs to know the relation between $\Lambda_{\rm PV}$ and $\Lambda_{\overline{\rm MS}}$.

In fact, the situation is slightly more complicated. Dimensional regularization *per se* cannot be used in SUSY theories, since it breaks the balance between the number of the fermionic and bosonic degrees of freedom. For instance, in SUSY gluodynamics, the standard dimensional regularization would effectively imply $d-2$ bosonic degrees of freedom and 2 fermionic. The problem is solved by elevating dimensional regularization to *dimensional reduction* (DRED), with the subsequent application of the $\overline{\rm MS}$ procedure. The supersymmetry is maintained because even in $d=4-\varepsilon$ dimensions, the numbers of the fermionic and bosonic degrees of freedom match. In the above example of SUSY gluodynamics it is 2 versus 2.

Our task here is to derive the relation between $\Lambda_{\rm PV}$ and $\Lambda_{\overline{\rm MS}}$. At the one-loop level, this is a rather simple exercise, a direct generalization of 't Hooft's argument [100]. An exact relation between two renormalized gauge couplings (one in PV, another in DRED + $\overline{\rm MS}$) at one loop will allow us to obtain the ratio $\Lambda_{\rm PV}/\Lambda_{\overline{\rm MS}}$ up to $O(\alpha)$ corrections. This is sufficient for practical purposes.

B.1. *Brief history*

Before delving into this simple calculation it would be in order to say a few words on history and literature, since it is rather confusing. The first analysis of $\alpha_{\rm PV}$ versus $\alpha_{\overline{\rm MS}}$ at one loop was carried out by 't Hooft, in connection with instantons; see Sect. XIII of his pioneering paper [100]. Unfortunately, the key expression (13.7) contained an error.[a] The error was noted and corrected by Hasenfratz and Hasenfratz [103], see also Ref. [104]. What is confusing is the fact that a later reprint of 't Hooft's paper (see [105]) reproduces the corrected derivation, and the updated result (13.9), without mentioning that the required corrections had been made.[b] A subtle point in the derivation, which in fact led to the above controversy, is present only in non-supersymmetric theories, such as pure Yang–Mills theory or QCD. It is absent in supersymmetric theories. This circumstance will immensely facilitate our analysis.

B.2. *Relating $\alpha_{\rm PV}$ and $\alpha_{\overline{\rm MS}}$ at one loop*

We will show that in supersymmetric gauge theories, at one loop,

$$\alpha_{\rm PV}^{\rm renorm} = \alpha_{\overline{\rm MS}}^{\rm renorm}\,, \tag{B.1}$$

where the left-hand side refers to the Pauli-Villars procedure, while the right-hand side to DRED+$\overline{\rm MS}$. The important aspect of this equality is that it applies to non-logarithmic terms. Equation (B.1) implies, in turn, that

$$\Lambda_{\rm PV} = \Lambda_{\overline{\rm MS}}\left(1 + O(\alpha_0)\right)\,. \tag{B.2}$$

To keep our derivation as simple as possible, we will limit ourselves to SQED. Generalization to non-Abelian theories is absolutely straightforward.

[a] Unfortunately, this error propagated in part in some reviews, e.g. Ref. [102].
[b] Equation (13.7) of the reprinted article still contains a typo, -1 on the right-hand side should be replaced by $-1/2$. This misprint has no impact on the subsequent expressions of the reprinted article.

The Lagrangian of SQED is

$$\mathcal{L} = \left(\frac{1}{8e_0^2} \int d^2\theta \, W^2 + \text{h.c.} \right) + \frac{1}{4} \int d^4\theta \left(\bar{S} e^V S + \bar{T} e^{-V} T \right) + \left(\frac{m_0}{2} \int d^2\theta \, S\, T + \text{h.c.} \right) , \tag{B.3}$$

where S and T are chiral superfields of the electric charge $+1$ and -1, respectively. The subscript 0 marks the bare coupling constant and mass. The Lagrangian (B.3) describes photon, electron and two selectrons. We will impose the Wess-Zumino gauge and will use the background field formalism. (For a detailed introduction see the review paper [106]).

The renormalized gauge coupling $1/e^2$ is defined as the coefficient in front of the operator $-\frac{1}{4}\, G_{\mu\nu} G^{\mu\nu}$ in the effective Lagrangian, presented as a sum in various Lorentz- and gauge-invariant operators (which are classified according to the number of derivatives).

The background field formalism assumes that a background field $(A)_{\rm bck}$ is introduced; the matter fields in the loop propagate in the given background. For instance, the contribution of one charged scalar field is given by

$$\left\{ \text{Det}(-D_\mu^2 - m^2) \right\}^{-1} , \tag{B.4}$$

where D_μ is the covariant derivative with respect to the background field,

$$D_\mu = \partial_\mu + i (A_\mu)_{\rm bck} . \tag{B.5}$$

The mass term m is assumed to be real; at one loop the distinction between m and m_0 is unimportant.

The contribution of one charged Dirac fermion can be written as

$$\left\{ \text{Det} \left[\left(-D_\mu^2 - m^2 \right) I - \frac{i}{2}\, \sigma^{\mu\nu} (G_{\mu\nu})_{\rm bck} \right] \right\}^{1/2} , \tag{B.6}$$

where I is the 4×4 unit matrix in the space of the spinorial Lorentz indices, and $\sigma^{\mu\nu} = \frac{1}{2}\, (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)$.

The second term in the square brackets is the magnetic interaction of the electron with the background field. The scalar field (selectron), having no spin, has no magnetic interaction; in this case the background field is coupled to the electric charge (through the covariant derivative).

Let us switch off, for a moment, the magnetic interaction of the electron. Then the electron contribution to the effective Lagrangian reduces to

$$\left\{ \text{Det} \left(-D_\mu^2 - m^2 \right) I \right\}^{1/2} = \left\{ \text{Det} \left(-D_\mu^2 - m^2 \right) \right\}^2 , \tag{B.7}$$

and is exactly canceled by the contribution of two selectrons (the square of Eq. (B.4)). This cancellation is due to the balance of the fermion and boson degrees of freedom inherent in supersymmetry. It is important that DRED does not destroy this balance — the cancellation will hold in $4-\varepsilon$ dimensions too.

Thus, the charge coupling drops out,[c] and it is only the magnetic interaction of the electron that determines the gauge-coupling renormalization at one loop. One obtains it by expanding Eq. (B.6) in $\sigma^{\mu\nu}(G_{\mu\nu})_{\rm bck}$. It is quite obvious that the linear term vanishes; the first non-vanishing term is quadratic in $\sigma^{\mu\nu}(G_{\mu\nu})_{\rm bck}$, and it is proportional to (after the Wick rotation of the integration momentum)

$$\mathrm{Tr}\left\{\sigma^{\mu\nu}(G_{\mu\nu})_{\rm bck}\,\sigma^{\alpha\beta}(G_{\alpha\beta})_{\rm bck}\right\}\int\frac{d^4p}{(2\pi)^4}\,\frac{1}{(p^2+m^2)^2}\bigg|_{\rm reg}\,, \tag{B.8}$$

where the subscript reg means that the expression must be regularized. In DRED we replace

$$\frac{d^4p}{(2\pi)^4}\to\frac{d^{4-\varepsilon}p}{(2\pi)^{4-\varepsilon}}\,, \tag{B.9}$$

while in the Pauli-Villars regularization we replace the integrand

$$\frac{1}{(p^2+m^2)^2}\to\frac{1}{(p^2+m^2)^2}-\frac{1}{(p^2+M_{\rm UV}^2)^2}\,. \tag{B.10}$$

As a result, expression (B.8) reduces to

$$-\frac{1}{\pi^2}\,(G_{\mu\nu})_{\rm bck}\,(G^{\mu\nu})_{\rm bck}\times\begin{cases}\frac{1}{\varepsilon}-\ln m-\frac{\gamma}{2}+\frac{\ln(4\pi)}{2}\,, & \text{DRED}\\ \ln\frac{M_{\rm UV}}{m}\,, & \text{PV}\end{cases} \tag{B.11}$$

where γ is Euler's constant.

The $\overline{\rm MS}$ renormalization prescribes one to replace $\frac{1}{\varepsilon}-\frac{\gamma}{2}+\frac{\ln(4\pi)}{2}$ by $\ln\mu$, while the PV renormalization replaces $M_{\rm UV}$ by the same $\ln\mu$. Thus, in these two schemes, at one loop, the renormalized gauge couplings are related to the bare coupling in one and the same way, i.e.

$$\alpha_{\overline{\rm MS}}^{-1}-\alpha_{\rm PV}^{-1}=0(\alpha)\,, \tag{B.12}$$

[c] The charge contribution is the one that is most labor-consuming. In terms of the instanton calculation of the gauge coupling renormalization, the charge contribution is in one-to-one correspondence with the non-zero modes, while the magnetic contribution corresponds to zero modes. For more details see the review paper [102].

implying, in turn, that

$$\Lambda_{\rm PV} = \Lambda_{\overline{\rm MS}} \,. \tag{B.13}$$

Appendix C: Getting two-loop $\boldsymbol{\beta}$ functions for free [d]

In our above considerations we have repeatedly used the values of the second coefficients in the β functions referring to various theories, which are derivatives of supersymmetric gluodynamics. These coefficients can be borrowed from the literature. They were calculated in the 1970's and 80's [107] through a direct computation of relevant two-loop graphs, exploiting the technique of dimensional regularization [108] — the only regularization scheme sufficiently developed to allow us to get practical results in the Yang–Mills theory beyond one loop. Calculation of two-loop diagrams in the Yang–Mills theory (let alone three loops) is a rather cumbersome task. It requires some experience, and usually, goes beyond the level accessible to field-theory students.

Here we will show how the observations we made in the body of the review allow us to reduce the calculation of the two-loop coefficients of the β functions, in a class of theories related to supersymmetric gluodynamics, to simple algebraic manipulations. The theories that can be treated in this way must have the same $T_{\rm eff}$ as in supersymmetric gluodynamics, i.e. $N/2$ (the definition of $T_{\rm eff}$ is given below). Besides practical simplicity, our analysis will allow us to address another issue, which is very hard (if at all possible) to study in a direct computation of two-loop graphs in dimensional regularization. The dimensional-regularization-based analysis makes no distinction between UV and IR contributions. In supersymmetric gluodynamics we know that the *first* coefficient of the β function is Wilsonian. The second and all higher coefficients in the β function *vanish* in the Wilsonian action; they occur when passing to the canonic action as a result of an anomaly (sometimes referred to as the *holomorphic anomaly* [109]). A detailed discussion of this topic can be found in the original publication [110] and later elaborations [111]. In non-supersymmetric Yang–Mills theories, the problem of what is Wilsonian and what is not has not been addressed so far, even at the two-loop level, let alone higher loops. We turn to this question at the end of this appendix.

In order to be able to follow the presentation below, the reader must be

[d] The above title is a Pickwickian joke — remember "The Posthumous Papers of the Pickwick Club", by Charles Dickens. The calculation of the two-loop $\boldsymbol{\beta}$ function outlined below does reduce to a few simple algebraic manipulations, and, thus, is indeed "for free", except that the price had been previously paid.

familiar with the background field method, as well as instanton calculus. We recommend the following reviews to study instanton calculus:

(a) *ABC of Instantons*, by V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Sov. Phys. Usp. **24**, 195 (1982); an updated and expanded version can be found in M. Shifman, *ITEP Lectures on Particle Physics and Field Theory* (World Scientific, Singapore, 1999), Vol. 1, pp. 201-300; especially Sects. 9.2, 9.3, and 9.4.

(b) *Instantons Versus Supersymmetry: Fifteen Years Later,* by M. Shifman and A. Vainshtein, in M. Shifman, *ITEP Lectures on Particle Physics and Field Theory* (World Scientific, Singapore, 1999) Vol. 2, pp. 485-647; especially Sect. 4.1 [hep-th/9902018].

As for the former, the background-field method goes back to Julian Schwinger's ideas. Extensive references can be found in the review article [106] as well as in a recent publication [112]. The former paper adapts various versions of the background-field technique to non-Abelian gauge-field theories. The latter presents the state-of-the-art of the background field technology. Building on the ideas of Ref. [113], the authors perform an economic and elegant calculation of the β function at *three* loops.

C.1. *The class of theories we will be dealing with*

The theories to be discussed below are SU(N) Yang–Mills theories with a judiciously chosen fermion sector. The composition of the fermion sector must be such that

$$T_{\rm eff} \equiv \sum_{\rm repr} T(R) = \frac{N}{2}\,, \tag{C.1}$$

where the sum runs over all fermions involved in the theory. For instance, in supersymmetric gluodynamics the only fermions are gluinos (the Weyl spinors) in the adjoint representation of SU(N) and, hence, the condition (C.1) is satisfied (the factor 1/2 is due to the "Weyl-ness" of the gluinos).

Another theory in which (C.1) is satisfied is $N_f = N_c$ QCD. For brevity, we refer to this theory as N^2- QCD. The number of colors is N, and so is the number of the quark flavors (the Dirac spinors in the fundamental representation of SU(N)). This theory is obviously non-supersymmetric, but $T_{\rm eff} = N \times (1/2)$, where 1/2 comes from each flavor.

Finally, the third example is Yang–Mills theory with one Dirac spinor in the antisymmetric two-index representation ("orienti") plus two fundamental quarks ("2f"). We refer to this theory as orienti/2f-QCD. At $N = 3$ it becomes three-flavor QCD. In this theory

$$T_{\rm eff} = \frac{N-2}{2} + 2 \times \frac{1}{2} = \frac{N}{2}\,, \tag{C.2}$$

i.e. the effective Casimir coefficient coincides with that in supersymmetric gluodynamics.

What remains to be added? The standard definition of the coefficients of the β function from the Particle Data Group (PDG) is (see (8.3))

$$\mu\,\frac{\partial\alpha}{\partial\mu} \equiv \beta(\alpha) = -\frac{\beta_0}{2\pi}\alpha^2 - \frac{\beta_1}{4\pi^2}\alpha^3 + \ldots\,. \tag{C.3}$$

The values of the first two coefficients borrowed from the literature are displayed in Table 6.

Table 6. The first two coefficients of the β function in N^2-QCD and orienti/2f-QCD (SUSY gluodynamics is shown for comparison).

Theory / Coeff.	N^2-QCD	Orienti/2f	SYM
β_0	$3\,N$	$3\,N$	$3\,N$
β_1	$\frac{7N^2+1}{2}$	$3\,N^2 + 2\,N - \frac{3}{N}$	$3N^2$

Equation (C.3) corresponds to the following effective action:

$$\mathcal{L}_{\rm eff} = -\frac{1}{4}\left\{\frac{1}{g_0^2} - \frac{\beta_0}{8\pi^2}\,\ln\frac{M_{\rm UV}}{\mu} - \frac{\beta_1}{64\pi^4}\,g_0^2\,\ln\frac{M_{\rm UV}}{\mu} + \ldots\right\} G^a_{\mu\nu}G^{\mu\nu,\,a}\,. \tag{C.4}$$

C.2. $T_{\rm eff}$

In SUSY gluodynamics, the gluino contribution is characterized by

$$T(G)_{\rm eff} = \frac{N}{2}\,, \tag{C.5}$$

where 1/2 reflects the fact that gluinos are the Majorana rather than Dirac spinors. In the daughter theories we define

$$T_{\rm eff} = \sum_R T(R)\,, \tag{C.6}$$

where the sum runs over all fermion representations present in the given theory, and it is assumed, for simplicity, that the daughter fermions are Dirac. Otherwise, we should have introduced a 1/2 factor for each Weyl fermion in the sum (C.6).

The class of daughter theories to be considered below is defined by the condition $T_{\rm eff} = N/2$. It is quite obvious that in any theory belonging to this class the first coefficient of the β function is the same as in SUSY gluodynamics, by construction. The second coefficient is almost the same, but not quite.

Why do we say "almost"? Because the vast majority of graphs relevant to the calculation of the β function at two loops are either completely insensitive to fermions (Fig. 12a–c) or, if sensitive, depend only on $T_{\rm eff}$ (Fig. 12d–f), and once $T_{\rm eff}$ is set to be the same as in SUSY gluodynamics, these graphs produce the same result. In fact, the only two diagrams that differentiate SUSY gluodynamics on the one hand, and daughter theories with $T_{\rm eff} = N/2$ on the other, are those depicted in Figs. 12g and h. Note that these two graphs are the simplest in the set.

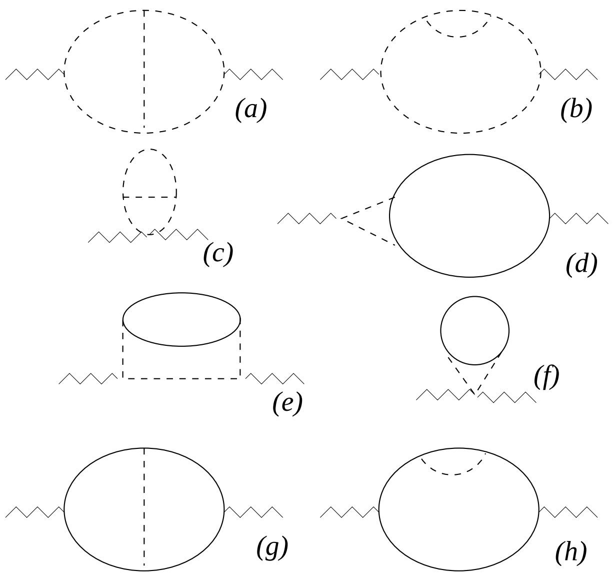


Figure 12. Two-loop contributions to the β function. The wavy line is the background (soft) gluon field, the dashed line stands for the quantum gluon field, the solid line for the fermion fields. The fermion contribution is represented by diagrams d through h.

C.3. *What is known about the last two graphs in Fig. 12?*

Let us compare these two graphs (Figs. 12g and h), with those determining the inclusive cross section of e^+e^- annihilation in hadrons. More exactly, we will truncate the e^+e^- vertex, which is irrelevant to our purposes, and will focus on the part describing the conversion of the virtual photon into quarks (Fig. 13).

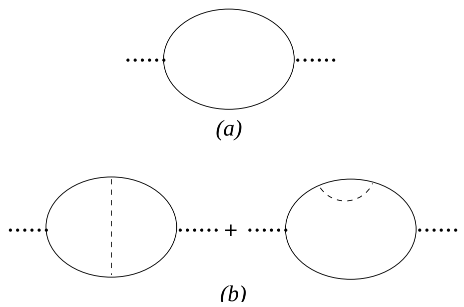


Figure 13. $\gamma^* \to \bar{\psi}\,\psi \to \gamma^*$ in one and two loops. A virtual photon is denoted by the dotted line.

These graphs are treated in textbooks (e.g. [114]), an easy exercise feasible in more than one way. The well-known answer is that the ratio of the diagrams in Fig. 13b to that in Fig. 13a is

$$3\,\frac{g^2}{16\pi^2}\,C(R) \tag{C.7}$$

(for quarks in SU(3) QCD, we have $C(R) = 4/3$, and Eq. (C.7) is nothing but the famous α_s/π correction!). The last two graphs in Fig. 12 differ from the last two graphs in Fig. 13 only by color factors.

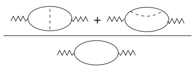
The color factor relevant to Fig. 12h is

$$\mathrm{Tr}\,\left(T^aT^cT^bT^c\right) \to \left[C(R) - \frac{N}{2}\right] T(R)\,\delta^{ab}\,. \tag{C.8}$$

We can drop the term $-NT(R)/2$, since this term is one and the same in SUSY gluodynamics and daughter theories satisfying the condition (C.1) — remember, we consider only such theories here. The color factor relevant to Fig. 12g is

$$\mathrm{Tr}\,\left(T^aT^cT^cT^b\right) \to C(R)T(R)\,\delta^{ab}\,. \tag{C.9}$$

Taking account of the previous remark, we conclude that the overall color factor in the last two diagrams of Fig. 12 is $C(R)T(R)$, and the ratio



$$= 3\,\frac{g^2}{16\pi^2}\,C(R)\,, \tag{C.10}$$

i.e. the same as in Eq. (C.7).

To summarize, the change of the two-loop coefficient in the β function in passing from SUSY gluodynamics to any daughter theory with $T_{\rm eff} = N/2$ is governed by the combination

$$\sum_R C(R)T(R)\,. \tag{C.11}$$

In SUSY gluodynamics the above combination reduces to $N^2/2$. If it were equal to $N^2/2$ in the daughter theory under consideration, this would have the same two-loop coefficient as the SYM theory, namely $3N^2$. With our judicious choice of the fermion sector the value of $\sum_R T(R)$ is fine-tuned, Eq. (C.1), but that of $\sum_R C(R)T(R)$ is not. The change in $\sum_R C(R)T(R)$ is very easy to take into account, however. Combining Eq. (C.4) with (C.10) we obtain [e]

$$\Delta\beta_1 = -2\,\Delta(CT)\,. \tag{C.12}$$

In deriving the above formula we also used the fact that

$$\beta_0 = (11/3)N - (4/3)\sum T(R)\,.$$

C.4. N^2- *QCD*

In QCD with N fundamental quark flavors, $T_{\rm eff}$ is $N/2$, so that this theory belongs to our class. At the same time

$$\sum C_{\rm fund}\,T_{\rm fund} = \frac{N}{2}\left(1 - \frac{1}{N^2}\right)\frac{N}{2} = -\frac{1}{4} + \frac{N^2}{4} \tag{C.13}$$

versus $N^2/2$ in SUSY gluodynamics. Hence, $\Delta(CT) = -(N^2+1)/4$, and

$$\Delta\beta_1 = (\beta_1)_{N^2{\rm QCD}} - (\beta_1)_{\rm SYM} = \frac{N^2+1}{2}\,, \tag{C.14}$$

in full accord with the value presented in Table 6.

[e] For the sign convention, see Eq. (C.30).

C.5. *Orienti/2f-QCD*

The analogue of Eq. (C.13) is

$$\sum_{A,\text{ fund}} C(R)T(R) = \frac{(N-2)^2\,(N+1)}{2N} + \frac{N}{2}\left(1 - \frac{1}{N^2}\right)$$
$$= \frac{N^2}{2} - N + \frac{3}{2N}\,, \tag{C.15}$$

to be compared with $N^2/2$ in SUSY gluodynamics. Equation (C.15) implies that $\Delta(CT) = -N + 3\,(2N)^{-1}$. Note that the leading term proportional to N^2 cancels in $\Delta(CT)$. This cancellation was certainly expected: as we know from the bulk of the review, in the planar limit, the β functions of SYM and orienti/2f theories must coincide. The difference emerges only at the $1/N$ level. From Eq. (C.12) we deduce that

$$\Delta\beta_1 = (\beta_1)_{\text{orienti/2f}} - (\beta_1)_{\text{SYM}} = 2N - \frac{3}{N}\,, \tag{C.16}$$

cf. Table 6.

C.6. *Wilsonian β*

To acquaint the reader with the issue, it is convenient to begin by outlining how the NSVZ β function is obtained in SU(N) SUSY gluodynamics from instanton calculus.[f] We will rephrase the standard argument in a slightly different form, which will, hopefully, make the presentation somewhat simpler.

Let us introduce a small mass term for the gluino field, m_{adj}; then the integration over the fermion moduli can be carried out explicitly. The instanton measure reduces to a product of an integral over the bosonic moduli, $\rho^{-5}\,d\rho\,d^4x_0\,d\Omega$, times the prefactor

$$M_{\text{UV}}^{3N}\,m_{\text{adj}}^{N}\left(\frac{1}{g_0^2}\right)^{2N}\exp\left(-\frac{8\pi^2}{g_0^2}\right)\,. \tag{C.17}$$

Here ρ is the instanton radius, x_0 its center, while Ω parametrizes the instanton color orientation in SU(N). Furthermore, M_{UV} is the ultraviolet (Pauli-Villars) cut-off, g_0^2 and m_{adj} are the gauge coupling and the gluino mass, respectively, normalized at M_{UV}. Supersymmetry endows the prefactor (C.17) with a crucial property: the bosonic and fermionic non-zero

[f] For a review see Refs. [24, 115]. *Note that here we limit ourselves to one and two loops.*

modes completely cancel each other since the instanton background field conserves two out of four supercharges. As usual, the zero modes are special. The fermion zero modes are represented in Eq. (C.17) by $m_{\rm adj}^N$, while the boson ones by $(g_0^2)^{-2N}$. The prefactor (C.17) is renormalization-group invariant (RGI). This means that a concerted variation of $M_{\rm UV}$ and g_0^2 must leave it intact. For what follows we also need to know that the renormalization-group running of $m_{\rm adj}$ coincides with that of g^2. Differentiating Eq. (C.17) with respect to $\ln M_{\rm UV}$, equating the result to zero, and recalling that $\partial\alpha_0/\partial\ln M_{\rm UV} = \beta(\alpha_0)$, we arrive at the NSVZ β function.

It is quite obvious that ignoring the zero modes (i.e. $m_{\rm adj}^N\,(g_0^2)^{-2N}$) in the instanton prefactor would result in the vanishing of β_1 and higher coefficients of the β function. Thus, in the instanton derivation, the non-vanishing β_1 in SUSY gluodynamics is a manifestation of the zero-mode effect. From the perturbative calculation side, the very same result emerges as an anomaly [109–111].

Now, what changes if we pass to a daughter theory belonging to the class (C.1)? Let us first examine examples, and then we will build from them a general formula.

N^2-QCD

Again, we introduce a small mass term $m_{\rm fund}$, one and the same for all flavors. The running law of $m_{\rm fund}$ no more coincides with that of g^2 (which was the case for SUSY gluodynamics). However, we can readily find this law at one loop, which is sufficient for the two-loop β function. The anomalous dimension of the relevant bifermion operator is determined by the diagrams of Fig. 14.

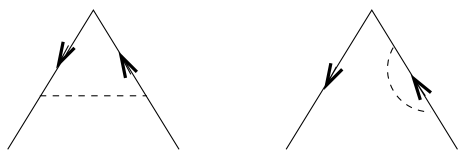


Figure 14. The one-loop anomalous dimension of the bifermion operator (or the mass term). The solid line depicts a fermion, the dashed line a gluon.

Actually, we do *not* have to calculate these graphs. All we have to do is to compare the color factors, which are determined by $C(R)$ in the corresponding representation. Namely, $C_{\rm adj} = N$ and $C_{\rm fund} = (N/2)(1 - N^{-2})$.

Inspecting the graphs of Fig. 14 we readily conclude that

$$m_{\rm fund} \sim \left(g^2\right)^{C_{\rm fund}/N} \sim \left(g^2\right)^{(1/2)(1-N^{-2})}\,, \tag{C.18}$$

where $\sim$ means "runs as." The running law (C.18) is valid at one loop.

Let us *falsely* assume that Eq. (C.18) is the only modification — in this way we ignore the non-zero mode contribution due to the absence of cancellation, that was inherent in SUSY gluodynamics, in non-supersymmetric daughters. At one loop the cancellation will still be operative; the condition $T_{\rm eff} = N/2$ takes care of this. At the two-loop level the non-zero modes do contribute. By focusing on the zero-mode-related part of the β function, we will separate the Wilsonian and anomaly-related parts in β_1.

The zero-mode-induced instanton prefactor in N^2-QCD is

$$\begin{aligned} & M_{\rm UV}^{3N}\, m_{\rm fund}^{N} \left(\frac{1}{g_0^2}\right)^{2N} \exp\left(-\frac{8\pi^2}{g_0^2}\right) \\ \sim\;& M_{\rm UV}^{3N} \left(g^2\right)^{2\,T_{\rm fund}\,C_{\rm fund}/N} \left(\frac{1}{g_0^2}\right)^{2N} \exp\left(-\frac{8\pi^2}{g_0^2}\right), \end{aligned} \tag{C.19}$$

where we used the fact that the power of $m_{\rm fund}$ is in fact $2T$ (cf. Eq. (C.17)). Performing the same procedure as in SUSY gluodynamics (RGI!), we would get a "zero-mode-β" at two loops

$$\text{zero-mode-}\beta = -\frac{3N\alpha^2}{2\pi}\left[1 + \frac{\alpha}{2\pi}\left(2N - \frac{2}{N}\,CT\right)\right]. \tag{C.20}$$

This result implies, in turn, that

$$(\beta_1)_{\rm anom} = (\beta_1)_{\rm SYM} - 6\Delta(CT)\,, \tag{C.21}$$

where

$$\Delta(CT) \equiv (CT)_{\rm fund} - (CT)_{\rm SYM}\,. \tag{C.22}$$

<u>*Orienti/2f*</u>

In addition to $m_{\rm fund}$, we introduce a small mass term m_A for the two-index antisymmetric Dirac spinor that is present in the theory at hand. Given the corresponding value of $C(R)$ and the anomalous dimension it entails (through the diagrams of Fig. 14) we arrive at the conclusion

$$m_A \sim \left(g^2\right)^{C_A/N} \sim \left(g^2\right)^{d}\,, \qquad d \equiv \frac{(N-2)(N+1)}{N^2}\,. \tag{C.23}$$

The zero-mode instanton prefactor takes the form

$$M_{\rm UV}^{3N}\, m_A^{N-2} m_{\rm fund}^2 \left(\frac{1}{g_0^2}\right)^{2N} \exp\left(-\frac{8\pi^2}{g_0^2}\right)$$
$$\sim\; M_{\rm UV}^{3N}\, \left(g^2\right)^{2T_A C_A/N}\, m_{\rm fund}^2 \left(\frac{1}{g_0^2}\right)^{2N} \exp\left(-\frac{8\pi^2}{g_0^2}\right). \tag{C.24}$$

Exploiting the RGI property of Eq. (C.24) we readily obtain the orienti/2f "zero-mode-β,"

$$\text{zero-mode-}\beta = -\frac{3N\alpha^2}{2\pi}\left[1+\frac{\alpha}{2\pi}\left(N+2-\frac{3}{N^2}\right)\right], \tag{C.25}$$

which leads to exactly the same formula as in Eq. (C.20),

$$(\beta_1)_{\rm anom} = (\beta_1)_{\rm SYM} - 6\Delta(CT)\,, \tag{C.26}$$

with

$$\Delta(CT) \equiv \sum_{A,\ {\rm fund}} C(R)T(R) - (CT)_{\rm SYM}\,. \tag{C.27}$$

Master formulae

Summarizing our considerations, we will make here a general statement that is valid for any daughter theory with $T_{\rm eff} = N/2$. In any theory satisfying the above condition the total second coefficient of the β function is

$$(\beta_1)_{\rm D} = (\beta_1)_{\rm SYM} - 2\Delta(CT)\,, \tag{C.28}$$

where D stands for daughter, while the Wilsonian part

$$(\beta_{1\ {\rm Wilsonian}})_{\rm D} = 4\,\Delta(CT)\,. \tag{C.29}$$

Here

$$\Delta(CT) = \sum_R C(R)T(R)_{\rm D} - CT_{\rm SYM}\,. \tag{C.30}$$

In the N^2-QCD and orienti/2f models

$$\Delta(CT) = \begin{cases} -\frac{N^2}{4} - \frac{1}{4}\,, \\ -N + \frac{3}{2N}\,, \end{cases} \tag{C.31}$$

respectively.

References

1. J. Bagger and J. Wess, *Supersymmetry and supergravity* , 2nd edition, (Princeton University Press, Princeton, N.J., 1992).
2. N. Seiberg, Proceedings 4th Int. Symposium on Particles, Strings, and Cosmology (PASCOS 94), Syracuse, New York, May 1994, Ed. K. C. Wali (World Scientific, Singapore, 1995), p. 183 [hep-th/9408013]; Int. J. Mod. Phys. A **12** (1997) 5171 [hep-th/9506077].
3. N. Seiberg and E. Witten, Nucl. Phys. **B426**, 19 (1994) (E) **B430**, 485 (1994) [hep-th/9407087]; Nucl. Phys. **B431**, 484 (1994) [hep-th/9408099].
4. S. Mandelstam, *Phys. Reports* **23**, 245 (1976).
5. G. 't Hooft, in *1981 Cargése Summer School Lecture Notes on Fundamental Interactions, NATO Adv. Study Inst. Series B: Phys.,* Vol. 85, ed. M. Lévy *et al.* (Plenum Press, New York, 1982) [reprinted in G. 't Hooft, *Under the Spell of the Gauge Principle* (World Scientific, Singapore, 1994), page 514]; *Nucl. Phys.* **B190**, 455 (1981).
6. A. Armoni, M. Shifman and G. Veneziano, Nucl. Phys. B **667**, 170 (2003) [hep-th/0302163].
7. A. Armoni, M. Shifman and G. Veneziano, Phys. Rev. Lett. **91**, 191601 (2003) [hep-th/0307097].
8. A. Armoni, M. Shifman and G. Veneziano, Phys. Lett. B **579**, 384 (2004) [hep-th/0309013].
9. S. Kachru and E. Silverstein, Phys. Rev. Lett. **80**, 4855 (1998) [hep-th/9802183].
10. A. E. Lawrence, N. Nekrasov and C. Vafa, Nucl. Phys. B **533**, 199 (1998) [hep-th/9803015].
11. M. Bershadsky, Z. Kakushadze and C. Vafa, Nucl. Phys. B **523**, 59 (1998) [hep-th/9803076].
12. G. 't Hooft, Nucl. Phys. B **72**, 461 (1974).
13. M. Bershadsky and A. Johansen, Nucl. Phys. B **536**, 141 (1998) [hep-th/9803249].
14. Z. Kakushadze, Nucl. Phys. B **529**, 157 (1998) [hep-th/9803214]; Phys. Rev. D **58**, 106003 (1998) [hep-th/9804184].
15. M. Schmaltz, Phys. Rev. D **59**, 105018 (1999) [hep-th/9805218].
16. A. Armoni and B. Kol, JHEP **9907**, 011 (1999) [hep-th/9906081].
17. M. J. Strassler, *On methods for extracting exact non-perturbative results in non-supersymmetric gauge theories,* hep-th/0104032, unpublished.
18. A. Gorsky and M. Shifman, Phys. Rev. D **67**, 022003 (2003) [hep-th/0208073].
19. D. Tong, JHEP **0303**, 022 (2003) [hep-th/0212235].
20. E. Witten, Nucl. Phys. B **202**, 253 (1982).
21. G. Veneziano and S. Yankielowicz, Phys. Lett. B **113**, 231 (1982).
22. G. Veneziano, Phys. Lett. B **124**, 357 (1983).
23. M. A. Shifman and A. I. Vainshtein, Nucl. Phys. B **296**, 445 (1988).
24. M. A. Shifman and A. I. Vainshtein, *Instantons versus supersymmetry: Fifteen years later,* in M. Shifman, *ITEP Lectures on Particle Physics and Field Theory*, (World Scientific, Singapore, 1999) Vol. 2, p. 485-647 [hep-th/9902018].
25. N. M. Davies, T. J. Hollowood, V. V. Khoze and M. P. Mattis, Nucl. Phys. B **559**, 123 (1999) [hep-th/9905015].
26. I. Hinchliffe (for the Particle Data Group (PDG)), Phys. Rev. **D66**, # 1-I, 010001-89 (2002).
27. V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B **229**, 381 (1983); Phys. Lett. B **166**, 329 (1986).
28. I. R. Klebanov, Phys. Lett. B **466**, 166 (1999) [hep-th/9906220].

29. A. Armoni and M. Shifman, Nucl. Phys. B **670**, 148 (2003) [hep-th/0303109].
30. S. Dimopoulos, Nucl. Phys. B **168**, 69 (1980);
M. E. Peskin, Nucl. Phys. B **175**, 197 (1980).
31. Y. I. Kogan, M. A. Shifman and M. I. Vysotsky, Yad. Fiz. **42**, 504 (1985) [Sov. J. Nucl. Phys. **42**, 318 (1985)].
32. J. J. Verbaarschot, Phys. Rev. Lett. **72**, 2531 (1994) [hep-th/9401059];
A. Smilga and J. J. Verbaarschot, Phys. Rev. D **51**, 829 (1995) [hep-th/9404031];
M. A. Halasz and J. J. Verbaarschot, Phys. Rev. D **52**, 2563 (1995) [hep-th/9502096].
33. E. Witten, Nucl. Phys. B **223**, 422 (1983).
34. E. Witten, Nucl. Phys. B **223**, 433 (1983).
35. A. Armoni and M. Shifman, Nucl. Phys. B **671**, 67 (2003) [hep-th/0307020].
36. E. Witten, Nucl. Phys. B **160**, 57 (1979).
37. E. Corrigan and P. Ramond, Phys. Lett. B **87**, 73 (1979).
38. V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B **191**, 301 (1981).
39. G. Veneziano, Nucl. Phys. B **117**, 519 (1976).
40. E. Witten, Nucl. Phys. B **156**, 269 (1979).
41. G. Veneziano, Nucl. Phys. B **159**, 213 (1979).
42. A. Belavin and A. Migdal, *Pis'ma ZhETF* **19** (1974) 317 [*JETP Lett.***19** (1974) 181]; *Scale Invariance and Bootstrap in the Non-Abelian Gauge Theories*, Landau Institute Preprint-74-0894, 1974 (unpublished).
43. T. Banks and A. Zaks, Nucl. Phys. **B196**, 189 (1982).
44. T. Schäfer and E. Shuryak, *Phys. Rev.* **D53**, 6522 (1996).
45. A. Casher, *Phys. Lett.* **B83**, 395 (1979).
46. G. Grunberg, Phys. Rev. D **65**, 021701 (2002) [hep-ph/0009272].
47. Y. Iwasaki *et al.*, *Z. Phys.* **C71**, 343 (1996).
48. D. Kutasov and A. Schwimmer, Nucl. Phys. B **442**, 447 (1995) [hep-th/9501024].
49. K. Demeterfi, I. R. Klebanov and G. Bhanot, Nucl. Phys. B **418**, 15 (1994) [hep-th/9311015].
50. G. Bhanot, K. Demeterfi and I. R. Klebanov, Phys. Rev. D **48**, 4980 (1993) [hep-th/9307111].
51. A. V. Smilga, Phys. Rev. D **49**, 6836 (1994) [hep-th/9402066]; Phys. Rev. D **54**, 7757 (1996) [hep-th/9607007].
52. D. J. Gross, I. R. Klebanov, A. V. Matytsin and A. V. Smilga, Nucl. Phys. B **461**, 109 (1996) [hep-th/9511104].
53. A. Armoni, Y. Frishman and J. Sonnenschein, Nucl. Phys. B **596**, 459 (2001) [hep-th/0011043].
54. A. Armoni, Y. Frishman and J. Sonnenschein, Phys. Lett. B **449**, 76 (1999) [hep-th/9807022].
55. E. Witten, *Supersymmetric index of three-dimensional gauge theory,* in *The Many Faces of the Superworld*, Yuri Golfand Memorial Volume, Ed. M. Shifman (World Scientific, Singapore, 2000), p. 156 [hep-th/9903005].
56. O. V. Tarasov and A. A. Vladimirov, *Three Loop Calculations In Non-Abelian Gauge Theories,* JINR-E2-80-483 (1980); Phys. Lett. B **93**, 429 (1980).
57. L. Giusti, C. Hoelbling and C. Rebbi, Phys. Rev. D **64**, 114508 (2001), (E) **D65**, 079903 (2002) [hep-lat/0108007].
58. M. Lüscher, R. Sommer, P. Weisz and U. Wolff, Nucl. Phys. B **413**, 481 (1994) [hep-lat/9309005]; M. Guagnelli, R. Sommer and H. Wittig [ALPHA collaboration], Nucl. Phys. B **535**, 389 (1998) [hep-lat/9806005].

59. G. R. Dvali and M. A. Shifman, Phys. Lett. B **396**, 64 (1997) (E) Phys. Lett. B **407**, 452 (1997)) [hep-th/9612128].
60. E. Witten, Nucl. Phys. B **507**, 658 (1997) [hep-th/9706109].
61. I. I. Kogan, A. Kovner and M. A. Shifman, Phys. Rev. D **57**, 5195 (1998) [hep-th/9712046].
62. G. R. Dvali, G. Gabadadze and Z. Kakushadze, Nucl. Phys. B **562**, 158 (1999) [hep-th/9901032]; G. Gabadadze and M. A. Shifman, Phys. Rev. D **61**, 075014 (2000) [hep-th/9910050].
63. J. Polchinski and M. J. Strassler, *The string dual of a confining four-dimensional gauge theory,* hep-th/0003136.
64. I. R. Klebanov and M. J. Strassler, JHEP **0008**, 052 (2000) [hep-th/0007191].
65. A. Loewy and J. Sonnenschein, JHEP **0108**, 007 (2001) [hep-th/0103163].
66. J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)] [hep-th/9711200].
67. C. Angelantonj and A. Armoni, Nucl. Phys. B **578**, 239 (2000) [hep-th/9912257].
68. J. M. Maldacena and C. Nuñez, Phys. Rev. Lett. **86**, 588 (2001) [hep-th/0008001].
69. J. Polchinski, Phys. Rev. Lett. **75**, 4724 (1995) [hep-th/9510017].
70. A. Ritz, *(Dis)assembling composite supersymmetric solitons*, in *Continuous Advances in QCD 2002/Arkadyfest*, Eds. K. Olive, M. Shifman, and M. Voloshin (World Scientific, Singapore, 2002), pages 345-368.
71. A. Ritz, M. Shifman and A. Vainshtein, Phys. Rev. D **66**, 065015 (2002) [hep-th/0205083].
72. A. A. Migdal and M. A. Shifman, Phys. Lett. B **114**, 445 (1982).
73. B. S. Acharya and C. Vafa, *On domain walls of $N = 1$ supersymmetric Yang–Mills in four dimensions,* hep-th/0103011.
74. E. Witten, Adv. Theor. Math. Phys. **2**, 505 (1998) [hep-th/9803131].
75. C. Csaki, H. Ooguri, Y. Oz and J. Terning, JHEP **9901**, 017 (1999) [hep-th/9806021].
76. I. R. Klebanov and A. A. Tseytlin, Nucl. Phys. B **546**, 155 (1999) [hep-th/9811035].
77. V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B **229**, 407 (1983).
78. F. Cachazo, M. R. Douglas, N. Seiberg and E. Witten, JHEP **0212**, 071 (2002) [hep-th/0211170].
79. J. Erlich and A. Naqvi, JHEP **0212**, 047 (2002) [hep-th/9808026].
80. K. Konishi, Phys. Lett. B **135**, 439 (1984).
81. R. Dijkgraaf and C. Vafa, Nucl. Phys. B **644**, 3 (2002) [hep-th/0206255], Nucl. Phys. B **644**, 21 (2002) [hep-th/0207106].
82. F. Cachazo, K. A. Intriligator and C. Vafa, Nucl. Phys. B **603**, 3 (2001) [hep-th/0103067].
83. A. Gorsky, Phys. Lett. B **554**, 185 (2003) [hep-th/0210281].
84. M. R. Douglas and S. H. Shenker, Nucl. Phys. B **447**, 271 (1995) [hep-th/9503163].
85. J. Polchinski, *String Theory*, Vols. 1 and 2, (Cambridge Univ. Press, Cambridge, 1998).
86. M. R. Douglas and G. W. Moore, *D-branes, Quivers, and ALE Instantons,* hep-th/9603167, unpublished.
87. Y. Oz and J. Terning, Nucl. Phys. B **532**, 163 (1998) [hep-th/9803167].
88. S. Gukov, Phys. Lett. B **439**, 23 (1998) [hep-th/9806180].
89. F. Gliozzi, J. Scherk and D. I. Olive, Phys. Lett. B **65**, 282 (1976).
90. R. Blumenhagen, A. Font and D. Lüst, Nucl. Phys. B **560**, 66 (1999) [hep-th/9906101].
91. A. Sagnotti, *Some properties of open string theories,* hep-th/9509080.

92. A. Sagnotti, Nucl. Phys. Proc. Suppl. **56B**, 332 (1997) [hep-th/9702093].
93. K. Hori, H. Ooguri and Y. Oz, Adv. Theor. Math. Phys. **1**, 1 (1998) [hep-th/9706082].
94. A. Brandhuber, N. Itzhaki, V. Kaplunovsky, J. Sonnenschein and S. Yankielowicz, Phys. Lett. B **410**, 27 (1997) [hep-th/9706127].
95. A. Giveon and D. Kutasov, Rev. Mod. Phys. **71**, 983 (1999) [hep-th/9802067].
96. O. Bergman and M. R. Gaberdiel, JHEP **9907**, 022 (1999) [hep-th/9906055].
97. V. A. Novikov *et al.*, Phys. Rept. **41**, 1 (1978), Sect. 1.3.
98. B. Alles, M. D'Elia and A. Di Giacomo, Nucl. Phys. B **494**, 281 (1997) [hep-lat/9605013].
99. M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B **147**, 385 (1979), Nucl. Phys. B **147**, 448 (1979).
100. G. 't Hooft, Phys. Rev. D **14**, 3432 (1976) (E) **D18**, 2199 (1978).
101. V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B **260**, 157 (1985);
102. V. Novikov et al., *ABC of Instantons*, in M. Shifman, *ITEP Lectures on Particle Physics and Field Theory*, (World Scientific, Singapore, 1999), Vol. 1. p. 201, Sects. 9.5 and 12.4.
103. A. Hasenfratz and P. Hasenfratz, Nucl. Phys. B **193**, 210 (1981).
104. G. M. Shore, Annals Phys. **122**, 321 (1979).
105. G. 't Hooft, in *Instantons in Gauge Theories*, Ed. M. Shifman (World Scientific, Singapore, 1994) p. 70. Note that Eq. (13.7) of the original paper becomes Eq. (13.6) in the reprint, while Eq. (13.7) of the reprint still contains a typo, -1 on the right-hand side should be replaced by -1/2. This misprint has no impact on Eq. (13.8) of the reprinted article.
106. V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Fortsch. Phys. **32**, 585 (1985).
107. The three-loop RGI functions were calculated in the Yang–Mills theory in O. V. Tarasov, A. A. Vladimirov and A. Y. Zharkov, Phys. Lett. B **93**, 429 (1980); S. A. Larin and J. A. M. Vermaseren, Phys. Lett. B **303**, 334 (1993) [hep-ph/9302208]. These calculations are extended to include scalar fields and Yukawa couplings in A. G. M. Pickering, J. A. Gracey and D. R. T. Jones, Phys. Lett. B **510**, 347 (2001) [Phys. Lett. B **512**, 230 (2001) (E) B535, 377 (2002)] [hep-ph/0104247].
108. C. G. Bollini and J. J. Giambiagi, Phys. Lett. B **40**, 566 (1972); Nuovo Cim. B **12**, 20 (1972); G. 't Hooft and M. J. G. Veltman, Nucl. Phys. B **44**, 189 (1972).
109. L. J. Dixon, V. Kaplunovsky and J. Louis, Nucl. Phys. B **355**, 649 (1991); M. A. Shifman and A. I. Vainshtein, Nucl. Phys. B **359**, 571 (1991);
110. M. A. Shifman and A. I. Vainshtein, Nucl. Phys. B **277**, 456 (1986).
111. N. Arkani-Hamed and H. Murayama, JHEP **0006**, 030 (2000) [hep-th/9707133].
112. J. P. Börnsen and A. E. M. van de Ven, Nucl. Phys. B **657**, 257 (2003) [hep-th/0211246].
113. M. Shifman and A. Vainshtein, Yad. Fiz. **44**, 498 (1986) [Sov. J. Nucl. Phys. **44**, 321 (1986)].
114. M. Peskin and D. Schroeder, *An Introduction to Quantum Field Theory*, (Addison-Wesley, 1995).
115. D. Amati, K. Konishi, Y. Meurice, G. C. Rossi and G. Veneziano, Phys. Rept. **162**, 169 (1988).