1.) (15 pts) The emission and absorption coefficients for atomic transitions depend on the statistical equilibrium of the quantum states. While we can sometimes assume a thermal equilibrium, this is often not a good assumption. In general the equilibrium depends on a detailed balancing of transitions among all the states including all processes. In this problem consider the simple case of a two level atom.

a) First, suppose the equilibrium is determined only by radiative transitions in the presence of a radiation field of color temperature, $T_c$, but with reduced intensity compared to the black body of the same temperature. That is, assume

$$ W = W_c \exp\left(-\frac{h\nu}{kT_c}\right), $$

where $W < 1$. (This might correspond, for example, to interstellar gas in the presence of light from distant stars.) Assuming detailed radiative balance, derive the ratio of the populations in the upper and lower levels, $n_u/n_l$. Express your answer as $n_u/n_l = d^*(g_u/g_l) \exp(-\Delta E/kT_c)$, where $d = d(W,h\nu/kT_c)$ measures the deviation from LTE at the temperature of the incident radiation field and $h\nu = \Delta E = E_u - E_l$ (Why does $d = 1$ correspond to LTE?). Show that $W < d < 1$, and indicate the relations between $h\nu$ and $kT_c$ that are appropriate to each limit (remember that $W < 1$). For conditions leading to the upper limit on $d$ compare the spontaneous and stimulated transition rates. If you were to ascribe an ‘excitation temperature’, $T_x$, to the atomic levels, how would $T_x$ compare qualitatively to $T_c$ for different values of $d$? Note that the excitation temperature can be defined as $n_u/n_l = (g_u/g_l) \exp(-\Delta E/kT_x)$. Why, physically, should we expect in this problem that $d < 1$ (i.e., $T_x < T_c$)?

b) Transitions between atomic levels can also result from collisions between the atoms and free electrons. Collisional de-excitation from the upper (u) to the lower (l) level can occur for any colliding electron energy. In that case the electron carries away an added energy, $\Delta E$. For a colliding electron to excite the atom from l to u it must have an initial kinetic energy exceeding $\Delta E$. The rates for these two processes will depend on the density and temperature of the electrons, $n_e$, $T_e$, as well as collision cross sections determined by the quantum properties of the atom. A convenient way to express the collisional excitation rate per unit volume is $n_u n_e q_{ul}$, where $q_{ul}$ is an excitation collision factor including the cross section and kinetic information about the electron energy distribution. The analogous collisional de-excitation rate would be $n_u n_e q_{ul}$. From detailed balancing of the upward and downward collision rates it is easy to demonstrate that $q_{ul} = (g_u/g_l) q_{ul} \exp(-\Delta E/kT_e)$. It turns out one can write the de-excitation collision factor $q_{ul}$ as $q_{ul} = (2\pi/kT_e)^{3/2} \left(\frac{h^2}{m_e^{1/2}}\right) \Omega_{ul}/g_u$, where $\Omega_{ul}$ is a quantum-mechanically determined “collision strength” for the transition that is analogous to the Einstein A coefficient for radiative transitions.

Consider an environment where the incident radiation field is very weak, so that only spontaneous radiative transitions from u to l can compete with collisional transitions. Write down the equation for transition rate balancing between u and l including spontaneous radiative transitions, collisional de-excitation and collisional excitation expressed in terms of $n_u$, $n_l$, $n_e$, $A_{ul}$, $q_{ul}$ and $q_{ul}$. Then solve this equation for the ratio $n_u/n_l$. Express a constraint on $n_e$ allowing neglect of spontaneous radiative transitions; what will be the value of $n_u/n_l$ under those circumstances? Defining the “excitation” temperature, $T_x$, according to $n_u/n_l = (g_u/g_l) \exp(-\Delta E/kT_x)$, compare $T_x$ to $T_c$. Why, physically, should we expect this comparison? Under what electron density conditions can you neglect collisional de-excitation compared to spontaneous radiative transitions, and what will be the value of $n_u/n_l$, expressed in terms of $T_c$, $n_e$ and $A_{ul}$?
2.) (15 pts) a) Consider a thin spherical shell of luminous gas expanding radially outward from a star at speed \( v << c \). Let the isotropic emissivity in the gas local rest frame of some spectral line be \( j_v = j_o \delta(v - v_o) \), where \( \delta \) is the Dirac delta function; that is, in its rest frame the gas emits isotropically at one frequency, \( v_o \). We use this intrinsic profile to simplify the math that follows compared to a Lorenz profile, for example. Assuming the shell to be optically thin and that it is spatially unresolved, compute the observed spectral flux profile of this line, \( F_\nu \), measured by a distant observer. Note that this means the entire shell is contained within your beam. (Hint: Add together contributions from all parts of the shell at appropriately Doppler-shifted frequencies. The motions are nonrelativistic, so the only important effect of the motion is the non-relativistic Doppler frequency shift, \( \nu' = \nu(1+\beta\cos(\theta)) \), where \( \theta \) is the angle between the velocity vector and the line of sight. It may also help to recall some properties of the Dirac delta function. In particular, \( \delta(x-a) = 0 \) if \( x \neq a \), and \( \int \delta(x-a) \, dx = 1 \) if \( a \) is inside the domain of integration. Further, \( \int \delta(f(x)) \, dx = 1/|df/dx|_{f=0} \).) The shell thickness, \( \Delta r/r << 1 \), but do not set it to zero; keep lowest order terms in \( \Delta r/r \). Sketch the shape of the resulting spectral line.

b) In part ‘a’ you ignored the star’s contribution to the problem. Now include its influence on the profile of the line assuming it radiates like a black body. You can assume that the star is hotter than the gas, so that Planck function for its photosphere exceeds the source function of the gas. You may also assume that the radius of the star is small compared to the shell, but not zero. (Hint: Not all lines of sight intercept the disk of the star.) Sketch the shape of the resulting spectral line.

c) Another star is surrounded by a thin ring (like a bike wheel) of gas with a similar temperature relationship. In this case the gas ring is rotating around the star at a constant speed \( v << c \). Compute the line profile in this case, including the contribution of the star. Assume the observer is in the plane of the rotating ring. Sketch the shape of the resulting spectral line.