Two-dimensional models on the world sheet of non-Abelian strings

with A. Yung ...
★ Hanany-Tong, 2003

★ ★ Auzzi et al., 2003

★ ★ ★ Shifman-Yung, 2003 - ...

❖ Gaiotto, 2012 & Gaiotto, Gukov, Seiberg, 2013 “surface defects”...

Outline: a) Non-Abelian BPS strings in SYM
b) World sheet models from $\mathcal{N}=2$ bulk
c) World sheet models from $\mathcal{N}=1$ bulk
\[ \mathcal{N} = 2 \rightarrow \mathcal{N} = (2,2) \]

\[ \mathcal{N} = 1 \rightarrow \mathcal{N} = (2,0) \text{ nonminimal (and minimal)} \]

\[ \mathcal{N} = 0 \rightarrow \mathcal{N} = (0,0) \text{ (no SUSY)} \]

bulk 2D world sheet
Prototype model: $N_f=N=2; \quad \mathcal{N}=2$

$$S = \int d^4x \left\{ \frac{1}{4g_2^2} (F_{\mu\nu}^a)^2 + \frac{1}{4g_1^2} (F_{\mu\nu})^2 + \frac{1}{g_2^2} |D_\mu a^a|_2^2 ight\}$$

$$+ \quad \text{Tr} (\nabla_\mu \Phi)^\dagger (\nabla^\mu \Phi) + \frac{g_2^2}{2} \left[ \text{Tr} \left( \Phi^\dagger T^a \Phi \right) \right]^2 + \frac{g_1^2}{8} \left[ \text{Tr} \left( \Phi^\dagger \Phi \right) - N \xi \right]^2$$

$$+ \quad \frac{1}{2} \text{Tr} \left| a^a T^a \Phi + \Phi \sqrt{2M} \right|^2 + \frac{i \theta}{32 \pi^2} F_{\mu\nu}^a \tilde{F}^{a \mu\nu} \right\},$$

$$\Phi = \begin{pmatrix} \Phi^{11} & \Phi^{12} \\ \Phi^{21} & \Phi^{22} \end{pmatrix},$$

$$M = \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix}.$$

U(2) gauge group, 2 flavors of (scalar) quarks
SU(2) Gluons $A^a_\mu + U(1)$ photon + gluinos+ photino

$\Phi = \sqrt{\xi} \times I$

Basic idea:
- Color-flavor locking in the bulk $\rightarrow$ Global symmetry $G$;
- $G$ is broken down to $H$ on the given string;
- $G/H$ coset; $G/H$ sigma model on the world sheet.
★ ANO strings are there because of U(1)!
★ New strings:
\[ \pi_1(SU(2) \times U(1)) = \mathbb{Z}_2: \] rotate by \( \pi \) around 3-d axis in SU(2)
\[ \rightarrow -1; \] another -1 rotate by \( \pi \) in U(1)
\[ \pi_1(U(1) \times SU(2)) \text{ nontrivial due to } \mathbb{Z}_2 \text{ center of } SU(2) \]

\[ \begin{array}{c}
\sqrt{\xi} e^{i\alpha} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
T=4\pi\xi
\end{array} \]

ANO

\[ \begin{array}{c}
\sqrt{\xi} \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & 1 \end{pmatrix} \\
T_{U(1) \pm T^3_{SU(2)}}
\end{array} \]

Non-Abelian

\[ T=2\pi\xi \]

\[ \text{SU}(2)/U(1) \leftarrow \text{orientational moduli; } O(3) \sigma \text{ model} \]
"Non-Abelian" string is formed if all non-Abelian degrees of freedom participate in dynamics at the scale of string formation.

\[ \text{SU}(2)/\text{U}(1) = \mathbb{CP}(1) \sim O(3) \text{ sigma model} \]
Versions of $\text{CP}(N-1)$ models in 2D: nonsupersymmetric and supersymmetric – with twisted mass and $\mathbb{Z}_N$ symmetry

$\mathcal{N} = (2.2)$ and $(0,2)$ (2.2)

★ Gauged formulation ★ (Witten, 1979)
1. Non-SUSY bulk → no SUSY in 2D

\[ S^{(1+1)} = \int dt \, dz \left\{ 2\beta |\nabla_\alpha n|^2 + \frac{1}{4e^2} F_{\alpha\gamma}^2 + \frac{1}{e^2} |\partial_\alpha \sigma|^2 \right\} \]

\[ + 4\beta \left| \left( \sigma - \frac{m\ell}{\sqrt{2}} \right) n^\ell \right|^2 + 2e^2\beta^2 \left( |n^\ell|^2 - 1 \right)^2 \]

\[ \nabla_\alpha = \partial_\alpha - iA_\alpha \]

\[ m/\Lambda \]

\[ Z_N \text{ symmetry} \]

\[ m \sim e^{2\pi i/N}, e^{4\pi i/N}, ..., e^{2(N-1)\pi i/N}, 1 \]
Coulomb/conf. phase, $Z_N$ unbroken, 1 true vac., N-1 "quasivacua"

Higgs phase, $Z_N$ spont. broken; N degenerate vacua

Phase transition

$E_{\text{vac}}$ vs. $m/\Lambda$
2. Introduction of 2D axion restores $Z_N$ and eliminates Coulomb/confinement phase

\[ \mathcal{L}_a = f_a^2 (\partial_\mu a)^2 + \frac{a}{2\pi} \varepsilon_{\alpha\gamma} \partial^\alpha A^\gamma. \]

Photon is (2D) Higgsed
3. $\mathcal{N} = 2$ SUSY bulk

$\mathcal{N} = (2,2)$ CP(N-1) model

$$\mathcal{L} = \frac{1}{e_0^2} \left( \frac{1}{4} F_{\mu\nu}^2 + |\partial_\mu \sigma|^2 + \frac{1}{2} D^2 \right) + i D \left( \bar{n}_i n^i - 2\beta \right)$$

$$+ \left| \nabla_\mu n^i \right|^2 + 2 \sum_i \left| \sigma - \frac{m_i}{\sqrt{2}} \right|^2 |n^i|^2$$

+ fermions
Figure 1: Plots of $n$ and $\sigma$ VEVs (thick lines) vs. $m$ in the $N=(2,2)$ CP($N-1$) model with twisted masses as in (2.2). Where we assumed for simplicity that $m \equiv m_0$ is real and positive. (This is by no means necessary; we will relax this assumption at the end of this section.) Note that the phase factor of $\sigma$ in (4.22) does not follow from (4.19). Rather, its emergence is explained by explicit breaking of the axial U(1)$_R$ symmetry down to $Z_2^N$ through the anomaly and non-zero masses (2.2), see Appendix D, with the subsequent spontaneous breaking of $Z_2^N$ down to $Z_2$. Once we have one solution to (4.19) with nonvanishing $\sigma$ we can generate all $N$ solutions (4.22) by the $Z_2^N$ transformation [6].

Although we derived Eq. (4.19) in the large-$N$ approximation, the complexified version of this equation, $N-1 \prod_i \left( \sqrt{2} \sigma - m_i \right) = \Lambda N$, (4.23) is in fact, exact, since this equation as well as the solution (4.22) follow from the Veneziano–Yankielowicz-type effective Lagrangian exactly in the $N=(2,2)$ CP($N-1$) model in [35, 36, 7, 37, 28]. The Veneziano–Yankielowicz Lagrangian implies (4.23) even at finite $N$.

$E_{\text{vac}}=0$ always, SUSY unbroken, $Z_N$ always broken, (N degenerate vacua)

Crossover instead of phase transition

Strong-coupling $\leftrightarrow$ weak coupling Higgs regime

Bifermion order parameter $\overline{\xi} \xi$
Direct (exact) large-N one-loop calculation:

\[
V_{\text{eff}} = \int d^2x \frac{N}{4\pi} \left\{ - (iD + 2|\sigma|^2) \ln \frac{iD + 2|\sigma|^2}{\Lambda^2} + iD \\
+ 2|\sigma|^2 \ln \frac{2|\sigma|^2}{\Lambda^2} + 2|\sigma|^2 u \right\},
\]

versus exact Veneziano-Yankielowicz superpotential of $\sigma \log \sigma$ type.
4. BPS Spectrum of SUSY CP(N-1) with $Z_N$ twisted masses (curves of marginal stability)

Figure 10: The decay curves of CP$^2$ in $m_0^3$ plane. The primary curve is shown in red. The two vertical whiskers are the initial coils of the two spirals.
5. $\mathcal{N} = 1$ SUSY bulk

$\mathcal{N} = (0,2) \text{ CP}(N-1)$ model

Supersymmetry is broken, generally speaking !!!
Phase transitions possible and do occur ✴✴✴

All phase transitions are of the second kind!
Break $\mathcal{N} = 2$ down to $\mathcal{N} = 1$ in the bulk

Deformation of the bulk: ADD $W = \mu (A^a)^2 + \mu' A^2$

Heterotic deformation the of the World-sheet theory:

$$(2,2)$$ supersymmetry is broken down to (0,2)

$$L_{\text{heterotic}} = \bar{\xi}_R i \partial_L \xi_R + \left[ \gamma \xi_R R (i \partial_L \phi^+) \psi_R + H.c. \right] - g_0^2 |\gamma|^2 \left( \xi_R^+ \xi_R \right) \left( R \psi_L^+ \psi_L \right)$$

at small $\gamma$

$\zeta_R$ is Goldstino

$$\mathcal{E}_{\text{vac}} = |\gamma|^2 \left| \langle R \psi_R^+ \psi_L \rangle \right|^2$$

$(0,2)$ supersymmetry is spontaneously broken!
At large $N$ heterotic $\text{CP}(N-1)$ is also solvable (à la Witten) and presents a wealth of various phases

We have two parameters, $\gamma$ and $m$, and a nontrivial phase diagram

With this choice of mass parameters we have $Z_N$ symmetry, and phases with broken/unbroken $Z_N$. SUSY is spontaneously broken.
\[ \gamma \gg 1 \quad (u \gg 1) \]

**Large deformation**

- **Strong coupling. Chiral \( Z_N \) spont. broken.**
- **No confinement**
- **Chiral \( Z_N \) unbroken**
- **SUSY restored here**

**Higgs phase**
- **Weak coupling**
- **Chiral \( Z_N \) broken**

- **Coulomb/confining. Chiral \( Z_N \) unbroken**

**Diagram**:}

\[ E_{\text{vac}} \]

\[ \Lambda^2 \]

\[ \Lambda e^{-u/2} \quad \Lambda \quad \Lambda \sqrt{u} \quad m \]

- **SUSY restored here**

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Wednesday, September 18, 13
Witten's point

Phase Diagram

- $Z_N$ unbroken
- Coulomb/Confining Phase
- Strongly Coupled Phase
- Higgs Phase

Witten's point
6. $\mathcal{N} = 1$ or $2$ SUSY bulk, @ Semilocal Strings

★ ★ Hanani-Tong model $\rightarrow$ Obtained from string/D brane consideration

★ ★ ★ From field theory we get $zn$ model: DIFFERENT

★ ★ ★ ★ Large-$N$ limit the same!!!

SY hep-th/0603134
SVY arXiv:1104/2077
KSVY arXiv:1107/3779
Hanani-Tong model

\[ \mathcal{L}_{\text{het}}^{N_F-1} = |\nabla_\mu n_i|^2 + |\tilde{\nabla}_\mu \rho_j|^2. \]

\[- \sum_{i=0}^{N-1} |\sigma - m_i|^2 |n_i|^2 - \sum_{j=0}^{\tilde{N}-1} |\sigma - \mu_j|^2 |\rho_j|^2 - D (|n_i|^2 - |\rho_j|^2 - r_0) \]

\[- 2|\omega|^2 |\sigma|^2. \]

\[ \nabla_\mu n_i = (\partial_\mu - iA_\mu)n_i, \quad \tilde{\nabla}_\mu \rho_j = (\partial_\mu + iA_\mu)\rho_j \]

\[ N_F = N + \tilde{N} \]

\[ m_k = me^{2\pi i \frac{k}{N}}, \quad k = 0, \ldots, N - 1 \]

\[ \mu_l = \mu e^{2\pi i \frac{l}{\tilde{N}}}, \quad l = 0, \ldots, \tilde{N} - 1. \]
zn Model \((\text{MS+Vinci+Yung})\)

\[
S_{\text{exact}} = \int d^2 x \left\{ |\partial_k (z_j n_i)|^2 + |\nabla_k n_i|^2 + \frac{1}{4e^2} F_{kl}^2 + \frac{1}{e^2} |\partial_k \sigma|^2 \\
+ |m_i - \tilde{m}_j|^2 |z_j|^2 |n_i|^2 + \left| \sqrt{2} \sigma + m_i \right|^2 |n_i|^2 + \frac{e^2}{2} (|n_i|^2 - r)^2 \right\},
\]

\(i = 1, \ldots, N\), \(j = 1, \ldots, \tilde{N}\), \(\nabla_k = \partial_k - i A_k\).

+ deform. + fermions

\(z_j\) of the opposite charge compared to \(n_i\) and unconstrained

Derived from the bulk theory in the limit \(\ln(\xi L^2) \gg 1\)
At $N \rightarrow \infty$ HT = zn

BPS sectors the same at any N

New type of renormalizability
Figure 4: Phase Diagram of the weighted \((2, 2) \mathbb{CP}^{N-1}\) model in the large-\(N\) approach. There are four domains with different VEVs for \(\sigma\): two Higgs branches \(H_\rho\) and \(H_n\), and two Coulomb branches \(C\). In the Coulomb phase \(C\ r = 0\). The curve \(\mu/\Lambda = (m/\Lambda)^{1/\alpha}\) together with horizontal and vertical lines starting from \(\mu = \Lambda\) and \(m = \Lambda\) respectively separates the \(C\) phases from the Higgs phases. In \(H_n\ r > 0\) and in \(H_\rho\ r < 0\). On the super-conformal line \(\mu/\Lambda = (m/\Lambda)^{1/\alpha}\) a new branch described by a super-conformal theory opens up.

From Koroteev-Monin-Vinci, 2010
In the (0,2) literature known as the heterotic minimal model at level k

with X. Cui
arXiv:1009.4421
arXiv:1105.5107
arXiv:1111.6350

\[ \mathcal{L}_A = \frac{1}{g^2} \int d^2 \theta_R \frac{A^\dagger i \partial_{\text{RR}A}}{1 + A^\dagger A} \]

= \[ G \left\{ \partial^\mu \phi \partial_\mu \phi^\dagger + i \psi_L^\dagger \partial_{\text{RR}} \psi_L - 2i \frac{1}{\chi} \psi_L^\dagger \psi_L \phi^\dagger \partial_{\text{RR}} \phi \right\} \]

\[ G = \frac{2}{g^2 \chi^2}, \]

\[ \chi \equiv 1 + \phi \phi^\dagger. \]
\[ \beta(g^2) = -\frac{g^4}{2\pi} \left( 1 + \frac{N_f}{2} \gamma(B_i) \right) \frac{1}{1 - \frac{1}{4\pi} g^2} \]

Full analog of NSVZ \( \beta \) function in 4D SYM

\[ \beta(g^2) = -\frac{g^4}{2\pi} \left( 1 + \frac{N_f}{2} \gamma(B_i) \right) \]

Anomaly: J. Chen, AV + MS, in progress
Applications in Condensed Matter

$^3$He$-$B example

$^3$He atoms

Spin 1/2

P-wave paring

$L=1, S=1 \Rightarrow$ Cooper pair order parameter $e_{\mu i}$ ← $3 \times 3$ matrix

Spin-orbit small, symmetry of $H$ is $G = U(1)_p \times SO_S(3) \times SO_L(3)$

In the ground state $U_p(1) \times SO_S(3) \times SO_L(3) \rightarrow H_B = SO(3)_{S+L}$

Vectorial order parameter broken on the vortex
\[ S_{\text{world sheet}} = (\mu^2/2\beta) \int d^2x \ (\partial_\mu S^i) (\partial_\mu S^i) \]

\[ S^i S^i = 1 \]

Classically two “rotational” zero modes.

QMechanically may be lifted

- Assume \( \chi^i \) is spin field!
- Add \( \Delta L = \varepsilon (\partial_i \chi^i)(\partial_k \chi^k) \)
If $\varepsilon \neq 0$ but small $\Rightarrow$

$$\Delta_{\text{CP}(1) S_{\text{world sheet}}} = \varepsilon \int d^2x \ ( (\partial_z S^3)^2 - M^2[1-(S^3)^2])$$

$$\mathcal{L}_{x_\perp} = \frac{T}{2} \left( \partial_a \vec{x}_\perp \right)^2 - \tilde{M}^2 \left( S^3 \right)^2 \left( \partial_z \vec{x}_\perp \right)^2,$$

★ EXTRA (quasi)gapless modes ★

★ ★ Translational (Kelvon) and orientational (spin) modes mix with each other ★ ★ ★
Instead of conclusions

- A treasure trove of novel 2D models with intriguing dynamics!
- 4D ↔ 2D Correspondence

World-sheet theory ↔ strongly coupled bulk theory inside

Dewar flask