Linear Algebra Detour

- The transpose of a matrix is formed by interchanging columns and rows

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

Symmetric Matrices

- It is possible for a matrix to be its own transpose, that is $A^T = A$. If so, A is a symmetric matrix.

$$A = \begin{pmatrix} a & e & f & g \\ e & b & h & i \\ f & h & c & j \\ g & i & j & d \end{pmatrix}$$

A 4x4 symmetric matrix has only 10 independent entries.

Adjoint Matrices

- If A is a matrix with only real coefficients, then the transpose operation and the adjoint operation are the same. If A is real and self-transpose, then A is also self-adjoint.
- More generally, the adjoint operation means taking the transpose and complex conjugation

Adjoint Matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A^{\dagger} = \begin{pmatrix} a* & c* \\ b* & d* \end{pmatrix}$$

If *A* is self-adjoint or Hermitian, then the diagonal elements must be real.

Self-Adjoint or Hermitian Matrices

- If a matrix A is self-adjoint or Hermitian, then it has real eigenvalues
- The eigenvalue equation is det $|A-\lambda I|=0$, where the λ's are numbers, which are called the eigenvalues

Inverse Matrix

- A matrix A generally has an inverse if $\det|A| \neq 0$. The inverse of matrix A is called A^{-1}.
- The product of $A^{-1} A$ (in either order) is the identity matrix I.

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Orthogonal Matrices

- A matrix is called **orthogonal**, if its adjoint is equal to its inverse, that is

$$A^T = A^{-1} \;\; so \;\; A^T A = I$$

- Since det$|A^{\dagger}|$=det$|A|$, if A is orthogonal, (det$|A|$)^2=1, so det$|A|=\pm 1$. Orthogonal matrices with det$|A|=+1$ are called rotation matrices. Matrices with det$|A|$=-1 are inversion matrices.

Orthogonal Matrices

- If a matrix is orthogonal, then its rows and columns are orthogonal to each other. Since the number of columns and rows is equal to the dimensionality of the vector space, the rows and columns each span the space and are therefore a set of basis vectors.
- The norms of the rows and columns are 1
- The rows/columns are orthonormal sets

Lorentz Transformation Matrix

$$\begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ ct_2 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & \mathrm{i}\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\mathrm{i}\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ ct_1 \end{pmatrix}$$

assuming motion v along the x axis,

$$\gamma = \sqrt{\frac{1}{1-\beta^2}}, \ \ \beta = v/c$$

Symmetric: No

Self-Adjoint or

Hermitian: Yes

Determinant: +1

Rows and columns orthonormal: Yes

Is Lorentz Matrix Orthogonal?

$$\begin{pmatrix} \gamma & 0 & 0 & \mathrm{i}\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\mathrm{i}\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} \gamma & 0 & 0 & -\mathrm{i}\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \mathrm{i}\gamma\beta & 0 & 0 & \gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Yes. Lorentz transformations are rotations in a 4-dimensional space where the *ct* axis is equivalent to a spatial dimension.

Applications

- A muon decays stochastically with a mean lifetime of 2.2 μsec in its rest frame. If a muon is created by a pion decay 30 km above the earth and travels at v=0.99 c, how long will it take to reach the earth in the LAB frame? In the muon rest frame?

LAB Frame

- The velocity is 0.99 c = 2.96 x 10^8 m/s
- The distance is 30 km
- The time is 1.01 x 10^{-4} s
- $\beta = 0.99$
- $\gamma = 50.25$

Muon Frame

- Muon velocity is zero
- How far does the earth and its atmosphere move?
- How long does it take?