Note: To solve the firewall problems, the website has been moved to www.physics.umn.edu/~marshak

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Physics 2403, Fall 2000, Notes on Lorentz 4-Vectors

1. Lorentz 4-Vectors

A Lorentz 4-vector **x** is defined as a 4-tuple having three spatial components and one time component, that is $\mathbf{x} = (x, y, z, ict)$. The purpose of the *i* in the last component is to allow use of the standard Euclidian definition of the scalar or dot product. When I use the usual definition that $i^2 = -1$, I will have a reasonable chance to put the minus signs into the right place.

A Lorentz 4-vector can represent an “event” in space-time. For example, if I am sitting at point that I define as the origin of coordinates and the beginning of a packet of electrons comes by me and I reset my watch at exactly that moment, then in my reference frame, the beginning of the packet of electrons is defined by the 4-vector (0,0,0,0). If I then wait until the end of the packet of electrons goes by, I can define a second event as the end of the packet. If the packet is 1 cm in length, then the end of the packet comes at (0,0,0,*ic*(1 cm/*v*)), where *v* is the velocity of the packet. If I then wanted a 4-vector to represent the length of the packet as I observe it, I could subtract the 4-vector for the beginning from the 4-vector for the end and get (0,0,0,*ic*(1 cm/*v*)).

Another possibility is to say that I can measure the 4-vectors for the beginning and the end of the packet at the same time. That is, as the beginning of the packet comes by me, I reset my watch and hold up a ruler. Then, I would say that I can represent the electron packet by the 4-vector (1 cm,0,0,0). This approach does not work. You need to measure times at the same place in order to get proper time. So, if the first component of the 4-vector is 1, the last component is not 0. See Ch. 39, problem 12.

Distance-time is not the only quantity that can be represented with a 4-vector. The other 4-vector that I will use here is the momentum-energy 4-vector defined as $\mathbf{p} = (p_x, p_y, p_z, iE/c)$. The first three components are the ordinary components of momentum. The fourth component is proportional to the total energy.

2. Manipulating 4-Vectors

The algebra of 4-vectors is mostly what you might expect. I can add or subtract 4-vectors and these operations are commutative.

The dot or scalar product of two 4-vectors vectors is given by

$$\mathbf{x_1} \cdot \mathbf{x}_2 = x_1 x_2 + y_1 y_2 + z_1 z_2 - c^2 t_1 t_2$$

The norm of a Lorentz vector $\mathbf{x_1}$ is given by

$$\|\mathbf{x_1}\|^2 = \mathbf{x_1} \cdot \mathbf{x_1}$$

Like any other Lorentz scalar, the norm of a Lorentz 4-vector is invariant under Lorentz transformations, which are defined below.

Note that unlike Euclidian vectors, Lorentz 4-vectors can have squared norms which are positive or negative and, therefore, norms which are real or imaginary. Also, unlike Euclidian vectors, Lorentz vectors with non-zero components can have zero norms. Consider two points in 4-space, defined by Lorentz vectors $\mathbf{x_1}$ and $\mathbf{x_2}$. Then, one can define a vector connecting them

$\mathbf{a} = \mathbf{x}_2 - \mathbf{x}_1$. **Q:** What can you say about the relationship between these two points if the norm squared of **a** is positive, negative, zero? **A:** For a spacetime 4-vector, the question of the polarity of the norm squared depends on the relative size of the spatial part and the time part. If the spatial distance is larger than *ct*, the norm will be positive. This kind of 4-vector is "outside the light cone," that is the distance is further than light can travel in the time stated. Such a 4-vector connects events that cannot be causally related. If *ct* is greater than the spatial distance, then the norm squared is negative and this 4-vector is within the light cone, that is, light could have traveled the spatial distance in the stated time. This type of 4-vector connects events that could be causally related. If the norm squared is zero, the 4-vector is "on the light cone" and represents two points in space-time that are exactly separated by space and time related by the speed of light. Causal connection in this case is just barely possible.

The norm squared of the momentum-energy 4-vector is $p^2 - E^2/c^2$. This norm squared must be negative or zero and is equal to $-m_0^2 c^2$. The mass m_0 is known as the rest mass and it is an intrinsic property of a particle. Note that because the rest mass is proportional to a norm squared, it is a Lorentz scalar and therefore must be Lorentz-invariant. The following equations may be useful:

$$p^2 - \frac{E^2}{c^2} = -m_0^2 c^2$$

$$E^2 = c^2 p^2 + m_0^2 c^4$$

$$\beta = \frac{cp}{E}$$

$$E^2 = \beta^2 E^2 + m_0^2 c^4$$

$$E^2 = \frac{m_0^2 c^4}{1-\beta^2} = \gamma^2 m_0^2 c^4 = m^2 c^4$$

$$E = mc^2$$

3. Lorentz Transformations

Lorentz 4-vectors can be transformed from one reference frame to another reference frame moving at a constant velocity *v* with respect to the first frame. The velocity *v* is usually written in terms of a dimensionless constant $\beta = (v/c)$, where *c* is the speed of light. Getting the sign of *v* and therefore β right is important in order to get the correct transformation. Here's one way to do it. Assume you have a 4-vector **x** defined in Reference Frame 1 and you want to transform it into Reference Frame 2. In Frame 1, Frame 2 moves in the positive-x direction. Then *v* is positive. If Frame 2 moves in the negative *x* direction, then *v* is negative. The transformation itself is done with a matrix multiplication. For example, assume that $\mathbf{x_1}$ is a Lorentz 4-vector as viewed in the first frame and $\mathbf{x_2}$ is the same Lorentz 4-vector as viewed in the second frame. Further, assume that Frame 2 is moving in the positive-x direction as seen in Frame 1. Then,

$$\begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ ict_2 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ ict_1 \end{pmatrix}$$

$$\beta = v/c, \;\; \gamma^2 = \frac{1}{1-\beta^2}$$

The Lorentz transformation matrix is self-adjoint or Hermitian. To calculate the adjoint of a matrix, you calculate its transpose by switching the rows and columns and then take the complex conjugate (or vice versa). A matrix is self-adjoint if its adjoint is equal to itself, which is true for this matrix. The adjoint of the Lorentz matrix is also equal to the inverse of the matrix. That is, the product of the Lorentz matrix and its adjoint is the identity matrix. Such matrices are known as orthogonal matrices. The squared determinant of an orthogonal matrix must be 1, implying that its determinant must be ±1. If the determinant is +1 (as is true for the Lorentz matrix), the matrix is known as a rotation matrix. The Lorentz matrix is a rotation matrix in 4-dimensional space-time. If the determinant is –1, the orthogonal matrix is an inversion matrix. For any orthogonal matrix, the columns (and rows separately) form an orthonormal basis or normal coordinate system for the vector space. That is, the scalar or dot products of different columns (or rows) is zero and the scalar or dot product of a column (or row) with itself is 1.

Problem 2: Assume that a particular pion dies in the mean lifetime of 2.6 x 10^{-8} s. Then, the 4-vector in the pion rest frame representing the pion's life is (0, 0, 0, *ic*2.6 x 10^{-8} s). If $\beta = 0.85$, then $\gamma = 1.898$ and $\gamma\beta = 1.614$. So, transforming this 4-vector to the LAB frame gives (-c(1.614)(2.6 x 10^{-8}), 0, 0, (-ic(1.898)(2.6 x 10^{-8})). So, the time is 4.9 x 10^{-8} s and the distance traveled is 12.6 m.

Problem 9: The 4-vector representing my passing of both ends of the stick in my reference frame is (0,0,0,i(x)/.8). $\beta = 0.80$, $\gamma = 1.667$ and $\gamma\beta = 1.333$. Transforming to the stick's frame, the 4-vector becomes (-1.66625x,0,0,2.08375ix). So, x = 0.6 m long and the time it takes to pass me is 2 ns in my frame.

Problem 12: The observer in ship A measures a 4-vector for Ship A as (100, 0, 0, 0) and Ship B as (0, 0, 0, i36/.92). $\beta = 0.92$, $\gamma = 2.552$ and $\gamma\beta = 2.347$.

Problem 13: The electron energy is 50 GeV or 8 x 10^{-9} J. The rest energy of an electron is (9.1 x 10^{-31} kg)(c^2)= 8.2x10^{-14} J. So, gamma= 97561 and beta=1. The 4-vector is (0,0,0,ic3.33x10^{-11}).

Problems For Next Week:

1. A rocket ship travels at v=0.95 c with respect to an observer on the earth. The ship fires bullets at 0.8 c both forward and backward with respect to its travel direction away from the earth. What velocities does the earthbound observer measure for both the forward-fired and backward-fired bullets?

2. Chapter 39: 31

3. Chapter 39: 61

4. Write a program in VBA that returns a sinusoidally-distributed random number between 0 and π.