

Multi-channel scattering and Feshbach resonances: Effective theory, phenomenology, and many-body effects

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A low energy effective theory based on a microscopic multi-channel description of the atom-atom interaction is derived for the scattering of alkali atoms in different hyperfine states. This theory describes all scattering properties, including medium effects, in terms of the singlet and triplet scattering lengths and the range of the atom-atom potential and provides a link between a microscopic description of Feshbach scattering and more phenomenological approaches. It permits the calculation of medium effects on the resonance coming from the occupation of closed channel states. The examination of such effects are demonstrated to be of particular relevance to an experimentally important Feshbach resonance for ^{40}K atoms. We analyze a recent rethermalization rate experiment on ^{40}K and demonstrate that a measurement of the temperature dependence of this rate can determine the magnetic moment of the Feshbach molecule. Finally, the energy dependence of the Feshbach interaction is shown to introduce a negative effective range inversely proportional to the width of the resonance. Since our theory is based on a microscopic multi-channel picture, it allows the explicit calculation of corrections to commonly used approximations such as the neglect of the effective range and the treatment of the Feshbach molecule as a point boson.

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I. INTRODUCTION

The use of Feshbach resonances to vary the interactions between atoms obeying Fermi statistics has become a matter of interest in the field of ultracold trapped atoms [1]. One reason is that these systems permit the experimental exploration of potentially strongly correlated Fermi systems in which the magnitude of the scattering length a is larger than the interparticle separation. The study of such strongly interacting Fermi systems is relevant to several fields including condensed matter and nuclear physics and astrophysics.

It is therefore important to understand in some detail the physics of atoms interacting via a Feshbach resonance. Theoretical studies of the two-body physics of Feshbach resonances are typically performed in the context of vacuum multi-channel scattering theory using a microscopic model of the atom-atom interaction [2]. On the other hand, studies of low energy many-body properties often require the use of effective theories with a simplified form of the atom-atom interaction in order to make calculations tractable. A precise link between such effective theories and microscopic multi-channel models is therefore crucial in order to incorporate the correct two-body physics in many-body calculations. In the atomic gas community, effective theories are of two types. In one, the presence of closed channels is simply neglected, and only the resultant (resonant) scattering length is used as an input parameter. This approximation is adequate for the calculation of certain many-body properties close to broad resonances where the wave function of the Feshbach molecule is dominated by atom pairs in the open channel [3]. In the other, the presence of a two-body

bound state in the closed channel (which leads to the resonance) is put in by hand in the form of a point boson [4, 5]. The parameters appearing in such theories are then the magnetic moment of the Feshbach molecule and the molecule-atom coupling.

The treatment of the bound state in the closed channels as a point boson is, however, incorrect in principle since it is really a composite two-body object. The main purpose of the present paper is therefore to develop a systematic, improved low energy effective theory for the multi-channel scattering of alkali atoms in which the presence of a closed channel molecular Feshbach state is not put in by hand as a point boson but rather emerges naturally as a composite object consisting of two interacting fermionic atoms. We will show that such a theory permits tractable calculations of low energy many-body properties including medium effects due to the occupation of states in the closed channels. The examination of such effects will be demonstrated to be of particular importance for the experimentally well-studied resonance of ^{40}K atoms. It will be demonstrated that our approach can be reduced to the well-known Bose-Fermi model when the two-fermion nature of the bare molecule in the closed channel can be ignored. This theory can be formulated in two equivalent ways: One uses the microscopic parameters describing the atom-atom interaction as input; the other takes physical parameters describing the low energy scattering as input including the position and width of the resonance and the magnetic moment of the Feshbach molecule (including the effects of dressing by the open channel). This provides an appealing link between the microscopic parameters often used in two-body multi-channel calculations and the physical

parameters used in low energy effective many-body theories. We will use this effective theory to analyze recent rethermalization experiments on ^{40}K atoms and propose that further experimental study of rethermalization rates at higher temperatures can provide a determination of the magnetic moment of the Feshbach molecule. Finally, the energy dependent Feshbach interaction will be shown to introduce an effective range inversely proportional to the width of the resonance. This effective range can be neglected (and universal behavior close to resonance expected) only in the case of broad resonances.

Note that we set $\hbar = 1$ throughout the paper.

II. LOW ENERGY EFFECTIVE THEORY

In this section, we develop a low-energy effective theory for the scattering of alkali atoms in different hyperfine states. Consider a gas of alkali atoms in a given magnetic field B along the z -direction. Incoming and outgoing scattering channels are characterized by the eigenstates of the single particle hyperfine Hamiltonian

$$\hat{H}_{\text{spin}} = A \mathbf{I} \cdot \mathbf{S} + C S_z + D I_z \quad (1)$$

where A is the hyperfine constant, $C = 2\mu_B B$, $D = -\mu_n B/I$ with μ_n the nuclear spin magnetic moment, and \mathbf{S} and \mathbf{I} are the electron and nuclear spin operators respectively. We denote the single particle eigenstates of the hyperfine Hamiltonian for a given B -field as $|\alpha\rangle$ with $\hat{H}_{\text{spin}}|\alpha\rangle = \epsilon_\alpha|\alpha\rangle$. The eigenstates can be labeled by their total spin, $\mathbf{F} = \mathbf{S} + \mathbf{I}$, $F = I \pm \frac{1}{2}$, and its z -projection, $m_F = -I - \frac{1}{2}, \dots, I + \frac{1}{2}$, e.g., $|\alpha\rangle \equiv |F, m_F\rangle$. The main contribution to the atom-atom interaction driving the scattering process is the electrostatic central potential

$$V(r) = \frac{V_s(r) + 3V_t(r)}{4} + [V_t(r) - V_s(r)] \mathbf{S}_1 \cdot \mathbf{S}_2 \quad (2)$$

where $V_s(r)$ and $V_t(r)$ are the singlet and triplet potentials and \mathbf{S}_1 and \mathbf{S}_2 are the spins of the valence electrons of the two alkali atoms [6]. We ignore here the small magnetostatic and magnetic dipole-dipole interactions between atoms. The potential (2) induces transitions among two-particle states with the same total z -projection of \mathbf{F} . The properly anti-symmetrized two-particle state with the lowest energy constitutes the open channel $|o\rangle = |\alpha_1, \alpha_2\rangle$. The states of higher energies which can be coupled to $|o\rangle$ by the potential (2) form a set of closed channels $|c^{(n)}\rangle = |\alpha_3^{(n)}, \alpha_4^{(n)}\rangle$. The threshold energies for the closed channels are then $E_{\text{th}}^{(n)}(B) = \epsilon_{\alpha_4^{(n)}} + \epsilon_{\alpha_3^{(n)}} - \epsilon_{\alpha_2} - \epsilon_{\alpha_1}$, and they depend on the magnetic field B through \hat{H}_{spin} .

As a practical illustration of our approach, we consider in the following the s-wave Feshbach resonance in a cold ^{40}K gas at $B_0 \simeq 201.6$ G, which has frequently been studied experimentally, e.g., Refs. [7, 8]. The open channel for this resonance has atoms in the states $|\frac{9}{2}, -\frac{9}{2}\rangle$ and

$|\frac{9}{2}, -\frac{7}{2}\rangle$. The only other state in the ground-state manifold to which the open channel can couple through the dominant electrostatic potential has atoms in the states $|\frac{9}{2}, -\frac{9}{2}\rangle$ and $|\frac{7}{2}, -\frac{7}{2}\rangle$. Thus, $|o\rangle = |\frac{9}{2}, -\frac{9}{2}, \frac{9}{2}, -\frac{7}{2}\rangle$, and the only closed channel is $|c\rangle = |\frac{9}{2}, -\frac{9}{2}, \frac{7}{2}, -\frac{7}{2}\rangle$. Note that the open and closed channels have the state $|\frac{9}{2}, -\frac{9}{2}\rangle$ in common. Thus, we can immediately conclude that any initial occupation of the state $|\frac{9}{2}, -\frac{9}{2}\rangle$ will have direct effects on the closed channel Feshbach state. Such effects cannot be described by the usual Bose-Fermi models which treat the Feshbach molecule as a point Boson. It was demonstrated in Ref. [10] using a schematic mean field theory that these effects can be important for BCS-BEC cross-over physics. Here, we will demonstrate that the occupation of the closed channel state can have significant effects on scattering properties using an effective theory based on a microscopic description of the atom-atom interaction. The threshold energy for this resonance is given by

$$\begin{aligned} E_{\text{th}} &= E_{\frac{7}{2}, -\frac{7}{2}} - E_{\frac{9}{2}, -\frac{7}{2}}, \\ E_{\frac{7}{2}, -\frac{7}{2}} &= -\frac{1}{4} [A + \sqrt{32A^2 + [2C - 2D - 7A]^2 + 14D}], \\ E_{\frac{9}{2}, -\frac{7}{2}} &= -\frac{1}{4} [A - \sqrt{32A^2 + [2C - 2D - 7A]^2 + 14D}]. \end{aligned} \quad (3)$$

With the values of $A = -1.37 \cdot 10^{-2}$ K, $C = 1.34 \cdot 10^{-4}$ B/G K, and $D = 1.19 \cdot 10^{-8}$ B/G K from Ref. [6], we can Taylor expand around $B_0 = 201.6$ G to obtain

$$E_{\text{th}} \simeq 0.084K + 1.78\mu_B(B - B_0) \left[1 + 0.0188 \frac{B - B_0}{100G} \right] \quad (4)$$

which is approximately linear in $B - B_0$ for fields near the resonance.

We emphasize that our effective theory is not limited to two-channel problems. It is readily generalized to more than two channels if appropriate for other atomic resonances.

The total Hamiltonian describing a system of particles in the scattering states $|o\rangle$ and $|c\rangle$ interacting via $V(r)$ is

$$\begin{aligned} \hat{H}(B) &= \sum_{\mathbf{k}, \alpha} \epsilon_{\alpha, k} \hat{a}_{\mathbf{k}\alpha}^\dagger \hat{a}_{\mathbf{k}\alpha} + \\ &\frac{1}{\mathcal{V}} \sum_{\mathbf{K}, \mathbf{q}} \hat{\Psi}_{\mathbf{q}}^\dagger(\mathbf{K}) \begin{bmatrix} V_{cc}(\mathbf{q}', \mathbf{q}) & V_{co}(\mathbf{q}', \mathbf{q}) \\ V_{oc}(\mathbf{q}', \mathbf{q}) & V_{oo}(\mathbf{q}', \mathbf{q}) \end{bmatrix} \hat{\Psi}_{\mathbf{q}}(\mathbf{K}) \end{aligned} \quad (5)$$

where $\hat{\Psi}_{\mathbf{q}}^\dagger(\mathbf{K}) = [\hat{a}_{\mathbf{K}/2+\mathbf{q}\alpha_4}^\dagger \hat{a}_{\mathbf{K}/2-\mathbf{q}\alpha_3}^\dagger, \hat{a}_{\mathbf{K}/2+\mathbf{q}\alpha_2}^\dagger \hat{a}_{\mathbf{K}/2-\mathbf{q}\alpha_1}^\dagger]$ describes the creation of an atomic pair in the closed and open channels with center of mass momentum \mathbf{K} and relative momentum \mathbf{q} . Here, $\epsilon_{\alpha, k} = \epsilon_\alpha + k^2/2m$ with m the mass of the atoms, and \mathcal{V} is the volume of the system. The interaction matrix elements of $V(r)$ in (2) between the different scattering channels are denoted $V_{ij}(\mathbf{q}', \mathbf{q})$ with $i, j = o, c$. They depend on B through the B -dependence of the states $|o\rangle$ and $|c\rangle$.

The scattering process is described by the scattering matrix, $T_{ij}(\omega, \mathbf{K}, \mathbf{q}', \mathbf{q})$, which depends on the center-of-mass frequency and momentum (ω, \mathbf{K}) and the relative

momenta \mathbf{q} and \mathbf{q}' in the incoming and outgoing channels respectively. The scattering matrix obeys the Lippmann-Schwinger equation

$$\begin{bmatrix} T_{cc} & T_{co} \\ T_{oc} & T_{oo} \end{bmatrix}^{-1} = \begin{bmatrix} V_{cc} & V_{co} \\ V_{oc} & V_{oo} \end{bmatrix}^{-1} - \begin{bmatrix} G_c & 0 \\ 0 & G_o \end{bmatrix} \quad (6)$$

where the propagator for two atoms in the open channel is

$$G_o(\omega, \mathbf{K}, \mathbf{q}) = \frac{1 - f_{\alpha_1}(\frac{1}{2}\mathbf{K} - \mathbf{q}) - f_{\alpha_2}(\frac{1}{2}\mathbf{K} + \mathbf{q})}{\omega + i\delta - \frac{K^2}{4m} - \frac{q^2}{m}}, \quad (7)$$

and

$$G_c(\omega, \mathbf{K}, \mathbf{q}, B) = \frac{1 - f_{\alpha_3}(\frac{1}{2}\mathbf{K} - \mathbf{q}) - f_{\alpha_4}(\frac{1}{2}\mathbf{K} + \mathbf{q})}{\omega + i\delta - E_{\text{th}}(B) - \frac{K^2}{4m} - \frac{q^2}{m}} \quad (8)$$

is the propagator for a pair of atoms in the closed channel. Here f_α is the Fermi distribution function appropriate for the hyperfine state $|\alpha\rangle$. Note that (6) is a matrix equation both in the relative momenta (\mathbf{q}', \mathbf{q}) and in the channels (i, j).

A. Effective interaction

We now define an effective interaction U appropriate for calculating the low energy scattering properties of the system. This is done by considering the Lippmann-Schwinger equation in vacuum for $\omega = 0$ and ignoring the hyperfine splitting of the atomic levels

$$\begin{bmatrix} U_{cc} & U_{co} \\ U_{oc} & U_{oo} \end{bmatrix}^{-1} = \begin{bmatrix} V_{cc} & V_{co} \\ V_{oc} & V_{oo} \end{bmatrix}^{-1} - \begin{bmatrix} G^{\text{vac}} & 0 \\ 0 & G^{\text{vac}} \end{bmatrix} \quad (9)$$

where $G^{\text{vac}}(q) = (i\delta - q^2/m)^{-1}$ is the vacuum pair propagator ignoring the hyperfine splitting. When the scattering is dominated by the presence of a single Feshbach state in a closed channel, the effective interaction mediated by the closed channel state is rank one separable: Its Fourier transform can be expressed in terms of the product of the Fourier transforms $\langle \phi_k | \hat{V} | \phi_m \rangle$, where $|\phi_m\rangle$ is the molecular state and $|\phi_k\rangle$ is a plane wave state of the two atoms [9]. For the low energies relevant for dilute atomic gases, the typical wavelength q^{-1} of the open channel scattering atoms is such that $qr_C \ll 1$ where r_C is the characteristic length scale of $\langle \mathbf{r} | \hat{V} | \phi_m \rangle$. For these long wavelengths, the solution of (9) can be written from (2) as

$$\hat{U}(\mathbf{q}', \mathbf{q}) = \frac{4\pi}{m} \left[\frac{a_s + 3a_t}{4} + (a_t - a_s)\mathbf{S}_1 \cdot \mathbf{S}_2 \right] g(q')g(q) \quad (10)$$

where a_s and a_t are the scattering lengths for the singlet $V_s(r)$ and triplet $V_t(r)$ potentials, respectively. Cut-off effects of the potential (for $r_C > 0$) are contained in the form factor $g(q) \rightarrow 0$ for $qr_C \rightarrow \infty$. The atom-atom potential is thus characterized by the three parameters: a_s ,

a_t , and (implicitly) r_C . Using (10), one can readily calculate the low energy effective interaction matrix elements $\langle i | U | j \rangle$ for the scattering channels $|i\rangle$. Note that these matrix elements depend on B through the B -dependence of the eigenstates of (1). For ^{40}K , a simple calculation expressing the hyperfine states $|9/2, -9/2\rangle$, $|9/2, -7/2\rangle$, and $|7/2, -7/2\rangle$ in terms of the eigenstates of the electron and nuclear spin for a given B -field yields

$$\begin{bmatrix} U_{cc} & U_{co} \\ U_{oc} & U_{oo} \end{bmatrix} = \frac{4\pi}{m} \begin{bmatrix} \frac{c_7 a_s + a_t}{1 + c_7} & \frac{a_t - a_s}{\sqrt{1 + c_7} \sqrt{1 + c_9}} \\ \frac{a_t - a_s}{\sqrt{1 + c_7} \sqrt{1 + c_9}} & \frac{c_9 a_s + a_t}{1 + c_9} \end{bmatrix} \quad (11)$$

where $c_7 = (2x - 7 - \sqrt{81 - 28x + 4x^2})^2/32$ and $c_9 = (2x - 7 + \sqrt{81 - 28x + 4x^2})^2/32$ with $x = (C - D)/A$ incorporate the B dependence of the effective interaction.

With (10) [or (11) for ^{40}K], we have obtained a low energy effective interaction. Since U is a solution to the scattering problem in vacuum, it includes the influence of the non-resonant coupling to high momentum states in both open and closed channels.

B. Scattering matrix

We can now use U as an effective low energy interaction for the exact solution of the scattering problem. The resonant scattering matrix can be expressed in terms of U using (6) and (9) with the result that

$$\begin{bmatrix} T_{cc} & T_{co} \\ T_{oc} & T_{oo} \end{bmatrix}^{-1} = \begin{bmatrix} U_{cc} & U_{co} \\ U_{oc} & U_{oo} \end{bmatrix}^{-1} - \begin{bmatrix} \Delta G_c & 0 \\ 0 & \Delta G_o \end{bmatrix} \quad (12)$$

where $\Delta G_o(\omega, \mathbf{K}, \mathbf{q}) = G_o(\omega, \mathbf{K}, \mathbf{q}) - G^{\text{vac}}(q)$ and $\Delta G_c(\omega, \mathbf{K}, \mathbf{q}, B) = G_c(\omega, \mathbf{K}, \mathbf{q}, B) - G^{\text{vac}}(q)$. The pair propagators ΔG_o and ΔG_c provide corrections to the scattering matrix arising from the hyperfine level structure which leads to the presence of a Feshbach resonance and corrections due to the presence of a medium in both channels. Note that since the effective interaction U includes the coupling to high energy states, our theory as defined in (12) is properly renormalized in the sense that it yields finite results even for $r_C = 0$ corresponding to a δ -function interaction.

With U given by (10), we can readily solve (12) to obtain the scattering matrix in the open channel

$$T_{oo}^{-1} = \left(U_{oo} + \frac{U_{oc}\Pi_c U_{co}}{1 - U_{cc}\Pi_c} \right)^{-1} - \Pi_o \quad (13)$$

where

$$\Pi_o(\omega, \mathbf{K}) = \int \frac{d^3q}{(2\pi)^3} [G_o(\omega, \mathbf{K}, \mathbf{q}) - G^{\text{vac}}(q)] g^2(q) \quad (14)$$

and

$$\Pi_c(\omega, \mathbf{K}, B) = \int \frac{d^3q}{(2\pi)^3} [G_c(\omega, \mathbf{K}, \mathbf{q}, B) - G^{\text{vac}}(q)] g^2(q). \quad (15)$$

The integrals in (14)-(15) can be performed straightforwardly in the vacuum case for a zero range ($r_C = 0$) potential with the result that $\Pi_o^{\text{vac}}(\omega, \mathbf{K}) = -im^{3/2}\sqrt{\tilde{\omega}}/(4\pi) + \mathcal{O}(\sqrt{m\tilde{\omega}}r_C)$ where $\tilde{\omega} = \omega - \mathbf{K}^2/(4m)$ is the energy in the center of mass frame. Similarly, $\Pi_c^{\text{vac}}(\omega, \mathbf{K}, B) = \Pi_o^{\text{vac}}[\omega - E_{\text{th}}(B), \mathbf{K}] + \mathcal{O}(\sqrt{mE_{\text{th}}r_C})$.

C. Feshbach resonance

Consider now the case when a Feshbach resonance is present at $B = B_0$. For fields close to B_0 , it is standard to parameterize the zero energy vacuum scattering matrix as

$$T_{oo}^{\text{vac}} = \frac{4\pi a_{\text{bg}}}{m} \left(1 - \frac{\Delta B}{B - B_0} \right) \quad (16)$$

where a_{bg} is the (non-resonant) background scattering length and ΔB the width of the resonance. We see from (13) that the zero energy ($\omega = 0$) scattering matrix in a vacuum is

$$T^{\text{vac}}(0) = U_{oo} + |U_{oc}|^2 \frac{\Pi_c^{\text{vac}}(0)}{1 - U_{cc}\Pi_c^{\text{vac}}(0)} \quad (17)$$

since $\Pi_o^{\text{vac}}(\omega = 0) = 0$. A comparison of (16) with (17) allows us to express the phenomenological low energy scattering parameters a_{bg} , ΔB , and B_0 in terms of the microscopic parameters a_s , a_t and r_C which characterize the atom-atom interaction. The resonant part of the scattering coming from the coupling between the open and closed channels is clearly given by the second term in (17), which depends on the B field through U and $\Pi_c^{\text{vac}}(0)$. The position of the resonance B_0 is thus determined by

$$1 - U_{cc}(B_0)\Pi_c^{\text{vac}}(B_0, \omega = 0) = 0. \quad (18)$$

A straightforward pole expansion in $B - B_0$ of (17) yields

$$T_{\text{bg}} = \frac{4\pi a_{\text{bg}}}{m} = \left[U_{oo} - \frac{U_{oc}^2}{U_{cc}} + \frac{2U_{oc}\partial_B U_{oc} - U_{oc}^2 \frac{\partial_B^2 (U_{cc} - U_{cc}^2 \Pi_c^{\text{vac}})}{2\partial_B (U_{cc} - U_{cc}^2 \Pi_c^{\text{vac}})}}{\partial_B (U_{cc} - U_{cc}^2 \Pi_c^{\text{vac}})} \right] \Big|_{\omega=0, B=B_0} \quad (19)$$

for the background scattering and

$$T_{\text{bg}}\Delta B = -\frac{U_{oc}(B_0)^2/U_{cc}(B_0)}{\partial_B [1 - U_{cc}(B)\Pi_c^{\text{vac}}(B, 0)]|_{B_0}} \quad (20)$$

for the width of the resonance. With (10), (13), (18), (19), and (20), we have finally arrived at our low energy ($kr_C \ll 1$) effective theory. Equation (13) gives the full low energy/momentum dependent multi-channel scattering matrix as a function of the parameters a_s and a_t (and r_C) through U in (10). Medium effects due to the occupation states in *both* open and closed channels are

included explicitly through Π_o and Π_c . Furthermore, (18), (19), and (20) express the physical parameters a_{bg} , B_0 , and ΔB describing zero energy scattering in vacuum in terms of a_s and a_t and r_C . Alternatively, these equations can be used to fix a_s and a_t and r_C from experimental data.

We emphasize that our theory is finite and well defined even for $r_C = 0$ (i.e., $g(q) = 1$). The form factors $g(q)$ have been retained in order to describe finite range effects. These can be important since the hyperfine splitting energy E_{th} can be comparable to $\hbar^2/(mr_C^2)$.

III. MAGNETIC MOMENT OF THE FESHBACH MOLECULE

In the spirit of Landau Fermi theory, it is often convenient to express the full energy- and momentum-dependent multi-channel scattering matrix in terms of physical observables only. To do this, it is necessary to introduce one more parameter: the magnetic moment of the Feshbach molecule in a vacuum. The energy of the molecule in a vacuum is determined by the poles of the full pair propagator in the closed channel:

$$\begin{aligned} D_{\text{vac}}^{-1}(\mathbf{K}, \omega) &= \Pi_c^{\text{vac}-1} - U_{cc} - U_{oc}^2 \frac{\Pi_o^{\text{vac}}}{1 - U_{oo}\Pi_o^{\text{vac}}} \\ &= D_{\text{bare}}^{-1} - U_{oc}^2 \frac{\Pi_o^{\text{vac}}}{1 - U_{oo}\Pi_o^{\text{vac}}} \end{aligned} \quad (21)$$

The first term, $D_{\text{bare}}^{-1} = \Pi_c^{\text{vac}-1} - U_{cc}$, describes the propagation of a pair of atoms interacting in the closed channel only (i.e., the ‘‘bare’’ molecule) while the final term describes the coupling to the open channel (‘‘dressing’’). The molecule propagator is illustrated diagrammatically in Fig. 1. To obtain an expression for the magnetic moment of the molecule for fields close to B_0 , we expand (21) about $B = B_0$ and $\omega = 0$ to obtain

$$\begin{aligned} z D_{\text{vac}}^{-1}(\mathbf{K}, \omega) &\simeq \omega - \Delta\mu(B - B_0) \\ &\quad + i \frac{\Delta B \Delta\mu}{1/a_{\text{bg}} + i\sqrt{m\omega}} \sqrt{m\omega}. \end{aligned} \quad (22)$$

with

$$\Delta\mu = -\frac{\partial_B [\Pi_c^{\text{vac}-1} - U_{cc}]|_{B_0}}{\partial_\omega \Pi_c^{\text{vac}-1} - U_{oc}^2 \partial_\omega \Delta\Pi_o} = z \Delta\mu_{\text{bare}} \quad (23)$$

the magnetic moment of the molecule. We have defined

$$\begin{aligned} \Delta\mu_{\text{bare}} &= -\frac{\partial_B [\Pi_c^{\text{vac}-1} - U_{cc}]|_{B_0}}{\partial_\omega \Pi_c^{\text{vac}-1}} \\ &\simeq \partial_B E_{\text{th}} + 2 E_{\text{th}} U_{cc}^{-1} \partial_B U_{cc} \end{aligned} \quad (24)$$

using $\Pi_c^{\text{vac}}(B, \omega) \propto [E_{\text{th}}(B) - \omega]^{1/2}$ for $r_C = 0$ to calculate the partial derivatives. In addition,

$$\begin{aligned} z^{-1} &= 1 - U_{oc}^2 \partial_\omega \Delta\Pi_o^{\text{vac}} / \partial_\omega \Pi_c^{\text{vac}-1} \\ &= 1 - T_{\text{bg}} \Delta B \Delta\mu_{\text{bare}} \partial_\omega \Delta\Pi_o^{\text{vac}}, \end{aligned} \quad (25)$$

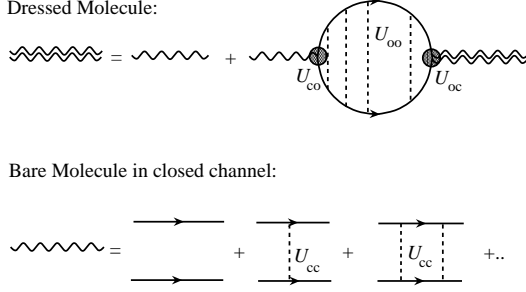


FIG. 1: The full (dressed) molecule given by (21) is indicated by the double wavy lines whereas the bare molecule in the closed channel, excluding coupling to the open channel, is given by the single wavy line. Fermions are indicated by straight lines, and the intrachannel couplings U_{oo} and U_{cc} are indicated by dashed lines. The coupling U_{oc} between the open and closed channels is indicated by \bullet .

where we have used $T_{\text{bg}}\Delta B\Delta\mu_{\text{bare}}\partial_{\omega}\Pi_c^{\text{vac}-1} = U_{oc}^2$ following from (20,24) and explicitly separated the threshold $\sqrt{\omega}$ term in Π_o by writing

$$\Pi_o^{\text{vac}}(\omega) = -i\frac{m^{3/2}}{4\pi}\sqrt{\omega} + \Delta\Pi_o(\omega). \quad (26)$$

The term $\Delta\Pi_o$ contains the corrections to the pair propagator in the open channel coming from the finite range ($r_C \neq 0$) of the potential. Equation (23)-(25) provides a clear physical interpretation of the various contributions to the magnetic moment of the molecule. Equation (24) gives the magnetic moment of the bare molecule when the interaction with atom pairs in the open channel is ignored. It consists of a contribution from the magnetic moment difference between the open and closed channels, $\partial_B E_{\text{th}}$ and a contribution from the magnetic field dependence of the interaction matrix elements in the closed channel. For the ^{40}K resonance discussed above, we have from (4) $\partial_B E_{\text{th}} = 1.78\mu_B$. The second term in (24) can be estimated as

$$2 E_{\text{th}} U_{cc}^{-1} \partial_B U_{cc} \approx 0.42 \mu_B \frac{a_s - a_t}{a_s + a_t} \left[1 + 0.89 \frac{a_s - a_t}{a_s + a_t} \right]^{-1}.$$

The factor z given by (25) gives the renormalization of the magnetic moment due to coupling of the molecule to high momentum atom pairs in the open channel (excluding the threshold effect given explicitly by the $\sqrt{\omega}$ -term). This renormalization was first included systematically in a low energy effective theory in Ref. [5] where it was argued that it can, in general, be significant. To see this, we use a simple cut-off form factor $g(q) = \Theta(r_C^{-1} - q)$ in

(14) to calculate $\Delta\Pi_o$ as

$$\begin{aligned} z^{-1} &= 1 - \frac{2}{\pi} \Delta\mu_{\text{bare}} \Delta B m a_{\text{bg}} r_C \\ &= 1 - 0.35 \frac{\Delta\mu_{\text{bare}}}{\mu_B} \frac{\Delta B}{10\text{G}} \frac{m}{40u} \frac{a_{\text{bg}}}{100\text{\AA}} \frac{r_C}{10\text{\AA}} \end{aligned} \quad (27)$$

where $1u$ is the atomic mass unit. For the Feshbach resonance of ^{40}K considered in this paper, $\Delta B \sim \mathcal{O}(10\text{G})$ and $a_{\text{bg}} \sim \mathcal{O}(100\text{\AA})$ (see next section). It then follows from (27) that the renormalization of the magnetic moment of the molecule due to the coupling to high momentum atom pairs in the open channel can be significant (i.e. $z < 1$) if $r_C \gtrsim 10\text{\AA}$. Likewise, the renormalization can be significant for the very broad resonance for ^6Li at $B \simeq 830\text{G}$ with $\Delta B \sim 300\text{G}$ and $a_{\text{bg}} \sim \mathcal{O}(1000\text{\AA})$ [12]. However, for the narrow ^6Li resonance at $B_0 \simeq 543\text{G}$ and $\Delta B = 0.23\text{G}$ studied in Ref. [11], (27) indicates that the renormalization of the magnetic moment away from threshold described by z can be ignored, i.e. $z \simeq 1$.

Having introduced the magnetic moment of the molecule, we can now express the full frequency- and momentum-dependent scattering matrix in a medium in terms of the physical parameters a_{bg} , B_0 , ΔB , and $\Delta\mu$. To do this, we rewrite the open-open channel scattering matrix given by (13) in the form

$$\begin{aligned} T_{oo} &= \frac{U_{oo}}{1 - U_{oo}\Pi_o} + \frac{U_{oc}^2/[1 - U_{oo}\Pi_o]^2}{\Pi_c^{-1} - U_{cc} - U_{oc}^2 \frac{\Pi_o}{1 - U_{oo}\Pi_o}} \\ &= \frac{U_{oo}}{1 - U_{oo}\Pi_o} + \frac{U_{oc}}{1 - U_{oo}\Pi_o} D \frac{U_{co}}{1 - U_{oo}\Pi_o}. \end{aligned} \quad (28)$$

The first term describes the scattering via the open channel interaction only, and the second term describes scattering via the closed channel. The in-medium molecule propagator D is given by (21) with the replacement $\Pi_c^{\text{vac}} \rightarrow \Pi$. This splitting of the scattering into open channel and closed channel processes is illustrated diagrammatically in Fig. 2. Comparison with (21) reveals that the second term in (28) describes the presence of the Feshbach molecule (including medium effects) leading to the resonance. Note that both terms in (28) contribute to the background scattering given by (19). Indeed, a Taylor expansion around $\omega = 0$ and $B = B_0$ and comparison with the vacuum expression for the scattering matrix (16) yields

$$T_{oo} = \frac{T_{\text{bg}}}{\left(1 + \frac{\Delta\mu\Delta B}{\tilde{\omega} + h(\omega) - \Delta\mu(B - B_0)}\right)^{-1} - T_{\text{bg}}\Pi_o^{\text{inf}}(\omega)} \quad (29)$$

where $\Pi_o^{\text{inf}}(\omega, \mathbf{K})$ is the pair propagator in the open channel assuming $r_C = 0$ and $\tilde{\omega} = \omega - K^2/4m$ is the energy in the center of mass frame as before. The term

$$\begin{aligned} h(\omega, \mathbf{K}) &= \frac{\Pi_c(\omega, \mathbf{K}, B) - \Pi_c^{\text{vac}}(B, 0) - \partial_{\omega}\Pi_c^{\text{vac}}(B_0, \omega)|_0 \tilde{\omega}}{\partial_{\omega}\Pi_c^{\text{vac}}(B_0, \omega)|_0 + \left(\frac{U_{oc}}{U_{cc}}\right)^2 \partial_{\omega}\Delta\Pi_o^{\text{vac}}} \\ &= z \left[\frac{\Pi_c(\omega, \mathbf{K}, B) - \Pi_c^{\text{vac}}(B, 0)}{\partial_{\omega}\Pi_c^{\text{vac}}(B_0, \omega)|_0} - \tilde{\omega} \right] \end{aligned} \quad (30)$$

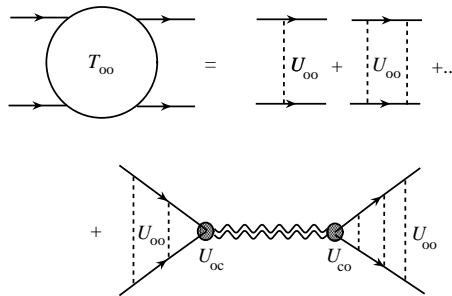


FIG. 2: The open-open channel scattering matrix split into scattering in the open channel only (first line) and scattering via the closed channel (second line) as given by (28).

in (29) describes effects coming from the fact that the Feshbach state is a composite two-fermion object including medium effects in the closed channel.

With (29), we have expressed the full scattering matrix in terms of the physical parameters a_{bg} , B_0 , ΔB , and $\Delta\mu$. These, in turn, can be calculated from a_s and a_t with the aid of (18)-(20) and (23). Thus, (29) and (13) are two formulations of the same result: The former expresses the in-medium scattering matrix in terms of the microscopic parameters characterizing the atom-atom interaction; the latter provides a formulation valid close to the resonance in terms of the physical observables characterizing the resonance. Our effective theory thus provides a link between these descriptions.

IV. MEDIUM EFFECTS

We now consider the shift of the resonance position due to the presence of a medium. To do this, we rewrite (29) as

$$T_{oo} = \frac{T_{\text{bg}}}{1 - T_{\text{bg}}\Pi_o^{\text{inf}}} + \frac{T_{\text{bg}}\Delta\mu\Delta B/[1 - T_{\text{bg}}\Pi_o^{\text{inf}}]^2}{\tilde{\omega} + h(\omega) - \Delta\mu(B - B_0) - T_{\text{bg}}\Delta\mu\Delta B \frac{\Pi_o^{\text{inf}}}{1 - T_{\text{bg}}\Pi_o^{\text{inf}}}}. \quad (31)$$

We see clearly that the term $T_{\text{bg}}\Delta\mu\Delta B\Pi_o^{\text{inf}}$ in the denominator describes the effects of the presence of atoms in the open channel whereas $h(\omega)$ describes the effects of the presence of atoms in the closed channel.

We first calculate medium effects due to the presence of open channel atoms. We see from (14) that medium effects in the open channel shift Π_o by

$$\int \frac{d^3q}{(2\pi)^3} \frac{f_{\alpha_1}(\frac{1}{2}\mathbf{K} - \mathbf{q}) + f_{\alpha_2}(\frac{1}{2}\mathbf{K} + \mathbf{q})}{\frac{K^2}{4m} + \frac{q^2}{m} - \omega} = \frac{mk_F}{\pi^2} \quad (32)$$

for $\mathbf{K} = 0$ and $\omega = 0$ where k_F is the Fermi momentum for both open channel hyperfine states (we have assumed equal population) and $k_B T \ll k_F^2/2m$. The resulting shift δB in the position of the center of mass $\mathbf{K} = 0$ resonance from its vacuum position B_0 is then readily determined from (31)

$$\delta B = -\frac{4}{\pi} k_F a_{\text{bg}} \Delta B. \quad (33)$$

Taking a density of $n = 2 \times 10^{12} \text{cm}^{-3}$, $a_{\text{bg}} = 290a_0$, and $\Delta B = 8\text{G}$ relevant for the ^{40}K resonance considered here, (33) yields $\delta B = -0.61\text{G}$ for the shift of the position of the resonance due to the occupation of states in the open channel. Of course, in a typical radio-frequency (RF) experiment which probes the scattering of pairs of atoms within the Fermi sea with varying center of mass momenta \mathbf{K} [12], it is necessary to average (14) and (31) appropriately over momenta in order to obtain the average medium shift of the resonance position.

The effects due to the population of the closed channel states can likewise be calculated from $h(\omega)$. The medium shift of $h(\omega)$ is obtained from (30) and (15) as

$$\int \frac{d^3q}{(2\pi)^3} \frac{f_{\alpha_3}(\frac{1}{2}\mathbf{K} - \mathbf{q}) + f_{\alpha_4}(\frac{1}{2}\mathbf{K} + \mathbf{q})}{E_{\text{th}} + \frac{K^2}{4m} + \frac{q^2}{m} - \omega} \simeq \frac{n_c}{E_{\text{th}}}, \quad (34)$$

where n_c is the density of closed channel atoms and $E_{\text{th}} \gg (3\pi^2 n_c)^{2/3}/2m$ by assumption. The shift of the resonance position due to the occupation of closed channel states follows from (31) as

$$\delta B = -\frac{8\pi}{m^{3/2}} \frac{n_c}{\sqrt{E_{\text{th}}}\Delta\mu_{\text{bare}}} = -\frac{2n_c U_{cc}}{\Delta\mu_{\text{bare}}}. \quad (35)$$

For the ^{40}K resonance, an assumed total density of $n = 2 \times 10^{12} \text{cm}^{-3}$ corresponds to $n_c = 10^{12} \text{cm}^{-3}$ since the open and the closed channels share one hyperfine state. Using $E_{\text{th}} = 0.084K$ from (4) and assuming $\Delta\mu_{\text{bare}} \sim \mu_B$ yields $\delta B = -1.7 \times 10^{-6} \text{G}$ for the shift of the position of the resonance due to the occupation of states in the closed channel. The shift δB is very small because $E_{\text{th}} \gg \epsilon_F$ for this resonance where ϵ_F is the Fermi energy.

V. COMPARISON WITH BOSE-FERMI EFFECTIVE THEORIES

A comparison with other recent low energy effective theories for atomic Feshbach resonances is now in order. Most of these theories are based on a Bose-Fermi model where the presence of the Feshbach molecule is put in by hand as a point boson [4, 5]. In the present approach, on the other hand, the Feshbach molecule emerges dynamically and is treated correctly as a composite two-fermion object consisting of atom pairs in both the open and the closed channels. This allows for the description of effects coming from the two-fermion nature of the molecule through Π_c in (13) or the term $h(\omega)$ in (29)-(30). The simpler Bose-Fermi theory, which describes the

Feshbach state as a point boson whose propagator has linear ω -dependence, is recovered only when $h(\omega) = 0$. Furthermore, the present theory provides expressions for all low energy scattering and bound state properties, including medium effects in both open *and* closed channels, through the parameters a_s , a_t , and r_C characterizing the atom-atom interaction and thus provides a nice link between a microscopic (coupled-channels) description of the scattering and more phenomenological descriptions. This includes a physically clear picture of the various contributions to the magnetic moment of the Feshbach molecule. Once a_s , a_t , and r_C are fixed, our approach affords a systematic description of Feshbach resonances in *all* the relevant channels.

VI. RETHERMALIZATION RATE

Finally, we analyze a recent experiment on the Feshbach resonance at $B_0 \simeq 201.6$ G for ^{40}K atoms in the hyperfine states $|9/2, -9/2\rangle$ and $|9/2, -7/2\rangle$. In refs. [7, 8], the elastic collision rate for ^{40}K atoms was measured near this Feshbach resonance using a rethermalization technique: The gas was ‘‘heated’’ preferentially in one spatial direction, and the relaxation toward equilibrium with a uniform temperature in all directions was then followed in the time evolution of the rms cloud radii. The rethermalization rate was then extracted from an exponential fit of the aspect ratio as a function of time.

A variational expression for the relaxation time of temperature anisotropies can be given [6] as

$$\frac{1}{\tau_T} = \frac{\langle\langle \Phi_T \Gamma[\Phi_T] \rangle\rangle}{\langle\langle \Phi_T^2 \rangle\rangle}. \quad (36)$$

Here,

$$\langle\langle \dots \rangle\rangle = \int d^3r \int \frac{d^3p}{(2\pi)^3} \dots f_0(1 - f_0) \quad (37)$$

where f_0 is the equilibrium distribution function for $|9/2, -9/2\rangle$ and $|9/2, -7/2\rangle$ atoms and where we treat the trap potential $V_{\text{trap}}(\mathbf{r})$ using the Thomas-Fermi approximation with a spatially dependent chemical potential $\mu(\mathbf{r}) = \mu - V_{\text{trap}}(\mathbf{r})$. The populations of $|9/2, -9/2\rangle$ and $|9/2, -7/2\rangle$ atoms are assumed to be equal. The deviation function appropriate for thermal anisotropy is

$$\Phi_T = p_z^2 - p^2/3 + \frac{m^2}{3}(2\omega_z^2 z^2 - \omega_\perp^2 r_\perp^2), \quad (38)$$

where ω_z and ω_\perp are the trapping frequencies in the axial and transverse directions (we assume an axially symmetric trap). The linearized collision operator is

$$f_0(1 - f_0)\Gamma[\Phi](\mathbf{r}, \mathbf{p}) = \int \frac{d^3p_1}{(2\pi\hbar)^3} \int d\Omega \frac{d\sigma}{d\Omega} |\mathbf{v} - \mathbf{v}_1| [\Phi + \Phi_1 - \Phi' - \Phi'_1] f_1^0 f_1^0 (1 - f_1^{0'}) (1 - f_1^{0'}), \quad (39)$$

where $d\sigma/d\Omega$ is the on-shell ($\omega = p^2/2m + p_1^2/2m$) differential cross section and Ω is the solid angle for the direction of the relative outgoing momentum $\mathbf{p}'_r = (\mathbf{p}' - \mathbf{p}'_1)/2$ with respect to the relative incoming momentum $\mathbf{p}_r = (\mathbf{p} - \mathbf{p}_1)/2$ [6]. The calculation of the integrals in (36) proceeds as in Ref. [13]. In the classical limit, these integrals can be evaluated analytically yielding

$$\frac{1}{\tau_T} = \frac{4}{5N} \left(\frac{k_B T}{\pi m} \right)^{1/2} \langle \sigma \rangle \int d^3r n(\mathbf{r})^2 \quad (40)$$

where N is the total number of trapped atoms (in both hyperfine states), n the total density of atoms, and

$$\langle \sigma \rangle = \frac{1}{3} \int dx x^7 \sigma(\sqrt{mk_B T} x) e^{-x^2}. \quad (41)$$

The factor of x^7 in (41) comes from the weight function Φ_T appropriate for temperature relaxation. Note that (40) differs from the averaged cross section arising from the collision relaxation time. The latter results from a thermal average of the cross section weighted with the relative velocity only, which corresponds to the factor x^3 in (41).

We now use $\sigma = m^2 |T_{oo}|^2 / 4\pi$ with T_{oo} given by (29) to analyze the experimental data. In Refs. [7, 8], the experiments are performed at temperatures $T \sim 2T_F$ and densities $n \sim 10^{12} \text{ cm}^{-3}$. We can therefore use the classical results (40) and (41). Likewise, T_{oo} in (29) can be approximated by the vacuum on-shell scattering matrix

$$T_{oo} = \frac{\frac{4\pi}{m} a_{\text{bg}}}{\left(1 + \frac{\Delta\mu\Delta B}{q^2/m - \Delta\mu(B - B_0)}\right)^{-1} + i a_{\text{bg}} q} \quad (42)$$

where \mathbf{q} is the relative momentum of the two scattering atoms. At present, knowledge of the experimental value of the magnetic moment of the Feshbach molecule is limited. We therefore consider two cases of a large and a small value of $\Delta\mu$. We take $\Delta\mu = \partial_B E_{\text{th}} = 1.78 \mu_B$ from (4) as a large value and $\Delta\mu = \Delta\mu_0 = 0.118 \mu_B$ from Ref. [5] as a small value. Using (41)-(42), we can now analyze the experimental results [7, 8]. For the two values of $\Delta\mu$ above, we tune the parameters ΔB and a_{bg} keeping the resonance position fixed at $B_0 = 201.6$ G. The results of our fits are presented in Fig. 3. For $\Delta\mu = 0.118 \mu_B$ we find

$$\Delta B = 6.5 \text{ Gauss} \quad \text{and} \quad a_{\text{bg}} = 286.7 a_0. \quad (43)$$

The fit with $\Delta\mu = 1.78 \mu_B$ gives

$$\Delta B = 8.1 \text{ Gauss} \quad \text{and} \quad a_{\text{bg}} = 296.7 a_0. \quad (44)$$

In Ref. [8], the authors give the fitted values of the singlet and triplet scattering lengths as $a_s \approx 104.8 a_0$ and $a_t \approx 174 a_0$. Substituting these values in (19), we find $a_{\text{bg}} \simeq 170 a_0$, which is close to the value $164 a_0$ reported in Ref. [14]. This value differs somewhat from ours ($a_{\text{bg}} \simeq 290 a_0$). The reason for this discrepancy can

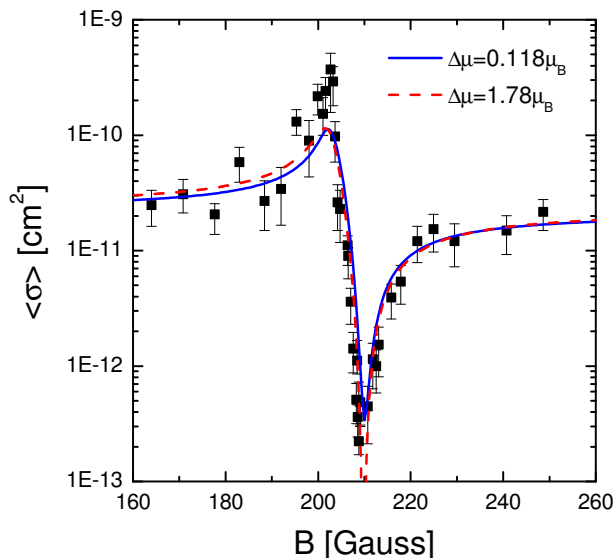


FIG. 3: Fit to the experimental rethermalization cross section [8] with (41)-(42) for two values of the molecule magnetic moment at $T = 4.4 \mu\text{K}$. The fit parameters are given in (43) for $\Delta\mu = 0.118 \mu_B$ and (44) for $\Delta\mu = 1.78 \mu_B$.

be the experimental uncertainty in determining the number of atoms trapped, and/or the fact that the authors in Ref. [8] use a different thermal averaging procedure than that given by (41) [15].

The fit parameters (43)-(44) do not depend strongly on the value of the molecule magnetic moment at the temperature $T = 4.4 \mu\text{K}$. This is because $\Delta\mu\Delta B \gg k_B T$ for both values of $\Delta\mu$: $\Delta\mu\Delta B/k_B = 55\mu\text{K}$ for $\Delta\mu = 0.118 \mu_B$ and $\Delta\mu\Delta B/k_B = 980\mu\text{K}$ for $\Delta\mu = 1.78 \mu_B$. In fact, the overall shape of the rethermalization curve as a function of $B - B_0$ (e.g. the position of the resonance and the magnitude of the maximum) becomes sensitive to $\Delta\mu$ only when

$$\Delta\mu\Delta B \sim k_B T. \quad (45)$$

To illustrate this, we show in Fig. 4 the cross section (41) evaluated for temperatures $T = 8 \mu\text{K}$ and $T = 20 \mu\text{K}$ and for the two values of $\Delta\mu$ above. We see a clear difference in the shape of the curve depending on the temperature.

The *magnitude* of the minimum of the rethermalization rate located at $B - B_0 \sim \Delta B$ (for $k_B T \ll \Delta\mu\Delta B$) is however sensitive to the temperature even for $k_B T \ll \Delta\mu\Delta B$. This can easily be understood as follows. If one ignores the q^2/m term in (42), the cross section is identically zero for $B - B_0 = \Delta B$ for all \mathbf{q} leading to a vanishing rethermalization rate. But from $q^2/m \sim k_B T$ it follows from (42) that the minimum of the rethermalization rate at $B - B_0 \sim \Delta B$ scales as $k_B T/\Delta\mu\Delta B$ for $k_B T \ll \Delta\mu\Delta B$. Thus, the minimum of the rethermalization rate is sensitive to the temperature or the value of the molecule magnetic moment even for $k_B T \ll \Delta\mu\Delta B$. To illustrate this, we plot in Fig. 5

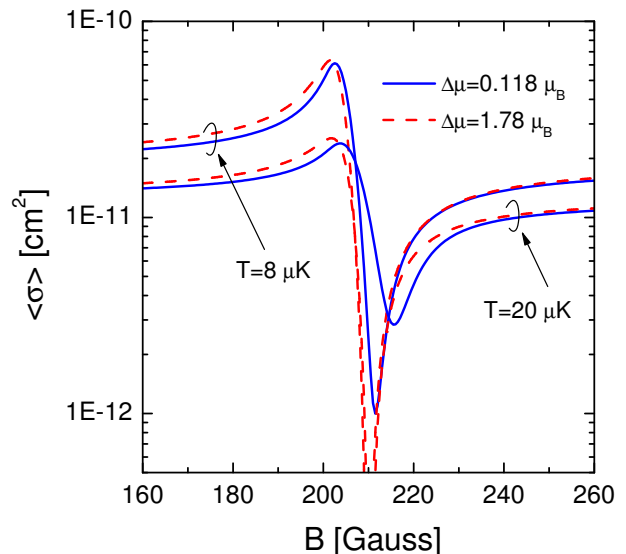


FIG. 4: The rethermalization cross section (41) with (42) for two values of the molecule magnetic moment and the temperatures $T = 8 \mu\text{K}$ and $20 \mu\text{K}$. The fit parameters are given in (43) for $\Delta\mu = 0.118 \mu_B$ and (44) for $\Delta\mu = 1.78 \mu_B$.

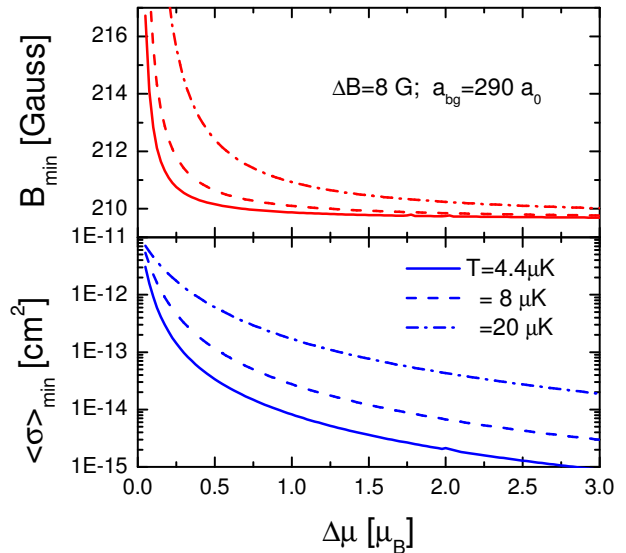


FIG. 5: The position $B - B_0 = B_{\min}$ and magnitude $\langle\sigma_{\min}\rangle$ of the minimal value of the rethermalization cross section as a function of the molecule magnetic moment $\Delta\mu$ for $T = 4.4 \mu\text{K}$, $T = 8 \mu\text{K}$, and $T = 20 \mu\text{K}$.

the position $B_{\min} = B - B_0$ and the magnitude $\langle\sigma_{\min}\rangle$ of the minimal value of the rethermalization cross section as a function of the molecule magnetic moment $\Delta\mu$ for $T = 4.4 \mu\text{K}$, $T = 8 \mu\text{K}$, and $T = 20 \mu\text{K}$. We see that the position of the minimum does not depend on $\Delta\mu$ when $k_B T \ll \Delta\mu\Delta B$ whereas the magnitude of the minimum value of the rethermalization rate does.

The temperature dependence of the rethermalization rate can be used to determine the magnetic moment of the Feshbach molecule: One can measure this mag-

netic moment by fitting the measured rethermalization cross section as a function of temperature using the theory described above. A scan of temperatures for which $k_B T \sim \Delta\mu\Delta B$ is likely to yield a fairly accurate value of $\Delta\mu$. If one is able to measure the magnitude of the minimum rethermalization rate which is sensitive to the temperature for all T , the size of the magnetic moment can even be determined for $k_B T \ll \Delta\mu\Delta B$.

VII. EFFECTIVE RANGE

We now demonstrate that the energy-dependent, Feshbach-mediated interaction introduces a new length scale: the effective range r_{eff} . The effective range expansion for the vacuum scattering matrix is given by [16]

$$T_{oo} = \frac{4\pi a}{m} \frac{1}{1 - \frac{1}{2}q^2 a r_{\text{eff}} + i q a} \quad (46)$$

where $a = a_{\text{bg}}[1 - \Delta B / (B - B_0)]$ is the resonant scattering length. This expansion can be obtained from (42) in the limit $|B - B_0| \ll \Delta B$ with the effective range given as

$$r_{\text{eff}} = -\frac{2}{m\Delta\mu\Delta B a_{\text{bg}}}. \quad (47)$$

The q^2/m term in (42) coming from the momentum dependence of the Feshbach molecule energy can therefore introduce a new length scale: An effective range given by (47). Equation (47) was also derived in Ref. [17] in the context of a Bose-Fermi model.

Note that the effective range given by (47) is negative as a direct consequence of the energy dependence of the multi-channel interaction. In a recent paper, finite range effects on the physics of cold atomic gases in the BEC-BCS crossover regime was examined using a single channel model [18]. The atom-atom potential used in Ref. [18] has a positive effective range and thus does not provide a correct description of finite range effects for resonant multi-channel atom-atom scattering. Similarly, the effective range of the single channel interaction used in a recent Monte-Carlo study also has $r_{\text{eff}} \geq 0$ [19]. In general, it is often difficult and sometimes impossible to obtain a negative effective range in single channel models of resonant scattering. This feature emerges naturally in multi-channel models.

It is common to neglect the effective range term by adopting a simple approximate form of the unitarized scattering matrix

$$T = \frac{4\pi a}{m} \frac{1}{1 + i q a}. \quad (48)$$

This approximation leads to the idea of universality. When $|a| \rightarrow \infty$, the only remaining length scale in the problem is the interparticle spacing, and the properties of the gas should therefore be independent of the microscopic details of the atom-atom interaction. The physics

of interacting gases in the universal limit has been examined by a number of authors [19, 20, 21, 22]. However, we have now demonstrated explicitly that there is an additional length scale in the problem given by (47). This length scale can be neglected only when calculating properties with a typical wave length q for which $q r_{\text{eff}} \ll 1$; the thermodynamic properties of the gas are independent of r_{eff} only when

$$|r_{\text{eff}}| \ll r_{12} \quad (49)$$

where r_{12} is the average interparticle spacing. For low temperatures where $q \sim k_F$, using (47), this condition can be written as

$$|\Delta\mu\Delta B| \gg \frac{\epsilon_F}{k_F |a_{\text{bg}}|}. \quad (50)$$

The simple unitarized form for the scattering given by (48) therefore holds only for broad resonances as defined by (50). In particular, one can expect universal behavior only for such broad resonances [17].

For the Feshbach resonance for ^{40}K at $B_0 \simeq 201.6\text{G}$, we find $\Delta B \sim 10\text{G}$ and $a_{\text{bg}} \sim 290a_0$ as explained above. Assuming $\Delta\mu \sim \mu_B$ and $\epsilon_F = 10^{-6}\text{K}$, this yields $\Delta\mu\Delta B \gg \epsilon_F/k_F a_{\text{bg}}$ and one would expect the gas to exhibit universal behavior close to resonance. This is also true for the very broad resonance for ^6Li at $B \simeq 830\text{G}$ with $\Delta B \sim 300\text{G}$. However, for the narrow ^6Li resonance at $B_0 \simeq 543\text{G}$, $\Delta B = 0.23\text{G}$, and $a_{\text{bg}} \sim 80a_0$ [11], we find $\epsilon_F \sim 3k_F a_{\text{bg}} \mu_B \Delta B$ for $\epsilon_F = 10^{-6}\text{K}$. One can therefore not expect the effective range to be an irrelevant length scale for this narrow resonance.

VIII. CONCLUSIONS

A low energy effective theory for the in-medium scattering of alkali atoms was obtained using a microscopic multi-channel description of the atom-atom interaction. Expressions for all scattering properties in terms of a few parameters characterizing the atom-atom potential were derived thereby providing a link between a microscopic multi-channel description of Feshbach resonances and more phenomenological approaches. The presence of a Feshbach resonance emerged naturally from the theory as a two-particle bound state, and the energy dependence of the Feshbach-mediated interaction was shown to introduce a negative effective range inversely proportional to the width of the resonance. Furthermore, medium effects arising from the occupation of both open and closed channel states were calculated. The theory was shown to allow the explicit calculation of corrections to commonly used approximations such as the neglect of the effective range and the treatment of the Feshbach molecule as a point boson. We also analyzed a recent rethermalization experiment on ^{40}K atoms and showed that measurement of the rethermalization rate as a function of temperature would permit the determination of the magnetic moment of the Feshbach molecule.

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