In the metro area, the Minnesota Twins are broadcast on the AM station 1500 (f = 1500 kHz). This frequency refers to the carrier wave of the signal; there is also the amplitude modulation of the signal, which has a frequency of about 10^4 Hz, such that many oscillations of the carrier wave occur while the modulation only changes slightly.

You can build a simple radio receiver to pick up the Twins/White Sox series by attaching an LRC circuit to a coil of wire that acts as an antenna. The incident radio waves will induce an oscillating voltage \( V_0 \cos(\omega t) \) in the antenna, where a reasonable \( V_0 \) is about 1 mV. Typically, the capacitance will be tunable, while the resistor and inductor are fixed (R = 5 ohms and L = 4*10^{-4} H are reasonable parts to pick up).

As a refresher, recall that the voltage drop across an inductor is L(di/dt), the drop across a resistor is IR, and the drop across a capacitor is \( q/C \). Also note that the frequencies given are not the angular frequencies.

\[
\omega = 2\pi f
\]

a) What capacitance should you tune your capacitor to to receive the Twins broadcast most clearly? Treat the resistance as negligible and find where the amplitude of some variable in your circuit (say, of \( q \)) becomes the largest.

b) Rewrite your equation (now including the resistance) in the form:

\[
\ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = f \cos(\omega t)
\]

and identify \( \gamma \) and \( \omega_0 \) in terms of R, L, etc. An equation of the above form has a particular solution given by:

\[
x_p(t) = \frac{f}{\omega} \cos(\omega t - \Theta)
\]

where:

\[
r^2 = (\omega_0^2 - \omega^2)^2 + 4\gamma^2 \omega^2
\]

and

\[
\tan \Theta = \frac{2\gamma \omega}{\omega_0^2 - \omega^2}
\]

(prove that the particular solution works if you have time after the other parts).

c) Show that you can add any transient solution to your particular solution to get another solution, where the transients are of the form:

\[
x_T(t) = A e^{-\gamma t} \cos(\omega t + \phi)
\]

As part of your proof, find \( A \).

d) Compute the value for the damping constant, \( \gamma \), and show that the transients die out much faster than the modulation varies, so that the carrier doesn't destroy the signal.

e) Find the maximum voltage across the capacitor in terms of \( V_0 \). This is the voltage amplification for the circuit.

f) The next station is at a frequency 20 kHz higher. How much will this station be amplified?
a) Going around the loop, we get
\[ V_0 \cos(\omega t) = L \frac{dI}{dt} + RI + \frac{q}{C} \]
\[ \dot{q} = I = \Rightarrow \quad V_0 \cos(\omega t) = L \ddot{q} + R \dot{q} + \frac{q}{C} \]

Ignoring \( R \) gives
\[ V_0 \cos(\omega t) = L \ddot{q} + \frac{q}{C} \]

Our solution for \( \ddot{q}(t) \) clearly needs a \( \cos(\omega t) \) part, so try
\[ \ddot{q}(t) = A \cos(\omega t), \]
\[ \dot{q}(t) = -\omega^2 A \cos(\omega t) \]

\[ \Rightarrow \quad V_0 \cos(\omega t) = AL \left\{ -\omega^2 \cos(\omega t) + \frac{1}{LC} \right\} \]

\[ \Rightarrow \quad A = \frac{V_0}{L \left\{ \frac{1}{LC} - \omega^2 \right\}} \]

which gives a resonance at
\[ \frac{1}{LC} = \omega^2 \]

\[ \Rightarrow \quad C = \frac{1}{L \omega^2} \]

\[ = \frac{1}{(4 \times 10^{-2} \text{H}/2\pi \times 1.5 \times 10^6 \text{Hz})^2} \]

\[ \Rightarrow \quad C = 2.89 \times 10^{-10} \text{F} \]

b) \[ L \dddot{q} + R \ddot{q} + \frac{q}{C} = V_0 \cos(\omega t) \]

\[ \Rightarrow \quad \dddot{q} + \frac{R}{L} \ddot{q} + \frac{q}{LC} = \frac{V_0}{L} \cos(\omega t) \]

\[ y = \frac{R}{2L}, \quad \omega_0^2 = \frac{1}{LC}, \quad \text{and} \quad f = \frac{V_0}{L} \]
c) If \( x_p \) solves \( \dot{x} + 2\omega x + \omega^2 x = f \cos(\omega t) \), then \( x_T + x_p \) should satisfy:

\[
\ddot{x}_T + 2\gamma \dot{x}_T + \omega^2 x_T = f \cos(\omega t)
\]

\[
= \gamma \ddot{x}_T + 2\gamma \dot{x}_T + \omega^2 x_T = 0
\]

\[
x_T(t) = A e^{-\gamma t} \cos(\omega' t + \alpha)
\]

\[
\dot{x}_T(t) = -A e^{-\gamma t} \left[ \gamma \cos(\omega' t + \alpha) + \omega' \sin(\omega' t + \alpha) \right]
\]

\[
\ddot{x}_T(t) = -A e^{-\gamma t} \left[ -\gamma^2 \cos(\omega' t + \alpha) - \omega'^2 \sin(\omega' t + \alpha) \right]
\]

\[
= Ae^{-\gamma t} \left\{ (\gamma^2 - \omega'^2) \cos(\omega' t + \alpha) + 2\gamma \omega' \sin(\omega' t + \alpha) \right\}
\]

So

\[
\ddot{x}_T + 2\gamma \dot{x}_T + \omega^2 x_T = Ae^{-\gamma t} \left\{ (\gamma^2 - \omega'^2) \cos(\omega' t + \alpha) + 2\gamma \omega' \sin(\omega' t + \alpha) \right. \left. - 2\gamma \omega' \cos(\omega' t + \alpha) - 2\gamma^2 \omega' \sin(\omega' t + \alpha) \right\}
\]

\[
= Ae^{-\gamma t} \left( \omega^2 - \gamma^2 \right) \cos(\omega' t + \alpha)
\]

\[
= 0 \quad \text{if} \quad \omega' = \omega^2 - \gamma^2
\]

\[
d) \quad \gamma = \frac{R}{2L} = \frac{5 \mu \Omega}{2(4 \times 10^{-8} F)} = 6.25 \times 10^4 \text{ Hz} > 10^4 \text{ Hz}
\]

so \( e^{-\gamma t} \) is small (less than 1\% over one amplitude modulation.

e) \[
V_{\text{max}} = \frac{q_{\text{max}} \cos(\omega t + \theta)}{C} \leq \frac{V_0}{\sqrt{\frac{1}{L} + \frac{1}{\omega^2 C}}} \leq V_0 \frac{\omega}{\dot{\omega}} \approx \frac{75 V_0}{\omega}
\]

\[
f) \quad V_{\text{max}} = \frac{1}{LC} \sqrt{(\omega_0 - \omega)^2 + 4 \gamma^2 \omega^2} \approx \frac{\sqrt{\frac{V_0 \omega}{\dot{\omega}}} \omega_0 \sqrt{\frac{1}{L} + \frac{1}{\omega^2 C}}} \approx \frac{V_0 (9.4 \times 10^4 \text{ Hz})}{2(14 \times 10^4 \text{ Hz})} \approx \frac{V_0}{2(14 \times 10^4 \text{ Hz})} \approx 34 V_0
\]