Two particles (one heavy and one light, with mass m) interact with each other under the influence of two forces.

One is attractive, and is given by the potential: \[ V_1(r) = V_o \left( \frac{r}{a_1} \right) \]

while the other is repulsive and given by the potential: \[ V_2(r) = V_o \left( \frac{a_2}{r} \right) \]

where \( a_1 \) and \( a_2 \) are constant distances.

1. Find the equilibrium distance for the two particle system.
2. Do a Taylor expansion to 2nd order about the equilibrium distance.
3. From the Taylor expansion and what you know about potential for a spring, find the frequency for vibrations between the two particles for distances not far from the equilibrium.

Note: The Taylor expansion of \( f(x) \) about a point \( x_0 \) is:

\[
 f(x) = \sum_{n=0}^{\infty} \frac{(x-x_0)^n}{n!} f^{(n)}(x_0)
\]
The Turkish bow in the 15th and 16th centuries was much more effective than standard western bows. The draw force, $F(x)$, approximately followed the elliptical formula

$$\left(\frac{F(x)}{F_{\text{max}}}\right)^2 + \left(\frac{x}{\xi}\right)^2 = 1$$

where $x$ is the pull distance and $l$ is the maximum length drawn.
Find the maximum range for the bow where $F_{\text{max}} = 360 \text{ N}$, $\xi = 0.7 \text{ m}$, and $m = 34 \text{ g}$, and compare it to a normal bow that acts like a simple spring with the same $F_{\text{max}}$ and $\xi$. 

1. \[ V(r) = V_0 \left[ \frac{C}{a_1} + \frac{a_2}{r} \right] \]

\[ V'(r_o) = 0 = V_0 \left[ \frac{1}{a_1} + \frac{-a_2}{r_o^2} \right] \]

\[ \Rightarrow \quad r_o = \sqrt{a_1 a_2} \]

2. \[ V(r) = \sum_{n=0}^{\infty} \frac{(r-r_o)^n}{n!} f^{(n)}(r_o) \]

\[ = V(r_o) + \frac{1}{2!} (r-r_o)^2 V''(r_o) + \frac{1}{3!} (r-r_o)^3 V'''(r_o) \]

\[ V''(r_o) = \frac{2V_0 a_2}{r_o^3} = \frac{2V_0}{\sqrt{a_1^3 a_2}} \]

\[ \Rightarrow \quad V(r) = V_0 \left[ \sqrt{\frac{a_2}{a_1}} + \frac{a_2}{\sqrt{a_1^3 a_2}} \right] + \frac{1}{2} (r-r_o)^2 \left[ \frac{2V_0}{\sqrt{a_1^3 a_2}} \right] \]

3. For a spring, \[ V(x) = \frac{1}{2} k (x-x_o)^2 \], and \[ \omega = \sqrt{\frac{k}{m}} \]

Comparing to a spring,

\[ k = \frac{2V_0}{\sqrt{a_1^3 a_2}} \]

So \[ \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{2V_0}{m \sqrt{a_1^3 a_2}}} \]
\[ \left( \frac{F(x)}{F_{\text{max}}} \right)^2 \cdot \left( \frac{l-x}{l} \right)^2 = 1 \]

\[ F(x) = F_{\text{max}} \left[ 1 - \left( \frac{l-x}{l} \right)^2 \right] \]

Energy (work) from \( x = 0 \) to \( x = l \):
\[ E = \int_0^l F(x) \, dx = F_{\text{max}} \int_0^l \left[ 1 - \left( \frac{l-x}{l} \right)^2 \right] \, dx \]
\[ = F_{\text{max}} \int_0^\pi \left[ 1 - \sin^2 u \right] (-l \cos u) \, du \]
\[ = -F_{\text{max}} l \int_0^\pi \cos^2 u \, du \]
\[ = -F_{\text{max}} l \left[ \frac{1}{2} \left( u + \frac{1}{2} \sin 2u \right) \right]_0^\pi \]
\[ = \left( \frac{\pi}{4} \right) F_{\text{max}} l \]

The max range is at \( 45^\circ \):

\[ R = \frac{2v_x y}{g} \]
\[ gt = y \]
\[ \Rightarrow R = \frac{2v_x y}{g} = \frac{v^2}{g} \quad \text{where} \quad v^2 = \frac{2E}{m} \]

\[ R = \frac{2E}{mg} = \frac{\pi F_{\text{max}} l}{2 mg} \approx 1200 \text{ m} \]

(actual range was about 430 m)

For a normal bow, \[ E = \frac{1}{2} k x^2 = \frac{1}{2} F_{\text{max}} l \Rightarrow R \approx 760 \text{ m} \]