I. CHARGED PENDULA IN EQUILIBRIUM

Two light, strong strings are attached to a horizontal support. At the other end of each string hangs an object. One of the objects is known to have weight 2.000 N; the other is unknown. A power supply is slowly turned on to give each object a charge, which causes them to move away from each other. When operating level is reached, the two objects hang at the same height, but at different angles. The string of the known mass makes an angle of 10° with the vertical, and the other makes an angle of 20°. Find the unknown mass and then estimate the net charge necessary for the observed deflection if the strings are 10 cm long and the objects are taken to be spherical. List your assumptions.

Solution. We begin by defining quantities. Let $d$ be the distance between the points where the strings are attached to the support and let $\ell$ be the length of the string. Then $x_1 = \ell \sin(\theta_1)$ is the horizontal distance the known mass is deflected from its initial position and $x_2 = \ell \sin(\theta_2)$ is the horizontal deflection of the unknown mass. $F_{ci}$ is the Coulomb force on the $i^{th}$ object.

Two methods will be demonstrated. The first is exact and relatively simple, but requires taking components of vectors. The second method makes a small approximation but demonstrates how the use of earlier concepts can dramatically simplify new problems.

Method 1. The pendula are in equilibrium, so if we directly consider the forces on the first we find

$$\sum F_x = T_1 \sin \theta_1 - F_{c1} = 0 \quad (1)$$
$$\sum F_y = T_1 \cos \theta_1 - m_1 g = 0 \quad (2)$$

Divide the first equation by the second to find

$$\tan \theta_1 = \frac{F_{c1}}{m_1 g} \quad (3)$$

We then apply the same analysis to the second pendulum and find that $\tan \theta_2 = F_{c2}/m_2 g$. 

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Since the Coulomb force of 1 on 2 is the same as the Coulomb force of 2 on 1, $F_{c1} = F_{c2}$. So

$$m_2 = m_1 \frac{\tan \theta_1}{\tan \theta_2} = 0.09877 \text{ kg} \quad (4)$$

We now assume the objects are spheres and turn to the Coulomb force on the first explicitly, defining $r$ as follows,

$$F = \frac{1}{4\pi \varepsilon_0} \frac{q_1 q_2}{(x_1 + x_2 + d)^2} = \frac{1}{4\pi \varepsilon_0} \frac{q_1 q_2}{r^2} \quad (5)$$

Using (3), we can now assume $q_1 q_2 = q^2$ and solve for $q$ to find

$$q = \sqrt{4\pi \varepsilon_0 r^2 m_1 g \tan \theta_1} \quad (6)$$

We’re not given $d$, but if we take it to be 1 cm, we get $q = 3.980 \times 10^{-7} \text{ C}$. If we take $d = 0$, then $q = 3.359 \times 10^{-7} \text{ C}$.

**Method 2.** In the case of small perturbations a pendulum can be thought of as a harmonic oscillator with spring constant $k = m\omega^2 = mg/\ell$. This is entailed by the standard pendulum approximation, that for small angles $\sin \theta \approx \theta$. In our case, $\sin 20^\circ = \sin 0.349 \text{ rad} = 0.342 \approx \theta$, so this approximation is fairly good. Hooke’s Law tells us that the restoring force $\vec{F} = -k\vec{x}$, so the magnitude of the spring force on mass $i$ is $F_i = m_i g x_i / \ell = m_i g \sin \theta_i$. Again, the Coulomb forces between the two masses are of the same magnitude. Since the only other force acting on each object on the horizontal direction is the Hooke’s Law force, and since both pendula are in equilibrium, the restoring forces for the two objects must be equal in magnitude.

$$m_1 g \sin \theta_1 = m_2 g \sin \theta_2 \implies m_2 = m_1 \frac{\sin \theta_1}{\sin \theta_2} = 0.1035 \text{ kg} \quad (7)$$

Compare this to (4). For small angles, $\sin \theta \approx \tan \theta$, and our answers differ by only 4.6%.

As before, the Coulomb force should be equal to the Hooke’s Law force,

$$\frac{1}{4\pi \varepsilon_0} \frac{q_1 q_2}{r^2} = m_1 g \sin \theta_1$$

We again assume that $q_1 q_2 = q^2$ and solve for $q$,

$$q = \sqrt{4\pi \varepsilon_0 r^2 m_1 g \sin \theta_1}$$

For $d = 1 \text{ cm}$, $q = 3.827 \times 10^{-7} \text{ C}$. If $d = 0$, then $q = 3.205 \times 10^{-7} \text{ C}$. Again these numbers differ from those found by Method 1 by about 4%, so our approximation that $\sin \theta \approx \theta$ is a good one.
II. ION MOTION THROUGH A CHARGED RING

A He$^{++}$ ion (often called an $\alpha$ particle) moves on axis toward a ring of charge $Q = -8.0 \ \mu C$ with radius $R = 3.0 \ \text{cm}$. It is on course to collide with a sample 2.5 mm from the ring, and has velocity $v_0 = 200 \ m/s$ when it passes through the center of the ring. Assume the radius of the ring is much larger than the distance the ion will travel, $R \gg x$. Will the ion reach the sample? Note that $e = 1.6 \times 10^{-19} \ C$, $m_{He} = 6.7 \times 10^{-27} \ kg$.

**Solution.** This problem can be approached from two directions. Consideration of kinematics and dynamics leads directly toward the answer, but there’s a fair amount of algebra and we have to make approximations. We will later consider energy, which uses concepts that have not been formally introduced, but produces an exact answer very quickly.

**Method 1.** If we define the axis of motion to be the x-axis, then Newton’s equations of motion describe the kinematics of the system:

\[
x = x_0 + v_0 t + \frac{1}{2} a t^2
\]

\[
v = v_0 + at
\]

Take as the initial condition the ion passing through the center of the ring and find $x$ as the particle comes to rest. Then $x_0 = 0$ and $v = 0$. Newton’s second law gives us $F = ma$, where the relevant force is Coulomb's law for a ring of charge, experienced by an on-axis charge $q = 2e$,

\[
F_x = \frac{qQ}{4\pi\varepsilon_0 \ (x^2 + R^2)^{3/2}} \approx \frac{qQ}{4\pi\varepsilon_0 R^3} \frac{x}{x^2 + R^2}
\]

where we have made the approximation that $x^2 + R^2 \approx R^2$ since $x \ll R$. We now solve the equations of motion and make the substitutions $t = -v_0/a$ and then $a = F/m$ into the $x$ equation to get

\[
x = \left[ \frac{4\pi\varepsilon_0 R^3 m v_0^2}{2qQ} \right]^{1/2} = 561 \ \text{nm}
\]

**Method 2.** Consider the potential energy $U$ of the ion near a ring of charge. If we know the on-axis electric potential $V$ of a ring of charge, then

\[
U = qV = \frac{1}{4\pi\varepsilon_0} \frac{qQ}{\sqrt{x^2 + R^2}}
\]

This scalar potential $V$ is related to the electric field by $\vec{E} = -\nabla V$. It will be introduced in Chapter 24 of Fishbane.
We know the initial total energy $E_i = T_i + U_i$ is the kinetic plus potential energy at the center of the ring. This should be equal to the total energy when the particle has stopped (when $T = 0$), so we have the conservation equation

$$T_i + U_i = U_f \implies \frac{1}{2}m_{He}v_0^2 + \frac{1}{4\pi\epsilon_0} \frac{qQ}{R} = \frac{1}{4\pi\epsilon_0} \frac{qQ}{\sqrt{x^2 + R^2}}$$

Solve this for $x$ and we eventually end up with

$$x = \sqrt{R^2 - \left( \frac{2\pi\epsilon_0 m v_0^2}{qQ} + \frac{1}{R} \right)^2} = 561 \text{ nm}$$

This is the same answer as we found with Method 1. This indicates that the approximation we made in Method 1 (that the radius of the ring is much greater than the distance the ion will travel past the center) is very good. In both cases, the ion will not reach the sample.