A satellite in a circular Earth orbit is subject to a very small constant friction force $f$, due to the atmosphere. As it spirals inward, it slowly decreases its orbital radius. Find the decrease in radius per revolution under the assumption that the orbit is approximately circular with radius $r$. Find the changes in potential energy, kinetic energy, and total energy per orbit. Does the satellite speed up or slow down as it spirals in?

**Solution.** For a circular orbit, the only radial (centripetal) force is gravity, so $F_g = F_c$ with $F_g = GMm/r^2$ and $F_c = mv^2/r$. This implies

$$\frac{GMm}{r^2} = \frac{mv^2}{r} \implies v = \sqrt{\frac{GM}{r}} \quad (1)$$

So as long as the satellite moves in a circular orbit, its velocity is inversely proportional to the square root of the orbital radius – it speeds up as it spirals inward.

Angular momentum is not constant $(\vec{r} \times \vec{f} = \tau$ is an external torque), so we instead consider energy. Using equation (1), the total energy is given by

$$E = T + U = \frac{mv^2}{2} - \frac{GMm}{r} = -\frac{GMm}{2r} \quad (2)$$

The only nonconservative force acting on the system is friction, so the amount of energy lost by the system in one revolution is equal to the amount of work done on the system by friction over an approximate circle of radius $r_{avg}$.

$$\Delta E = -W_f = -2\pi r_{avg}f \quad (3)$$

Energy is neither created nor destroyed, so we can write the conservation equation

$$-\frac{GMm}{2r_1} - 2\pi r_{avg}f = -\frac{GMm}{2r_2} \quad (4)$$

This gives us a relation between the initial radius $r_1$ and the radius after one revolution $r_2$. We will approximate $r_{avg}$ two different ways. The first method is a bit more difficult, but it allows us to input a starting radius and find the difference after one revolution. The second method is simpler, but does not immediately allow for useful numerical results.
Method 1. Since the magnitude of the dissipated energy term is small compared to that of the total energy and since $\Delta r$ is also small, we can approximate $r_{\text{avg}}$ with $r_1$ and solve for $r_2$ in terms of $r_1$ as follows:

$$r_2 = r_1 \left( \frac{4\pi r_1^2 f}{GMm} + 1 \right)^{-1}$$  \hspace{1cm} (5)

Note that if $r_1$ were at the surface of the Earth $R_E$, we’d have the factor $GM/R_E^2 = g = 9.81 \text{ m/s}^2$ in the denominator above, so the value at $r_1$ should be of roughly the same order of magnitude. For a force of friction $f$ whose numerical value in N is much smaller than the numerical value of the mass $m$ in kg, we see that $r_2 = r_1(\epsilon + 1)^{-1}$ where $\epsilon \ll 1$. This can be expanded in a Taylor series so that

$$r_2 = r_1 \left( 1 - \frac{4\pi r_1^2 f}{GMm} + \ldots \right)$$  \hspace{1cm} (6)

In this form we can easily see that $\Delta r = r_2 - r_1 < 0$.

The difference in kinetic energy $\Delta T = T_2 - T_1 = mv_2^2/2 - mv_1^2/2$ so if we substitute in equation (5) for $r_1/r_2$ and cancel a factor of $GMm/r_1$ we see that

$$\Delta T = \frac{GMm}{2r_2} - \frac{GMm}{2r_1} = \frac{GMm}{2r_1} \left( \frac{r_1}{r_2} - 1 \right) = 2\pi r_1 f$$  \hspace{1cm} (7)

The difference in potential energy $\Delta U = U_2 - U_1$ is

$$\Delta U = \left( -\frac{GMm}{r_2} \right) - \left( -\frac{GMm}{r_1} \right) = \frac{GMm}{r_1} \left( \frac{r_1}{r_2} - 1 \right) = -4\pi r_1 f$$  \hspace{1cm} (8)

So, as expected, $\Delta E = \Delta T + \Delta U = -2\pi r_1 f$.

Method 2. If we instead make the approximation $r_{\text{avg}} = \sqrt{r_1r_2}$ (the geometric mean of the two radii) we can write equation (4) as

$$-\frac{GMm}{2} \left( \frac{r_2 - r_1}{r_1r_2} \right) = -\frac{GMm}{2} \left( \frac{\Delta r}{r_{\text{avg}}^2} \right) = 2\pi r_{\text{avg}} f$$  \hspace{1cm} (9)

$$\Delta r = -\frac{4\pi r_{\text{avg}}^3 f}{GMm}$$  \hspace{1cm} (10)

This is actually a different statement, since our expression for $r_{\text{avg}}$ now contains $r_2$. Unlike the first method, calculating $\Delta r$ now requires $r_{\text{avg}}$, which itself requires $r_1$ and $r_2$, so if we had $r_{\text{avg}}$ we’d already have $\Delta r = r_2 - r_1$. To escape from this difficulty, we can further approximate $r_{\text{avg}} \approx r_1$. Then $\Delta r$ in equation (10) becomes equivalent to taking the first two terms in the Taylor expansion (6) and we recover our results from Method 1.