Consider a series ac circuit consisting of a voltage supply oscillating at 600 Hz and whose emf has magnitude $V_0 = 4 \text{ V}$, two capacitors in series with $C_1 = 4 \mu\text{F}$ and $C_2 = 9 \mu\text{F}$, and an inductor with $L = 70 \mu\text{H}$. Find the maximum current and the resonance frequency.

**Solution.** First combine the capacitors into an equivalent capacitance of magnitude $C = C_1 C_2 / (C_1 + C_2)$. Application of the Kirchhoff Loop Rule yields

$$V_0 \sin \omega t - L \frac{dI}{dt} - \frac{Q}{C} = 0 \quad (1)$$

This is much more difficult to solve explicitly than the equation from last week’s problem, but there is a way to dramatically simplify the process. We know that eventually, the oscillatory behavior of the circuit will be entirely governed by the driving voltage, and so the final solution for the current in the circuit (or the charge on a capacitor) will have the same general form as the driving voltage,

$$Q(t) = Q_{\text{max}} \sin \omega t \quad (2)$$

If we plug this trial solution (2) back into (1), realizing that $I = dQ/dt$, we get

$$V_0 \sin \omega t - L \left(-\omega^2 Q_{\text{max}} \sin \omega t\right) - \frac{Q_{\text{max}}}{C} \sin \omega t$$

Divide out the $\sin \omega t$ factor and solve for $Q_{\text{max}}$,

$$Q_{\text{max}} = \frac{V_0}{1/C - L \omega^2} \quad (3)$$

Then we can find

$$I(t) = \frac{dQ(t)}{dt} = \omega Q_{\text{max}} \cos \omega t$$

So $I_{\text{max}} = \omega Q_{\text{max}} = 41.9 \text{ mA}$.

The resonance (natural) frequency in a simple LC circuit is independent of the driving voltage and so is easy to remember, but given our previous solution, it’s straightforward to find. Resonance is characterized by a sharp peak in the response (current, charge, etc.) of
a circuit at a finite frequency. To make the expression for $Q_{\text{max}}$ blow up, we want to make the denominator 0. Solving for the unique positive $\omega$ that does this results in,

$$\omega_0 = \frac{1}{\sqrt{LC}} = 71814 \text{ s}^{-1}$$

Note that we can now rewrite (3) as follows:

$$Q_{\text{max}} = \frac{V_0}{L(\omega_0^2 - \omega^2)}$$

This clearly shows the expected resonance behavior: as $\omega$ gets arbitrarily close to $\omega_0$, the charge on the capacitor gets arbitrarily large. This puzzling situation is resolved if you add a small resistor (as there must be in any real situation), which adds another term to the denominator.