Physics 1302W.500, Week 11

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I. MAGNETIC FIELD FROM A PART-CIRCULAR CURRENT LOOP

A current runs counter-clockwise along a circular wire segment of radius \( b \) beginning at point \( D \) and centered on point \( P \), forming an arc of 120°. The wire then bends and current flows radially outward for a length \( a \). Current then flows clockwise along a 120° arc of radius \( a + b \) centered at \( P \), returning to the same angular position as \( D \), then moving radially inward to meet up with \( D \). Find the magnetic field produced at \( P \).

Solution. For this problem we will make use of the Biot-Savart Law,

\[
d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{\ell} \times \vec{r}}{r^3}
\]

(1)

The line segments \( AB \) and \( CD \) are parallel to the radius vector from point \( P \), so for these segments, \( d\vec{\ell} \times \vec{r} = 0 \). On the circular portions of the loop, \( |d\vec{\ell} \times \vec{r}| = rd\ell = r(d\theta) \). Consider the inner loop of radius \( b \). For convenience, define the angles relative to the line \( PC \).

\[
B_{\text{inner}} = \int_0^{2\pi/3} d\vec{B}_{\text{inner}} = \frac{\mu_0 I}{4\pi b^2} \int_0^{2\pi/3} d\ell = \frac{\mu_0 I}{6b}
\]

By the right hand rule, this field is pointing out of the page. Now consider the outer loop,

\[
B_{\text{outer}} = \int_{2\pi/3}^0 d\vec{B}_{\text{outer}} = -\frac{\mu_0 I}{4\pi(a + b)^2} \int_0^{2\pi/3} d\ell = -\frac{\mu_0 I}{6(a + b)}
\]

where the negative sign means the field points into the page, by the angular convention we’ve adopted for this problem. Use a common denominator to add the fields and find that

\[
B = B_{\text{inner}} + B_{\text{outer}} = \frac{\mu_0 Ia}{6b(a + b)}
\]

The sign indicates that field points out of the page. This is as we should have expected, since the inner arc has a smaller radius and hence a larger contribution to the field.

We could have done this problem more carefully by checking the sign of \( d\vec{\ell} \times \vec{r} \) for each arc and making sure we’re integrating in the correct direction, while taking account of the direction of current, but it’s much easier to just look at magnitudes and deduce the final direction from what we know the result has to look like.
II. MAGNETIC FIELD FROM A COAXIAL CABLE

A coaxial cable consists of an inner wire surrounded by an insulator, followed by a conducting shell. The wire and shell both carry the same current, but in the opposite direction. Find the magnetic field at a radius larger than that of the shell and at a radius smaller than that of the shell. Then take the shell to have non-negligible thickness and find the magnetic field between inner radius \( R_1 \) and outer radius \( R_2 \) of the shell.

Solution. For this problem we will make use of Ampère’s Law,

\[
\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{\text{enclosed}}
\]  

Outside the conducting shell, the total enclosed current \( I_{\text{enc}} = 0 \), so \( B = 0 \). In the region between the central wire and the outer shell, draw an Ampèrian loop of radius \( r \). Then

\[
B(2\pi r) = \mu_0 I \quad \implies \quad B = \frac{\mu_0 I}{2\pi r}
\]

Now we wish to consider the magnetic field inside the outer shell, between inner radius \( R_1 \) and outer radius \( R_2 \). The problem is trivial if we assume that all the current moves on the outer surface of the conductor, and the previous answer applies.

If we make it interesting by assuming the current is uniformly distributed throughout the shell, then our loop of radius \( r \) such that \( R_1 < r < R_2 \) will enclose a total current of \( I - fI \) where \( f \) accounts for the fact that only part of the current in the outer shell is enclosed, and is the ratio of the current-carrying area of a circle of outer radius \( r \) to that of a circle of outer radius \( R_2 \) (both current-carrying areas have inner radius \( R_1 \)). That is,

\[
f = \frac{\pi r^2 - \pi R_1^2}{\pi R_2^2 - \pi R_1^2}
\]

Then the amount of current enclosed is

\[
I_{\text{enc}} = I \left( 1 - \frac{r^2 - R_1^2}{R_2^2 - R_1^2} \right)
\]

Note that this gives us the correct boundary conditions: at \( r = R_1 \), \( I_{\text{enc}} = I \) and at \( r = R_2 \), \( I_{\text{enc}} = 0 \).

Finally, we plug back into (2) to find that

\[
B = \frac{\mu_0 I_{\text{enc}}}{2\pi r} = \frac{\mu_0 I \left( 1 - \frac{r^2 - R_1^2}{R_2^2 - R_1^2} \right)}{2\pi r}
\]