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BASIC CONCEPTS OF TERNARY HOPF ALGEBRAS

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The theory of ternary semigroups, groups and algebras is reformulated in the abstract arrow language. Then using the reversing arrow ansatz we define ternary comultiplication, bialgebras and Hopf algebras and investigate their properties. The main property "to be binary derived"is considered in detail. The co-analog of Post theorem is formulated. It is shown that there exist 3 types of ternary coassociativity, 3 types of ternary counits and 2 types of ternary antipodes. Some examples are also presented.

KEYWORDS : ternary operation, derived operation, skew element, ternary antipod, ternary Hopf algebra

Ternary and n-ary generalizations of algebraic structures is the most natural way for further development and deeper understanding of their fundamental properties. Firstly ternary algebraic operations were introduced already in the XIX-th century by A. Cayley. As the development of Cayley's ideas it were considered n-ary generalization of matrices and their determinants [1, 2] and general theory of n-ary algebras [3, 4] and ternary rings [5] (for physical applications in Nambu mechanics, supersymmetry, Yang-Baxter equation, etc. see [6, 7] as surveys). The notion of an n-ary group was introduced in 1928 by W. Dörnte [8] (inspired by E. Nöther) which is a natural generalization of the notion of a group and a ternary group considered by Certaine [9] and Kasner [10]. For many applications of n-ary groups and quasigroups see [11, 12] and [13] respectively. From another side, Hopf algebras [14, 15] and their generalizations [16, 17, 18, 19] play a basic role in the quantum group theory (see e.g. [20, 21, 22]).

In the first part of this paper we reformulate necessary material on ternary semigroups, groups and algebras [13, 11] in the abstract arrow language. Then according to the general scheme [14] using systematic reversing order of arrows, we define ternary bialgebras and Hopf algebras, investigate their properties and present examples.

TERNARY SEMIGROUPS

A non-empty set G with one *ternary* operation $[\ ] : G \times G \times G \to G$ is called a *ternary groupoid* and is denoted by $(G, [\ ])$ or $\left(G, m^{(3)}\right)$. We will present some results using second notation, because it allows to reverse arrows in the most clear way. In proofs we will mostly use the first notation due to convenience and for short.

If on G there is a binary operation $\odot$ (or $m^{(2)}$) such that $[xyz] = (x \odot y) \odot z$ or

$$m^{(3)} = m^{(3)}_{der} = m^{(2)} \circ \left(m^{(2)} \times \mathrm{id}\right) \tag{1}$$

for all $x, y, z \in G$, then we say that $[\ ]$ or $m^{(3)}_{der}$ is *derived* from $\odot$ or $m^{(2)}$and denote this fact by $(G, [\ ]) = der(G, \odot)$. If

$$[xyz] = ((x \odot y) \odot z) \odot b$$

holds for all $x, y, z \in G$ and some fixed $b \in G$, then a groupoid $(G, [\ ]$ is b-*derived* from $(G, \odot)$. In this case we write $(G, [\ ]) = der_b(G, \odot)$ (cf. [23, 24]).

We say that $(G, [\ ]$ is a *ternary semigroup* if the operation $[\ ]$ is *associative*, i.e. if

$$[[xyz]\,uv] = [x\,[yzu]\,v] = [xy\,[zuv]] \tag{2}$$

holds for all $x, y, z, u, v \in G$, or

$$m^{(3)} \circ \left(m^{(3)} \times \mathrm{id} \times \mathrm{id}\right) = m^{(3)} \circ \left(\mathrm{id} \times m^{(3)} \times \mathrm{id}\right) = m^{(3)} \circ \left(\mathrm{id} \times \mathrm{id} \times m^{(3)}\right) \tag{3}$$

Obviously, a ternary operation $m^{(3)}_{der}$ derived from a binary associative operation $m^{(2)}$ is also associative in the above sense, but a ternary groupoid $(G, [\ ])$ b-derived (b is a cansellative element) from a semigroup $(G, \odot)$ is a ternary semigroup if and only if b lies in the center of $(G, \odot)$.

Fixing in a ternary operation $m^{(3)}$ one element a we obtain a binary operation $m_a^{(2)}$. A binary groupoid $(G, \odot)$ or $\left(G, m_a^{(2)}\right)$, where $x \odot y = [xay]$ or

$$m_a^{(2)} = m^{(3)} \circ (\mathrm{id} \times a \times \mathrm{id}) \tag{4}$$

for some fixed $a \in G$ is called a *retract* of $(G, [\ ])$ and is denoted by $ret_a(G, [\ ])$. In some special cases described in [23, 24] we have $(G, \odot) = ret_a(der_b(G, \odot))$ or $(G, \odot) = der_c(ret_d(G, [\ ]))$.

Lemma 1. *If in the ternary semigroup $(G, [\ ])$ or $\left(G, m^{(3)}\right)$ there exists an element e such that for all $y \in G$ we have $[eye] = y$, then this semigroup is derived from the binary semigroup $\left(G, m_e^{(2)}\right)$, where*

$$m_e^{(2)} = m^{(3)} \circ (\mathrm{id} \times e \times \mathrm{id}) \tag{5}$$

In this case $(G, [\ ]) = der(ret_e(G, [\ ])$.

Proof. Indeed, if we put $x \circledast y = [xey]$, then $(x \circledast y) \circledast z = [[xey]ez] = [x[eye]z] = [xyz]$ and $x \circledast (y \circledast z) = [xe[yez]] = [x[eye]z] = [xyz]$, which completes the proof. □

The same ternary semigroup $\left(G, m^{(3)}\right)$ can be derived from two different semigroups $(G, \circledast)$ or $\left(G, m_e^{(2)}\right)$ and $(G, \diamond)$ or $\left(G, m_a^{(2)}\right)$. Indeed, if in G there exists $a \neq e$ such that $[aya] = y$ for all $y \in G$, then by the same argumentation we obtain $[xyz] = x \diamond y \diamond z$ for $x \diamond y = [xay]$. In this case for $\varphi(x) = x \diamond e = [xae]$ we have

$$x \circledast y = [xey] = [x[aea]y] = [[xae]ay] = (x \diamond e) \diamond y = \varphi(x) \diamond y$$

and

$$\varphi(x \circledast y) = [[xey]ae] = [[x[aea]y]ae] = [[xae]a[yae]] = \varphi(x) \diamond \varphi(y).$$

Thus φ is a binary homomorphism such that $\varphi(e) = a$. Moreover for $\psi(x) = [eax]$ we have

$$\psi(\varphi(x)) = [ea[xae]] = [e[axa]e] = x,$$
$$\varphi(\psi(x)) = [[eax]ae] = [e[axa]e] = x$$

and

$$\psi(x \diamond y) = [ea[xay]] = [ea[x[eae]y]] = [[eax]e[aey]] = \psi(x) \circledast \psi(y).$$

Hence semigroups $(G, \circledast)$ and $(G, \diamond)$ are isomorphic.

Definition 2. An element $e \in G$ is called a *middle identity* or a *middle neutral element* of $(G, [\ ])$ if for all $x \in G$ we have $[exe] = x$ or

$$m^{(3)} \circ (e \times \mathrm{id} \times e) = \mathrm{id}\,. \tag{6}$$

An element $e \in G$ satisfying the identity $[eex] = x$ or

$$m^{(3)} \circ (e \times e \times \mathrm{id}) = \mathrm{id}\,. \tag{7}$$

is called a *left identity* or a *left neutral element* of $(G, [\ ])$. By the symmetry we define a *right identity*. An element which is a left, middle and right identity is called a *ternary identity* (briefly: identity).

There are ternary semigroups without left (middle, right) neutral elements, but there are also ternary semigroups in which all elements are identities [11, 27].

Example 3. In ternary semigroups derived from the symmetric group S_3 all elements of order 2 are left and right (but no middle) identities.

Example 4. In ternary semigroup derived from Boolean group all elements are ternary identities, but ternary semigroup 1-derived from the additive group $\mathbb{Z}_4$ has no left (right, middle) identities.

Lemma 5. *For any ternary semigroup $(G, [\ ])$ with a left (right) identity there exists a binary semigroup $(G, \odot)$ and its endomorphism μ such that*

$$[xyz] = x \odot \mu(y) \odot z$$

for all $x, y, z \in G$.

Proof. Let e be a left identity of $(G, [\ ])$. It is not difficult to see that the operation $x \odot y = [xey]$ is associative. Moreover, for $\mu(x) = [exe]$, we have

$$\mu(x) \odot \mu(y) = [[exe]e[eye]] = [[exe][eey]e] = [e[xey]e] = \mu(x \odot y)$$

and

$$[xyz] = [x[eey][eez]] = [[xe[eye]]ez] = x \odot \mu(y) \odot z.$$

The case of right identity the proof is analogous. □

Definition 6. We say that a ternary groupoid $(G, [\ ])$ is:
a *left cancellative* if $[abx] = [aby] \Longrightarrow x = y$,
a *middle cancellative* if $[axb] = [ayb] \Longrightarrow x = y$,
a *right cancellative* if $[xab] = [yab] \Longrightarrow x = y$
holds for all $a, b \in G$.

A ternary groupoid which is left, middle and right cancellative is called *cancellative*.

Theorem 7. *A ternary groupoid is cancellative if and only if it is a middle cancellative, or equivalently, if and only if it is a left and right cancellative.*

Proof. Assume that a ternary semigroup $(G, [\ ])$ is a middle cancellative and $[xab] = [yab]$. Then $[ab[xab]] = [ab[yab]]$ and in the consequence $[a[bxa]b] = [a[bya]b]$ which implies $x = y$.

Conversely if $(G, [\ ])$ is a left and right cancellative and $[axb] = [ayb]$ then $[a[axb]b] = [a[ayb]b]$ and $[[aax]bb] = [[aay]bb]$ which gives $x = y$. □

The above theorem is a consequence of the general result proved in [25].

Definition 8. A ternary groupoid $(G, [\ ])$ is *semicommutative* if $[xyz] = [zyx]$ for all $x, y, z \in G$. If the value of $[xyz]$ is independent on the permutation of elements x, y, z, viz.

$$[x_1 x_2 x_3] = \left[x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)}\right] \tag{8}$$

or $m^{(3)} = m^{(3)} \circ \sigma$, then $(G, [\ ])$ is a *commutative* ternary groupoid. If σ is fixed, then a ternary groupoid satisfying (8) is called σ-commutative.

The group S_3 is generated by two transpositions; (12) and (23). This means that $(G, [\ ])$ is commutative if and only if $[xyz] = [yxz] = [xzy]$ holds for all $x, y, z \in G$.

As a simple consequence of Theorem 7 from [26] we obtain

Corollary 9. If in a ternary semigroup $(G, [\ ])$ satisfying the identity $[xyz] = [yxz]$ there are a, b such that $[axb] = x$ for all $x \in G$, then $(G, [\ ])$ is commutative.

Proof. According to the above remark it is sufficient to prove that $[xyz] = [xzy]$. We have

$$[xyz] = [a[xyz]b] = [ax[yzb]] = [ax[zyb]] = [a[xzy]b] = [xzy].$$

□

Mediality in the binary case $(x \odot y) \odot (z \odot u) = (x \odot z) \odot (y \odot u)$ can be presented as a matrix $\begin{matrix} & \Downarrow & \Downarrow \\ \Rightarrow & x & y \\ \Rightarrow & z & u \end{matrix}$ and for groups coincides with commutativity.

Definition 10. A ternary groupoid $(G, [\ ])$ is *medial* if it satisfies the identity

$$[[x_{11}x_{12}x_{13}][x_{21}x_{22}x_{23}][x_{31}x_{32}x_{33}]] = [[x_{11}x_{21}x_{31}][x_{12}x_{22}x_{32}][x_{13}x_{23}x_{33}]]$$

or

$$m^{(3)} \circ \left(m^{(3)} \times m^{(3)} \times m^{(3)}\right) = m^{(3)} \circ \left(m^{(3)} \times m^{(3)} \times m^{(3)}\right) \circ \sigma_{medial}, \tag{9}$$

where $\sigma_{medial} = \binom{123456789}{147258369} \in S_9$.

It is not difficult to see that a semicommutative ternary semigroup is medial.

An element x such that $[xxx] = x$ is called an *idempotent*. A groupoid in which all elements are idempotents is called an *idempotent groupoid*. A left (right, middle) identity is an idempotent.

Definition 11. A ternary semigroup $(G, [\ ])$ is a *ternary group* if for all $a, b, c \in G$ there are $x, y, z \in G$ such that

$$[xab] = [ayb] = [abz] = c.$$

One can prove [27] that elements x, y, z are uniquely determined. Moreover, according to the suggestion of [27] one can prove (cf. [28]) that in the above definition, under the assumption of the associativity, it suffices only to postulate the existence of a solution of $[ayb] = c$, or equivalently, of $[xab] = [abz] = c$.

In a ternary group the equation $[xxz] = x$ has a unique solution which is denoted by $z = \overline{x}$ and called *skew element* (cf. [8]), or equivalently

$$m^{(3)} \circ (\mathrm{id} \times \mathrm{id} \times \bar{\cdot}) \circ D^{(3)} = \mathrm{id},$$

where $D^{(3)}(x) = (x, x, x)$ is a ternary diagonal map. As a consequence of results obtained in [8] we have

Theorem 12. *In any ternary group $(G, [\ ])$ for all $x, y, z \in G$ the following relations take place*

$$[xx\,\overline{x}] = [x\,\overline{x}\,x] = [\,\overline{x}\,xx] = x,$$
$$[yx\,\overline{x}] = [y\,\overline{x}\,x] = [x\,\overline{x}\,y] = [\,\overline{x}\,xy] = y,$$
$$\overline{[xyz]} = [\,\overline{z}\,\overline{y}\,\overline{x}],$$
$$\overline{\overline{x}} = x$$

Since in an idempotent ternary group $\overline{x} = x$ for all x, an idempotent ternary group is semicommutative. From results obtained in [28] (see also [26]) for $n = 3$ we obtain

Theorem 13. *A ternary semigroup $(G, [\ ])$ with a unary operation $^{-} : x \to \overline{x}$ is a ternary group if and only if it satisfies identities*

$$[yx\,\overline{x}\,] = [x\,\overline{x}\,y] = y,$$

or

$$m^{(3)} \circ (\mathrm{id} \times \bar{\cdot} \times \mathrm{id}) \circ \left(D^{(2)} \times \mathrm{id}\right) = \mathrm{Pr}_2,$$
$$m^{(3)} \circ (\mathrm{id} \times \mathrm{id} \times \bar{\cdot}) \circ \left(\mathrm{id} \times D^{(2)}\right) = \mathrm{Pr}_1,$$

where $D^{(2)}(x) = (x, x)$ and $\mathrm{Pr}_1(x, y) = x$, $\mathrm{Pr}_2(x, y) = y$.

Corollary 14. A ternary semigroup $(G, [\ ])$ is an idempotent ternary group if and only if it satisfies identities

$$[yxx] = [xxy] = y$$

A ternary group with an identity is derived from a binary group.

REMARK. The set $A_3 \subset S_3$ with ternary operation $[\ ]$ defined as composition of three permutations is an example of a ternary group which is not derived from any group (all groups with three elements are commutative and isomorphic to $\mathbb{Z}_3$).

From results proved in [26] follows

Theorem 15. *A ternary group $(G, [\ ])$ satisfying the identity*

$$[xy\overline{x}] = y$$

or

$$[\overline{x}yx] = y$$

is commutative.

Theorem 16 (Gluskin-Hosszú). *For a ternary group $(G, [\ ])$ there exists a binary group $(G, \circledast)$, its automorphism φ and fixed element $b \in G$ such that*

$$[xyz] = x \circledast \varphi(y) \circledast \varphi^2(z) \circledast b. \tag{10}$$

Proof. Let $a \in G$ be fixed. Then the binary operation $x \circledast y = [xay]$ is associative, because

$$(x \circledast y) \circledast z = [[xay]az] = [xa[yaz]] = x \circledast (y \circledast z).$$

An element $\overline{a}$ is its identity. x^{-1} (in $(G, \circledast)$ is $[\overline{a},\ \overline{x}\,\overline{a}]$. $\varphi(x) = [\overline{a}xa]$ is an automorphism of $(G, \circledast)$. The easy calculation proves that the above formula holds for $b = [\overline{a}\,\overline{a}\,\overline{a}\,]$. (see [29]). □

One can prove that the group $(G, \circledast)$ is unique up to isomorphism. From the proof of Theorem 3 in [30] it follows that any medial ternary group satisfies the identity

$$\overline{[xyz]} = [\overline{x}\,\overline{y}\,\overline{z}\,],$$

which together with our previous results shows that in such groups we have

$$\overline{[xyz]} = \overline{[xyz]}.$$

But $\overline{\overline{x}} = x$. Hence, any medial ternary group is semicommutative. Thus any retract of such group is a commutative group. Moreover, for φ from the proof of the above theorem we have

$$\varphi(\varphi(x)) = [\overline{a}\,[\overline{a}xa]a] = [\overline{a}\,a\,[x\overline{a}\,a]] = x$$

Corollary 17. Any medial ternary group $(G, [\ ])$ has the form

$$[xyz] = x \odot \varphi(y) \odot z \odot b,$$

where $(G, \odot)$ is a commutative group, φ its automorphism such that $\varphi^2 = \mathrm{id}$ and $b \in G$ is fixed.

Corollary 18. A ternary group is medial if and only if it is semicommutative.

Corollary 19. A ternary group is semicommutative (medial) if and only if $[xay] = [yax]$ holds for all $x, y \in G$ and some fixed $a \in G$.

Corollary 20. A commutative ternary group is b-derived from some commutative group.

Indeed, $\varphi(x) = [\overline{a}\,xa] = [xa\,\overline{a}] = x$.

Theorem 21 (Post). *For any ternary group $(G, [\ ])$ there exists a binary group $(G^*, \circledast)$ and $H \lhd G^*$, such that $G^* \diagup H \simeq \mathbb{Z}_2$ and*

$$[xyz] = x \circledast y \circledast z$$

for all $x, y, z \in G$.

Proof. Let c be a fixed element in G and let $G^* = G \times \mathbb{Z}_2$. In G^* we define binary operation $\circledast$ putting

$$(x, 0) \circledast (y, 0) = ([xy\overline{c}], 1)$$

$$(x, 0) \circledast (y, 1) = ([xyc], 0)$$

$$(x, 1) \circledast (y, 0) = ([xcy], 0)$$

$$(x, 1) \circledast (y, 1) = ([xcy], 1).$$

It is not difficult to see that this operation is associative and $(\overline{c}, 1)$ is its neutral element. The inverse element (in G^*) has the form:

$$(x, 0)^{-1} = (\overline{x}, 0)$$

$$(x, 1)^{-1} = ([\overline{c}\,\overline{x}\,\overline{c}], 1)$$

Thus G^* is a group such that $H = \{(x, 1) : x \in G\} \lhd G^*$. Obviously the set G can be identified with $G \times \{0\}$ and

$$[xyz] = ((x, 0) \circledast (y, 0)) \circledast (z, 0) = ([xy\overline{c}], 1) \circledast (z, 0) =$$

$$([[xy\overline{c}]cz], 0) = ([xy[\overline{c}cz]], 0) = ([xyz], 0)$$

which completes the proof. □

Proposition 22. *All retracts of a ternary group are isomorphic*

$$ret_a\,(G, [\ ]) \simeq ret_b\,(G, [\ ])\,.$$

Definition 23. Autodistributivity in a ternary group is

$$[[xyz]\,ab] = [[xab]\,[yab]\,[zab]]\,.$$

Let us consider ternary algebras. Take 2 ternary operations $\{\ ,\ ,\ \}$ and $[\ ,\ ,\ ]$, then distributivity is

$$\{[xyz]\,ab\} = [\{xab\}\,\{yab\}\,\{zab\}]\,,$$

and additivity is

$$[\{x+z\}\,ab] = [xab] + [zab]\,.$$

Definition 24. Ternary algebra is a pair $\left(A, m^{(3)}\right)$, where A is a linear space and $m^{(3)}$ is a linear map

$$m^{(3)} : A \otimes A \otimes A \to A$$

called *ternary multiplication* which is associative

$$m^{(3)} \circ \left(m^{(3)} \otimes \mathrm{id} \otimes \mathrm{id}\right) = m^{(3)} \circ \left(\mathrm{id} \otimes m^{(3)} \otimes \mathrm{id}\right) = m^{(3)} \circ \left(\mathrm{id} \otimes \mathrm{id} \otimes m^{(3)}\right).$$

TERNARY COALGEBRAS

Let C is a linear space over a field $\mathbb{K}$.

Definition 25. Ternary comultiplication $\Delta^{(3)}$ is a linear map over a fixed field $\mathbb{K}$

$$\Delta^{(3)} : C \to C \otimes C \otimes C.$$

For convenience we also use the short-cut Sweedler-type notations [15]

$$\Delta^{(3)}(a) = \sum_{i=1}^{n} a_i' \otimes a_i'' \otimes a_i''' = a_{(1)} \otimes a_{(2)} \otimes a_{(3)}. \tag{11}$$

Now we discuss various properties of $\Delta^{(3)}$ which are in sense (dual) analog of the above ternary multiplication $m^{(3)}$. First consider different possible types of ternary coassociativity.

1. *Standard* ternary coassociativity

$$(\Delta^{(3)} \otimes \mathrm{id} \otimes \mathrm{id}) \circ \Delta^{(3)} = (\mathrm{id} \otimes \Delta^{(3)} \otimes \mathrm{id}) \circ \Delta^{(3)} = (\mathrm{id} \otimes \mathrm{id} \otimes \Delta^{(3)}) \circ \Delta^{(3)}, \tag{12}$$

In the Sweedler notations

$$\left(a_{(1)}\right)_{(1)} \otimes \left(a_{(1)}\right)_{(2)} \otimes \left(a_{(1)}\right)_{(3)} \otimes a_{(2)} \otimes a_{(3)} = a_{(1)} \otimes \left(a_{(2)}\right)_{(1)} \otimes \left(a_{(2)}\right)_{(2)} \otimes \left(a_{(2)}\right)_{(3)} \otimes a_{(3)}$$
$$= a_{(1)} \otimes a_{(2)} \otimes \left(a_{(3)}\right)_{(1)} \otimes \left(a_{(3)}\right)_{(2)} \otimes \left(a_{(3)}\right)_{(3)} \equiv a_{(1)} \otimes a_{(2)} \otimes \otimes a_{(3)} \otimes a_{(4)} \otimes a_{(5)}.$$

2. *Nonstandard* ternary Σ-coassociativity (Gluskin-type — positional operatives)

$$(\Delta^{(3)} \otimes \mathrm{id} \otimes \mathrm{id}) \circ \Delta^{(3)} = (\mathrm{id} \otimes \left(\sigma \circ \Delta^{(3)}\right) \otimes \mathrm{id}) \circ \Delta^{(3)},$$

where

$$\sigma \circ \Delta^{(3)}(a) = \Delta^{(3)}_{\sigma}(a) = a_{(\sigma(1))} \otimes a_{(\sigma(2))} \otimes a_{(\sigma(3))}$$

and $\sigma \in \Sigma \subset S_3$.

3. *Permutational* ternary coassociativity

$$(\Delta^{(3)} \otimes \mathrm{id} \otimes \mathrm{id}) \circ \Delta^{(3)} = \pi \circ (\mathrm{id} \otimes \Delta^{(3)} \otimes \mathrm{id}) \circ \Delta^{(3)},$$

where $\pi \in \Pi \subset S_5$.

Ternary comediality is

$$\left(\Delta^{(3)} \otimes \Delta^{(3)} \otimes \Delta^{(3)}\right) \circ \Delta^{(3)} = \sigma_{medial} \circ \left(\Delta^{(3)} \otimes \Delta^{(3)} \otimes \Delta^{(3)}\right) \circ \Delta^{(3)},$$

where σ_{medial} is defined in (9).

Ternary counit is defined as a map $\varepsilon^{(3)} : C \to \mathbb{K}$. In general, $\varepsilon^{(3)} \neq \varepsilon^{(2)}$ satisfying one of the conditions below. If $\Delta^{(3)}$ is derived, then maybe $\varepsilon^{(3)} = \varepsilon^{(2)}$, but another counits may exist.

Example 26. Define $[xyz] = (x + y + z)\,|_{\mathrm{mod}\ 2}$ for $x, y, z \in \mathbb{Z}_2$. It is seen that here there are 2 ternary counits $\varepsilon^{(3)} = 0, 1$.

There are 3 types of ternary counits:

1. Standard (strong) ternary counit

$$(\varepsilon^{(3)} \otimes \varepsilon^{(3)} \otimes \mathrm{id}) \circ \Delta^{(3)} = (\varepsilon^{(3)} \otimes \mathrm{id} \otimes \varepsilon^{(3)}) \circ \Delta^{(3)} = (\mathrm{id} \otimes \varepsilon^{(3)} \otimes \varepsilon^{(3)}) \circ \Delta^{(3)} = \mathrm{id}, \tag{13}$$

2. Two sequensional (polyadic) counits $\varepsilon_1^{(3)}$ and $\varepsilon_2^{(3)}$

$$(\varepsilon_1^{(3)} \otimes \varepsilon_2^{(3)} \otimes \mathrm{id}) \circ \Delta = (\varepsilon_1^{(3)} \otimes \mathrm{id} \otimes \varepsilon_2^{(3)}) \circ \Delta = (\mathrm{id} \otimes \varepsilon_1^{(3)} \otimes \varepsilon_2^{(3)}) \circ \Delta = \mathrm{id}, \tag{14}$$

3. Four long ternary counits $\varepsilon_1^{(3)}$–$\varepsilon_4^{(3)}$ satisfying

$$\left(\left(\mathrm{id} \otimes \varepsilon_3^{(3)} \otimes \varepsilon_4^{(3)}\right) \circ \Delta^{(3)} \circ \left((\mathrm{id} \otimes \varepsilon_1^{(3)} \otimes \varepsilon_2^{(3)}) \circ \Delta^{(3)}\right)\right) = \mathrm{id} \tag{15}$$

By analogy with (8) *σ-cocommutativity* is defined as $\sigma \circ \Delta^{(3)} = \Delta^{(3)}$.

Definition 27. Ternary coalgebra is a pair $\left(C, \Delta^{(3)}\right)$, where C is a linear space and $\Delta^{(3)}$ is a ternary comultiplication which is coassociative in one of the above senses.

We will consider below only first standard type of associativity (12).
Let $\left(A, m^{(3)}\right)$ is a ternary algebra and $\left(C, \Delta^{(3)}\right)$ is a ternary coalgebra and $f, g, h \in \mathrm{Hom}_{\mathbb{K}}(C, A)$.

Definition 28. Ternary convolution product is

$$[f, g, h]_* = m^{(3)} \circ (f \otimes g \otimes h) \circ \Delta^{(3)} \tag{16}$$

or $[f, g, h]_*(a) = \left[f\left(a_{(1)}\right) g\left(a_{(2)}\right) h\left(a_{(3)}\right)\right]$.

Definition 29. Ternary coalgebra is called *derived*, if there exists a binary (usual, see e.g. [14, 15]) coalgebra $\Delta^{(2)} : C \to C \otimes C$ such that (cf. 1))

$$\Delta_{der}^{(3)} = \left(\mathrm{id} \otimes \Delta^{(2)}\right) \otimes \Delta^{(2)}. \tag{17}$$

The derived ternary and n-ary coalgebras were considered e.g. in [31] and [32] respectively.

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Definition 30. Ternary bialgebra B is triple $\left(B, m^{(3)}, \Delta^{(3)}\right)$ for which $\left(B, m^{(3)}\right)$ is a ternary algebra and $\left(B, \Delta^{(3)}\right)$ is a ternary coalgebra and

$$\Delta^{(3)} \circ m^{(3)} = m^{(3)} \circ \Delta^{(3)} \tag{18}$$

One can distinguish four kinds of ternary bialgebrs with respect to a ”being derived” property”:

1. Δ-derived ternary bialgebra

$$\Delta^{(3)} = \Delta_{der}^{(3)} = \left(\mathrm{id} \otimes \Delta^{(2)}\right) \circ \Delta^{(2)}$$

2. m-derived ternary bialgebra

$$m_{der}^{(3)} = m_{der}^{(3)} = m^{(2)} \circ \left(m^{(2)} \otimes \mathrm{id}\right)$$

3. Derived ternary bialgebra is simultaneously m-derived and Δ-derived ternary bialgebra.

4. Full ternary bialgebra is not derived.

Now we define possible types of ternary antipodes using analogy with binary coalgebras.

Definition 31. *Skew ternary antipod* is

$$\begin{aligned} m^{(3)} \circ (S_{skew}^{(3)} \otimes \mathrm{id} \otimes \mathrm{id}) \circ \Delta^{(3)} &= m^{(3)} \circ (\mathrm{id} \otimes S_{skew}^{(3)} \otimes \mathrm{id}) \circ \Delta^{(3)} \\ &= m^{(3)} \circ (\mathrm{id} \otimes \mathrm{id} \otimes S_{skew}^{(3)}) \circ \Delta^{(3)} = \mathrm{id} \end{aligned}$$

or in terms of the ternary convolution product (16)

$$\left[S_{skew}^{(3)}, \mathrm{id}, \mathrm{id}\right]_* = \left[\mathrm{id}, S_{skew}^{(3)}, \mathrm{id}\right]_* = \left[\mathrm{id}, \mathrm{id}, S_{skew}^{(3)}\right]_* = \mathrm{id}\,.$$

Definition 32. *Strong ternary antipod* is

$$\left(m^{(2)} \otimes \mathrm{id}\right) \circ (\mathrm{id} \otimes S^{(3)}_{strong} \otimes \mathrm{id}) \circ \Delta^{(3)} = 1 \otimes \mathrm{id},$$
$$\left(\mathrm{id} \otimes m^{(2)}\right) \circ (\mathrm{id} \otimes \mathrm{id} \otimes S^{(3)}_{strong}) \circ \Delta^{(3)} = \mathrm{id} \otimes 1,$$

where 1 is a unit of algebra.

Definition 33. Ternary coalgebra is derived, if $\Delta^{(3)}$ is derived.

Lemma 34. *If in a ternary coalgebra* $\left(C, \Delta^{(3)}\right)$ *there exists a linear map* $\varepsilon^{(3)} : C \to \mathbb{K}$ *satisfying*

$$\left(\varepsilon^{(3)} \otimes \mathrm{id} \otimes \varepsilon^{(3)}\right) \circ \Delta^{(3)} = \mathrm{id}, \tag{19}$$

then $\exists \Delta^{(2)}$ *such that*

$$\Delta^{(3)} = \Delta^{(3)}_{der} = \left(\mathrm{id} \otimes \Delta^{(2)}\right) \otimes \Delta^{(2)}$$

Definition 35. If in ternary coalgebra

$$\Delta^{(3)} \circ S = \tau_{13} \circ (S \otimes S \otimes S) \circ \Delta^{(3)},$$

where $\tau_{13} = \binom{123}{321}$, then it is called *skew-involutive*.

Definition 36. Ternary Hopf algebra is a ternary bialgebra with a ternary antipod of the corresponding type, i.e. $\left(H, m^{(3)}, e^{(3)}, \Delta^{(3)}, S^{(3)}\right)$.

REMARK. There are 8 types of associative ternary Hopf algebras and 4 types of medial Hopf algebras. Also it can happen that there are several ternary units $e^{(3)}_i$ and several ternary counits $\varepsilon^{(3)}_i$ (see (13)–(15)), which makes number of possible ternary Hopf algebras enormous.

Theorem 37. *For any a ternary Hopf algebra there exists a binary Hopf algebra, automorphism* ϕ *and a linear map* λ, *such that*

$$\Delta^{(3)} = (\mathrm{id} \otimes \phi \otimes \mathrm{id}) \circ \left(\Delta^{(2)} \otimes \mathrm{id}\right) \tag{20}$$

Proof. The binary coproduct is $\Delta^{(2)} = (\mathrm{id} \otimes \lambda \otimes \mathrm{id}) \circ \Delta^{(3)}$ and $\Delta^{(3)} = (\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \lambda) \circ \left(\Delta^{(2)} \otimes \Delta^{(2)}\right) \circ \Delta^{(2)}$. □

The co-analog of the Post Theorem 21 is

Theorem 38. *For any ternary Hopf algebra* $\left(H, \Delta^{(3)}\right)$ *there exists a binary Hopf algebra* $\left(H^*, \Delta^{(2)}\right)$*and* $\Delta^{(3)} = \Delta^{(3)}_{der}|_H$, *such that* $H \diagup H^* \simeq k(\mathbb{Z}_2)$ *and*

$$(\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id}) \circ \Delta^{(3)} = \left(\mathrm{id} \otimes \Delta^{(2)}\right) \circ \Delta^{(2)}. \tag{21}$$

EXAMPLES

Example 39. Ternary dual pair $k(G)$ (push-forward) and $\mathcal{F}(G)$ (pull-back) which are related by $k^*(G) \cong \mathcal{F}(G)$. Here $k(G) = span(G)$ is a ternary group (G has a ternary product $[\ ]_G$ or $m^{(3)}_G$) algebra over a field k. If $u \in k(G)$ ($u = u^i x_i, x_i \in G$), then $[uvw]_k = u^i v^j w^l [x_i x_j x_l]_G$ is associative, and so $(k(G), [\ ]_k)$ becomes a ternary algebra. Define a ternary coproduct $\Delta^{(3)}_k : k(G) \to k(G) \otimes k(G) \otimes k(G)$ by $\Delta^{(3)}_k(u) = u^i x_i \otimes x_i \otimes x_i$ (derive and associative), then $\Delta^{(3)}_k([uvw]_k) = \left[\Delta^{(3)}_k(u) \Delta^{(3)}_k(v) \Delta^{(3)}_k(w)\right]_k$, and $k(G)$ is a ternary bialgebra. If we define a ternary antipod by $S^{(3)}_k = u^i \bar{x}_i$, where $\bar{x}_i$ is a skew element of x_i, then $k(G)$ becomes a ternary Hopf algebra. In the dual case of functions $\mathcal{F}(G) : \{\varphi : G \to k\}$ a ternary product $[\ ]_{\mathcal{F}}$ or $m^{(3)}_{\mathcal{F}}$ (derive and associative) acts on $\psi(x, y, z)$ as $\left(m^{(3)}_{\mathcal{F}} \psi\right)(x) = \psi(x, x, x)$, and so $\mathcal{F}(G)$ is a ternary algebra. Let $\mathcal{F}(G) \otimes \mathcal{F}(G) \otimes \mathcal{F}(G) \cong \mathcal{F}(G \times G \times G)$, then we define a ternary coproduct $\Delta^{(3)}_{\mathcal{F}} : \mathcal{F}(G) \to \mathcal{F}(G) \otimes \mathcal{F}(G) \otimes \mathcal{F}(G)$ as $\left(\Delta^{(3)}_{\mathcal{F}} \varphi\right)(x, y, z) = \varphi([xyz]_{\mathcal{F}})$, which is derive and associative. Thus we can obtain $\Delta^{(3)}_{\mathcal{F}}([\varphi_1 \varphi_2 \varphi_3]_{\mathcal{F}}) = \left[\Delta^{(3)}_{\mathcal{F}}(\varphi_1) \Delta^{(3)}_{\mathcal{F}}(\varphi_2) \Delta^{(3)}_{\mathcal{F}}(\varphi_3)\right]_{\mathcal{F}}$, and therefore $\mathcal{F}(G)$ is a ternary bialgebra. If we define a ternary antipod by $S^{(3)}_{\mathcal{F}}(\varphi) = \varphi(\bar{x})$, where $\bar{x}$ is a skew element of x, then $\mathcal{F}(G)$ becomes a ternary Hopf algebra.

Example 40. Matrix representation. Possible non-derived matrix representations of the ternary product can be done only by four-rank tensors: twicely covariant and twicely contravariant and allow only 2 possibilities $A^{oi}_{jk} B^{jl}_{oo} C^{ko}_{il}$ and $A^{ij}_{ok} B^{ol}_{io} C^{ko}_{il}$ (where o is any index).

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ОСНОВНЫЕ КОНЦЕПЦИИ ТЕРНАРНЫХ АЛГЕБР ХОПФА

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Теория тернарных полугрупп, групп и алгебр переформулирована на абстрактном языке стрелок. Используя анзац обращения стрелок, определяются тернарные коумножение, биалгебры и алгебры Хопфа, и исследуются их свойства. Основная характеристика “быть бинарно выводимым” исследуется детально. Сформулирован ко-аналог теоремы Поста. Показано, что имеется 3 типа тернарной коассоциативности, 3 типа тернарных коединиц и 2 типа тернарных антиподов. Представлены примеры.

КЛЮЧЕВЫЕ СЛОВА: тернарная операция, выводимая операция, косой элемент, антипод, тернарная алгебра Хопфа