Weak Hopf algebras as generalizations of Hopf algebras [1] were introduced in [2], where its characterizations and applications were also studied. A $k$-bialgebra $H = (H, \mu, \eta, \Delta, \varepsilon)$ with multiplication $\mu$, unity $\eta$, counity $\varepsilon$, comultiplication $\Delta$, is called a weak Hopf algebra if there exists $T \in \text{Hom}_k(H, H)$ such that

$$id \ast T \ast id = id, \quad T \ast id \ast T = T,$$

where $T$ is called a weak antipode of $H$. One of its aims is to construct some singular solutions of the quantum Yang-Baxter equation (QYBE) [2].

We study here generalization of Hopf algebra $sl_q(2)$ by weakening the invertibility of the generator $K$, i.e. exchanging its invertibility $KK^{-1} = 1$ to the regularity $K\overline{K} = K$. Here we investigate a weak Hopf algebra $wsl_q(2)$ and a $J$-weak Hopf algebra $vsl_q(2)$ as generalizations of $sl_q(2)$ and non-trivial examples of weak Hopf algebras [2]. A quasi-braided weak Hopf algebra $U_q^w$ from $wsl_q(2)$ is constructed whose quasi-$R$-matrix is regular [3].

Let $q \in C$ and $q \neq \pm 1, 0$. The quantum enveloping algebra $U_q = U_q(sl_q(2))$ (see [6]) is generated by four variables (Chevalley generators) $E, F, K, K^{-1}$ with the relations

$K^{-1}K = KK^{-1} = 1$, 
$KEK = q^2E$, 
$KFK = q^2F$, 
$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$.

Now we try to weaken the invertibility of $K$ to regularity, as usually in semigroup theory [4] (see also [5] for higher regularity). It can be done in two different ways.

Define $U_q^w = wsl_q(2)$, which is called a weak quantum algebra, as the algebra generated by the four variables $E_w, F_w, K_w, \overline{K}_w$ with the relations:

$K_w \overline{K}_w = \overline{K}_w K_w$, 
$K_w \overline{K}_w K_w = K_w$, 
$\overline{K}_w K_w \overline{K}_w = \overline{K}_w$, 
$K_w E_w = q^2 E_w K_w$, 
$\overline{K}_w E_w = q^2 E_w \overline{K}_w$, 
$K_w F_w = q^2 F_w K_w$, 
$\overline{K}_w F_w = q^2 F_w \overline{K}_w$, 

$E_w F_w - F_w E_w = \frac{K_w - \overline{K}_w}{q - q^{-1}}$.

Define $U_q^v = vsl_q(2)$, which is called a $J$-weak quantum algebra, as the algebra generated by the four variables $E_v, F_v, K_v, \overline{K}_v$ with the relations ($J_v = K_v \overline{K}_v$):

$K_v \overline{K}_v = \overline{K}_v K_v$, 
$K_v \overline{K}_v K_v = K_v$, 
$\overline{K}_v K_v \overline{K}_v = \overline{K}_v$, 
$K_v E_v \overline{K}_v = q^2 E_v$, 
$K_v F_v \overline{K}_v = q^2 F_v$, 
$E_v J_v F_v - F_v J_v E_v = \frac{K_v - \overline{K}_v}{q - q^{-1}}$.

Let $J_w = K_w \overline{K}_w$. List some useful properties of $J_w$ which will be needed below. Firstly, $J_w^2 = J_w$, which means that $J_w$ is a projector. For any variable $X$, define “$J$-conjugation” as $X_{J_w} = J_w X J_w$, and
the corresponding mapping will be written as $e_w(X) : X \to X_{J_w}$. Note that the mapping $e_w$ is idempotent.

**Proposition** (i) $wsl_q(2)/(J_w - 1) \cong sl_q(2)$; (ii) Quantum algebras $wsl_q(2)$ and $vsl_q(2)$ possess zero divisors, one of which is $(J_{J_w} - 1)$ which annihilates all generators.

**Lemma** (i) The idempotent $J_w$ is in the center of $wsl_q(2)$; (ii) There are unique algebra automorphisms $\omega_w$ and $\omega_v$ (called the weak Cartan automorphisms) of $U^w_q$ and $U^v_q$ respectively such that $\omega_w(K_{w,v}) = \overline{K}_{w,v}$, $\omega_v(K_{w,v}) = K_{w,v}$, $\omega_v(E_{w,v}) = F_{w,v}$, $\omega_v(F_{w,v}) = E_{w,v}$.

Let $R$ be an algebra over $k$ and $R[t]$ be the free left $R$-module consisting of all polynomials of the form $P = \sum_{i=0}^{n} a_i t_i$ with coefficients in $R$. If $a_n \neq 0$, define $\deg(P) = n$; say $\deg(0) = -\infty$. Let $\alpha$ be an algebra morphism of $R$. An $\alpha$-derivation of $R$ is a left (resp. right) $R$-linear endomorphism $\delta$ of $R$ such that $\delta(ab) = \alpha(a)\delta(b) + \delta(a)b$ for all $a, b \in R$. It follows that $\delta(1) = 0$.

**Theorem** (i) Assume that $R[t]$ has an algebra structure such that the natural inclusion of $R$ into $R[t]$ is a morphism of algebras and $\deg(PQ) \leq \deg(P) + \deg(Q)$ for any pair $(P, Q)$ of elements of $R[t]$. Then there exists a unique injective algebra endomorphism $\alpha$ of $R$ and a unique $\alpha$-derivation $\delta$ of $R$ such that $\delta(ab) = \alpha(a)\delta(b) + \delta(a)b$ for all $a \in R$.

(ii) Conversely, given an algebra endomorphism $\alpha$ of $R$ and an $\alpha$-derivation $\delta$ of $R$, there exists a unique algebra structure on $R[t]$ such that the inclusion of $R$ into $R[t]$ is an algebra morphism and $\delta(ta) = \alpha(a)t + \delta(a)$ for all $a \in R$.

It is recognized as a generalization of Theorem I.7.1 in [6]. We call the algebra constructed from $\alpha$ and $\delta$ a weak Ore extension of $R$, denoted as $R_n[t, \alpha, \delta]$. Let $R$ be an algebra, $\alpha$ be an algebra automorphism and $\delta$ be an $\alpha$-derivation of $R$. If $R$ is a left (resp. right) Noetherian, then so is the weak Ore extension $R_n[t, \alpha, \delta]$.

**Theorem** The algebra $wsl_q(2)$ is Noetherian with the basis

$$P_w = \{E_w^iF_w^jK_w^l, E_w^iF_w^j\overline{K}_w^l, E_w^iF_w^jJ_w^l\}$$

where $i, j, l$ are any non-negative integers, $m$ is any positive integer.

The similar theorem can be obtained for $vsl_q(2)$ as well. Define $U_q^{w'}$ as the algebra generated by the five variables $E_w, F_w, K_w, \overline{K}_w, L_w$ with the relations:

$$K_w\overline{K}_w = \overline{K}_wK_w = K_w, \quad K_wE_w = q^2E_wK_w, \quad K_wF_w = q^2F_wK_w, \quad [L_w, E_w] = q(E_wK_w + \overline{K}_wE_w), \quad [L_w, F_w] = -q^{-1}(F_wK_w + \overline{K}_wF_w),$$

$$E_wF_w - F_wE_w = L_w, \quad (q - q^{-1})L_w = (K_w - \overline{K}_w).$$

Then $U_q^{w'}$ is isomorphic to the algebra $U_q^{w''}$ with $\varphi_w$ satisfying $\varphi_w(E_w) = E_w, \varphi_w(F_w) = F_w, \varphi_w(K_w) = K_w, \varphi_w(\overline{K}_w) = \overline{K}_w$. And, the relationship between $U_q^{w''}$ and $U(sl(2))$ is that for $q = 1$, (i) the algebra isomorphism $U(sl(2)) \cong U_q^{w''} / (K_w - 1)$ holds; (ii) there exists an injective algebra morphism $\pi$ from $U_q^{w''}$ to $U(sl(2))[K_w] / (K_3^3 - K_w)$ satisfying $\pi(E_w) = XK_w, \pi(F_w) = Y, \pi(K_w) = K_w, \pi(L) = HK_w$.

For $wsl_q(2)$, define the maps $\Delta_w : wsl_q(2) \to wsl_q(2) \otimes wsl_q(2)$, $\varepsilon_w : wsl_q(2) \to k$ and $T_w : wsl_q(2) \to wsl_q(2)$ satisfying respectively
\[ \Delta_w(E_w) = 1 \otimes E_w + E_w \otimes K_w, \quad \Delta(F_w) = F_w \otimes 1 + \overline{K_w} \otimes F_w, \]
\[ \Delta_w(K_w) = K_w \otimes K_w, \quad \Delta_w(\overline{K_w}) = \overline{K_w} \otimes \overline{K_w}, \]
\[ \varepsilon_w(E_w) = \varepsilon_w(F_w) = 0, \quad \varepsilon_w(K_w) = \varepsilon_w(\overline{K_w}) = 1, \]
\[ T_w(E_w) = -E_w \overline{K_w}, \quad T_w(F_w) = -K_w F_w, \]
\[ T(K_w) = \overline{K_w}, \quad T_w(\overline{K_w}) = K_w. \]

**Proposition** The relations above endow \( wsl_q(2) \) with a bialgebra structure possessing a weak antipode \( T_w \).

**Proposition** \( T_w^2 \) is an inner endomorphism of the algebra \( wsl_q(2) \) satisfying \( T_w^2(X) = K_w X \overline{K_w} \) for any \( X \in wsl_q(2) \).

It can be shown that about the operations above, it is not possible \( wsl_q(2) \) would possess an antipode \( S \) so as to become a Hopf algebra. Hence, \( wsl_q(2) \) is an example for a non-commutative and non-cocommutative weak Hopf algebra which is *not* a Hopf algebra. For \( J \)-weak quantum algebra \( vsl_q(2) \), a thorough analysis gives the following nontrivial definitions
\[ \Delta_v(E_v) = J_v \otimes E_v J_v + J_v E_v J_v \otimes K_v, \]
\[ \Delta_v(F_v) = J_v F_v J_v \otimes J_v + \overline{K_v} \otimes J_v F_v J_v, \]
\[ \Delta_v(K_v) = K_v \otimes K_v, \quad \Delta_v(\overline{K_v}) = \overline{K_v} \otimes \overline{K_v}, \]
\[ \varepsilon_v(E_v) = \varepsilon_v(F_v) = 0, \quad \varepsilon_v(K_v) = \varepsilon_v(\overline{K_v}) = 1, \]
\[ T_v(E_v) = -J_v E_v \overline{K_v}, \quad T_v(F_v) = -K_v F_v J_v, \]
\[ T_v(K_v) = \overline{K_v}, \quad T_v(\overline{K_v}) = K_v. \]

These relations endow \( vsl_q(2) \) with a bialgebra structure with a \( J \)-weak antipode \( T_v \), i.e. satisfying the regularity conditions
\[ (e_v \ast^\# T_v \ast^\# e_v)(X) = e_v(X), \quad (T_v \ast^\# e_v \ast^\# T_v)(X) = T_v(X), \]
for any \( X \) in \( vsl_q(2) \). From the difference between id and \( e_v \), \( vsl_q(2) \) is not a weak Hopf algebra in the definition of [2]. So we will call it \( J \)-weak Hopf algebra and \( T_v \) a \( J \)-weak antipode. Remark the variable \( e_v \) can be treated as \( n = 2 \) example of the "tower identity" \( e_v^{(n)} \) introduced for semisupermanifolds or the "obstructor" \( e_v^{(n)} \) for general mappings, categories and Yang-Baxter equation in [5].

Now, we discuss the set \( G(wsl_q(2)) \) of all group-like elements of \( wsl_q(2) \). The concept of inverse monoid can be found in [4].

**Proposition** The set of all group-like elements \( G(wsl_q(2)) = \{ J^{(i,j)} = K_w^{i} \overline{K_w}^{j} : i, j \ \text{run over all non-negative integers} \} \), which forms a regular monoid under the multiplication of \( wsl_q(2) \).

For \( vsl_q(2) \) we can get a similar statement.

**Theorem** \( wsl_q(2) \) possesses an ideal \( W \) and a sub-algebra \( Y \) satisfying \( wsl_q(2) = Y \oplus W \) and \( W \cong sl_q(2) \) as Hopf algebras.

Let us assume here that \( q \) is a root of unity of order \( d \) in the field \( k \) where \( d \) is an odd integer and \( d > 1 \). Set \( I = (E_w^d, F_w^d, K_w^d - J_w) \) the two-sided ideal of \( U_q^w \) and the algebra \( U_q^w / I \) is also a coideal of \( U_q^w \) and \( T_w(I) \subseteq I \). Then \( I \) is a weak Hopf ideal and \( U_q^w / I \) has a unique weak Hopf algebra structure with the same operations of \( U_q^w \).
We have $\widehat{U}_q = U_q / I = Y / I \oplus W / I \cong Y / (E^d, F^d) \oplus \widehat{U}_q$ where $\widehat{U}_q = sl_q(2)(E^d, F^d, K^d - 1)$ is a finite dimensional Hopf algebra. The sub-algebra $\widehat{B}_q$ of $\widehat{U}_q$ generated by $\{E^m K^n : 0 \leq m, n \leq d - 1\}$ is a finite dimensional Hopf sub-algebra and $\tilde{U}_q$ is a braided Hopf algebra as a quotient of the quantum double of $\widehat{B}_q$ [6]. The $R$-matrix of $\widehat{U}_q$ is

$$\tilde{R} = \frac{1}{d} \sum_{0 \leq k \leq d-1; 1 \leq i, j \leq d} \frac{(q - q^{-1})^k}{[k]!} q^{k(k-1)/2 + 2k(i-j) - 2j} E^k K^i \otimes F^k K^j.$$ 

Since $sl_q(2) \cong W$ and $(E^d, F^d, K^d - 1) \cong I$, we get $\tilde{U}_q \cong W / I$ under the induced morphism of $\rho$. Then $W / I$ possesses also a $R$-matrix

$$R^w = \frac{1}{d} \sum_{0 \leq k \leq d-1; 1 \leq i, j \leq d} \frac{(q - q^{-1})^k}{[k]!} q^{k(k-1)/2 + 2k(i-j) - 2j} E^k K^i \otimes F^k K^j.$$ 

So, we get

**Theorem** $\tilde{U}_q$ is a quasi-braided weak Hopf algebra with

$$R^w = \frac{1}{d} \sum_{0 \leq k \leq d-1; 1 \leq i, j \leq d} \frac{(q - q^{-1})^k}{[k]!} q^{k(k-1)/2 + 2k(i-j) - 2j} E^k K^i \otimes F^k K^j$$

as its quasi-$R$-matrix, which is von Neumann's regular.

Because the identity of $W / I$ is $J$, there exists the inverse $\tilde{R}^w$ of $R^w$ such that $\tilde{R}^w R^w = \tilde{R}^w = J$ (the identity). Then we have

$$R^w \tilde{R}^w R^w = R^w, \quad \tilde{R}^w R^w \tilde{R}^w = \tilde{R}^w,$$

which shows that this $R$-matrix is regular in $\tilde{U}_q$. It obeys the following relations

$$\Delta^\rho(x) R^w = R^w \Delta(x)$$

for any $x \in W / I$ and

$$(\Delta \otimes \text{id})(R^w) = R^w_{13} R^w_{23}, \quad (\text{id} \otimes \Delta)(R^w) = R^w_{13} R^w_{12},$$

which are also satisfied in $\tilde{U}_q$. Therefore $R^w$ is a von Neumann's regular quasi-$R$-matrix of $\tilde{U}_q$.

A further interesting work is to study our weak Hopf algebras through the similar objects and methods for the non-unital weak Hopf algebras [7] (their class and the class of weak Hopf algebras [2,3] are not included each other) and to find applications in the theory of quantum chain models and other relative areas.