Regular solutions of quantum Yang–Baxter equation from weak Hopf algebras *)

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Generalization of Hopf algebra $\mathfrak{sl}_q(2)$ by weakening the invertibility of the generator K, i.e., exchanging its invertibility $KK^{-1} = 1$ to the regularity $K\overline{K}K = K$ is studied. Two weak Hopf algebras are introduced: a weak Hopf algebra $w\mathfrak{sl}_q(2)$ and a J-weak Hopf algebra $v\mathfrak{sl}_q(2)$ which are investigated in detail. The monoids of group-like elements of $w\mathfrak{sl}_q(2)$ and $v\mathfrak{sl}_q(2)$ are regular monoids, which supports the general conjucture on the connection betweek weak Hopf algebras and regular monoids. A quasi-braided weak Hopf algebra $\overline{U}_q^w$ is constructed from $w\mathfrak{sl}_q(2)$. It is shown that the corresponding quasi-R-matrix is regular $R^w\hat{R}^wR^w = R^w$.

A k-bialgebra[1]) $H = (H, \mu, \eta, \Delta, \varepsilon)$ is called a *weak Hopf algebra* if there exists $T \in \mathrm{Hom}_k(H, H)$ such that $id \star T \star \mathrm{id} = \mathrm{id}$ and $T \star \mathrm{id} \star T = T$ where T is called a *weak antipode* of H. The concept of weak Hopf algebra as a generalization of a Hopf algebra [1] was introduced and studied in [2]. One of its aims is to construct some singular solutions of the quantum Yang-Baxter equation (QYBE) and to study QYBE in a larger scope, e.g. [3]. We investigate a weak Hopf algebra $w\mathfrak{sl}_q(2)$ and a J-weak Hopf algebra $v\mathfrak{sl}_q(2)$ as generalizations of $\mathfrak{sl}_q(2)$ and non-trivial examples of weak Hopf algebras. The fact that the monoids of group-like elements of $w\mathfrak{sl}_q(2)$ and $v\mathfrak{sl}_q(2)$ are regular, supports the general conjucture on the connection between weak Hopf algebras and regular monoids. A quasi-braided weak Hopf algebra $\overline{U}_q^w$ from $w\mathfrak{sl}_q(2)$ is constructed whose quasi-R-matrix is regular.

Let $q \in \mathbb{C}$ and $q \neq \pm 1, 0$. The quantum enveloping algebra $U_q = U_q(\mathfrak{sl}_q(2))$ (see [4]) is generated by four variables (Chevalley generators) E, F, K, K^{-1} with the relations $K^{-1}K = KK^{-1} = 1$, $KEK^{-1} = q^2E$, $KFK^{-1} = q^{-2}F$, $EF - FE = (K - K^{-1})/(q - q^{-1})$. Now we try to weaken the invertibility of K to regularity, as usual in the semigroup theory [5] (see also [6, 7] for higher regularity). It can be done in two different ways.

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1) In this paper, k always denotes a field.

(I) Define $U_q^w = w\mathfrak{sl}_q(2)$, which is called a *weak quantum algebra*, as the algebra generated by the four variables E_w, F_w, K_w, $\overline{K}_w$ with the relations:

$$K_w\overline{K}_w = \overline{K}_w K_w, \quad K_w\overline{K}_w K_w = K_w, \quad \overline{K}_w K_w \overline{K}_w = \overline{K}_w, \tag{1}$$

$$K_w E_w = q^2 E_w K_w, \quad \overline{K}_w E_w = q^{-2} E_w \overline{K}_w, \tag{2}$$

$$K_w F_w = q^{-2} F_w K_w, \quad \overline{K}_w F_w = q^2 F_w \overline{K}_w, \tag{3}$$

$$E_w F_w - F_w E_w = \frac{K_w - \overline{K}_w}{q - q^{-1}}. \tag{4}$$

(II) Define $U_q^v = v\mathfrak{sl}_q(2)$, which is called a *J-weak quantum algebra*, as the algebra generated by the four variables E_v, F_w, K_v, $\overline{K}_v$ with the relations ($J_v = K_v\overline{K}_v$):

$$K_v\overline{K}_v = \overline{K}_v K_v, \quad K_v\overline{K}_v K_v = K_v, \quad \overline{K}_v K_v \overline{K}_v = \overline{K}_v, \tag{5}$$

$$K_v E_v \overline{K}_v = q^2 E_v, \quad K_v F_v \overline{K}_v = q^{-2} F_v, \quad E_v J_v F_v - F_v J_v E_v = \frac{K_v - \overline{K}_v}{q - q^{-1}}. \tag{6}$$

Let $J_w = K_w\overline{K}_w$. List some useful properties of J_w which will be needed below. Firstly, $J_w^2 = J_w$, which means that J_w is a projector. For any variable X, define "J-conjugation" as $X_{J_w} = J_w X J_w$, and the corresponding mapping will be written as $\mathbf{e}_w(X) : X \to X_{J_w}$. Note that the mapping $\mathbf{e}_w$ is idempotent.

Proposition 1. *(i)* $w\mathfrak{sl}_q(2)/(J_w - 1) \cong \mathfrak{sl}_q(2)$; $v\mathfrak{sl}_q(2)/(J_v - 1) \cong \mathfrak{sl}_q(2)$; *(ii) Quantum algebras* $w\mathfrak{sl}_q(2)$ *and* $v\mathfrak{sl}_q(2)$ *possess zero divisors, one of which is*[2] $(J_{w,v} - 1)$ *which annihilates all generators.*

Since $\mathfrak{sl}_q(2)$ is an algebra without zero divisors, some properties of $\mathfrak{sl}_q(2)$ cannot be upgraded to $w\mathfrak{sl}_q(2)$ and $v\mathfrak{sl}_q(2)$, e.g. the standard theorem of Ore extensions and its proof (see Theorem I.7.1 in [4]).

Lemma 2. *(i) The idempotent* J_w *is in the center of* $w\mathfrak{sl}_q(2)$; *(ii) There are unique algebra automorphisms* ω_w *and* ω_v *(called the* weak Cartan automorphisms*) of* U_q^w *and* U_q^v*, respectively, such that* $\omega_{w,v}(K_{w,v}) = \overline{K}_{w,v}$, $\omega_{w,v}(\overline{K}_{w,v}) = K_{w,v}$, $\omega_{w,v}(E_{w,v}) = F_{w,v}$, $\omega_{w,v}(F_{w,v}) = E_{w,v}$.

In general, $\omega_w \neq \omega$ and $\omega_v \neq \omega$ for the automorphism ω of $\mathfrak{sl}_q(2)$ [4]. According to their definitions, some (but not all) properties of $w\mathfrak{sl}_q(2)$ can be extended on $v\mathfrak{sl}_q(2)$ as well, and below we mostly will consider $w\mathfrak{sl}_q(2)$ in detail.

Let R be an algebra over k and R$[t]$ be the free left R-module consisting of all polynomials of the form $P = \sum_{i=0}^{n} a_i t^i$ with coefficients in R. If $a_n \neq 0$, define $\deg(P) = n$; say $\deg(0) = -\infty$. Let α be an algebra morphism of R. An *α-derivation* of R is a k-linear endomorphism δ of R such that $\delta(ab) = \alpha(a)\delta(b) + \delta(a)b$ for all $a, b \in$ R. It follows that $\delta(1) = 0$.

[2]) We denote by $X_{w,v}$ one of the variables X_w or X_v.

Theorem 3. (i) *Assume that* $\mathsf{R}[t]$ *has an algebra structure such that the natural inclusion of* R *into* $\mathsf{R}[t]$ *is a morphism of algebras and* $\deg(PQ) \leq \deg(P) + \deg(Q)$ *for any pair* (P, Q) *of elements of* $\mathsf{R}[t]$. *Then there exists a unique injective algebra endomorphism* α *of* R *and a unique* α*-derivation* δ *of* R *such that* $ta = \alpha(a)t + \delta(a)$ *for all* $a \in \mathsf{R}$;

(ii) *Conversely, given an algebra endomorphism* α *of* R *and an* α*-derivation* δ *of* R, *there exists a unique algebra structure on* $\mathsf{R}[t]$ *such that the inclusion of* R *into* $\mathsf{R}[t]$ *is an algebra morphism and* $ta = \alpha(a)t + \delta(a)$ *for all* $a \in \mathsf{R}$.

Proof. (Schema) (i) Take any $0 \neq a \in \mathsf{R}$ and consider the product ta. We have $\deg(ta) \leq \deg(t) + \deg(a) = 1$. By the definition of $\mathsf{R}[t]$, there exist uniquely determined elements $\alpha(a)$ and $\delta(a)$ of R such that $ta = \alpha(a)t + \delta(a)$. The left multiplication by t is linear and so are α and δ. Expanding both sides of the equality $(ta)b = t(ab)$ in $\mathsf{R}[t]$ using $ta = \alpha(a)t + \delta(a)$ for $a, b \in \mathsf{R}$, we get $\alpha(ab) = \alpha(a)\alpha(b)$ and $\delta(ab) = \alpha(a)\delta(b) + \delta(a)b$. Moreover, α is an algebra endomorphism and δ is an α-derivation whose uniqueness follows from the freeness of $\mathsf{R}[t]$ over R.

(ii) To construct the multiplication on $\mathsf{R}[t]$ as an extension of that on R such that $ta = \alpha(a)t + \delta(a)$, only needs to determine the multiplication ta for any $a \in \mathsf{R}$. Let $M = \{(f_{ij})_{i,j\geq 1} : f_{ij} \in \mathrm{End}_k(\mathsf{R})$ and each row and each column has only finitely many $f_{ij} \neq 0\}$ and I is the identity of M. For $a \in \mathsf{R}$, let $\widehat{a} : \mathsf{R} \to \mathsf{R}$ satisfying $\widehat{a}(r) = ar$. And, let $T = \begin{pmatrix} \delta & & & \\ \alpha & \delta & & \\ & \alpha & \ddots & \\ & & \ddots & \end{pmatrix} \in M$ and define $\Phi : \mathsf{R}[t] \to M$, satisfying $\Phi(\sum_{i=0}^{n} a_i t^i) = \sum_{i=0}^{n} (\widehat{a}_i I) T^i$. Let S denote the subalgebra generated by T and $\widehat{a}I$ (all $a \in \mathsf{R}$) in M. It can be shown that $\mathsf{R}[t]$ and S are linearly isomorphic.

Define $ta = \Phi^{-1}(T(\widehat{a}I))$, which can be extended to define the multiplication of $\mathsf{R}[t]$ with $fg = \Phi^{-1}(xy)$ for any $f, g \in \mathsf{R}[t]$ and $x = \Phi(f)$, $y = \Phi(g)$. Thus $\mathsf{R}[t]$ becomes an algebra and Φ is an algebra isomorphism from $\mathsf{R}[t]$ to S. And, $ta = \Phi^{-1}(T(\widehat{a}I)) = \Phi^{-1}((\widehat{\alpha(a)})I)T + \widehat{\delta(a)}I) = \alpha(a)t + \delta(a)$ for all $a \in \mathsf{R}$. □

It is recognized as a generalization of Theorem I.7.1 in [4]. We call the algebra constructed from α and δ a *weak Ore extension* of R, denoted as $\mathsf{R}_w[t, \alpha, \delta]$.

Under the condition of Theorem 3(ii), $\mathsf{R}_w[t, \alpha, \delta]$ is free with basis $\{t^i\}_{i\geq 0}$ as a left R-module; moreover, if α is an automorphism, then $\mathsf{R}_w[t, \alpha, \delta]$ is also a right free R-module with the same basis $\{t^i\}_{i\geq 0}$.

Let R be an algebra, α be an algebra automorphism and δ be an α-derivation of R. If R is a left (resp. right) Noetherian, then so is the weak Ore extension $\mathsf{R}_w[t, \alpha, \delta]$.

Theorem 4. *The algebra* $w\mathfrak{sl}_q(2)$ *is Noetherian with the basis*

$$\mathsf{P}_w = \{E_w^i F_w^j K_w^l, E_w^i F_w^j \overline{K}_w^m, E_w^i F_w^j J_w\}$$

where i, j, l *are any non-negative integers,* m *is any positive integer.*

Proof. (Schema) $k[K_w, \overline{K}_w]$ is Noetherian. Let α_1 satisfy $\alpha_1(K_w) = q^2 K_w$ and $\alpha_1(\overline{K}_w) = q^{-2}\overline{K}_w$. Let α_2 satisfy $\alpha_2(F_w^j K_w^l) = q^{-2l} F_w^j K_w^l$, $\alpha_2(F_w^j \overline{K}_w^m) = q^{2m} F_w^j \overline{K}_w^m$, $\alpha_2(F_w^j J_w) = F_w^j J_w$. Let δ satisfy $\delta(F_w^j K_w^l) = \sum_{i=0}^{j-1} F_w^{j-1}(q^{-2i}K_w - q^{2i}\overline{K}_w)K_w^l/(q-q^{-1})$, $\delta(F_w^j \overline{K}_w^l) = \sum_{i=0}^{j-1} F_w^{j-1}(q^{-2i}K_w - q^{2i}\overline{K}_w)\overline{K}_w^l/(q-q^{-1})$, $\delta\left(F_w^j J_w\right) = \sum_{i=0}^{j-1} F_w^{j-1}(q^{-2i}K_w - q^{2i}\overline{K}_w)J_w/(q-q^{-1})$ for $j > 0, l \geq 0$, and $\delta(1) = \delta(K_w) = \delta(\overline{K}_w) = 0$.

Then $A_0 = k[K_w, \overline{K}_w]/(J_w K_w - K_w, \overline{K}_w J_w - \overline{K}_w)$, $A_1 = A_0[F_w, \alpha_1, 0]$, $U_q^w \cong A_2 = A_1[E_w, \alpha_2, \delta]$ such that A_{i+1} is a weak Ore extension of A_i. It follows that U_q^w is Noetherian and is free with basis $\{E_w^i\}_{i\geq 0}$ as a left A_1-module. Moreover, as a k-linear space, U_q^w has the basis P_w. □

The similar theorem can be obtained for $v\mathfrak{sl}_q(2)$ as well.

Let $q \in \mathbb{C}$ and $q \neq \pm 1, 0$. Define $U_q^{w\prime}$ as the algebra generated by the five variables E_w, F_w, K_w, $\overline{K}_w$, L_v with the relations:

$$K_w\overline{K}_w = \overline{K}_w K_w, \quad K_w\overline{K}_w K_w = K_w, \quad \overline{K}_w K_w \overline{K}_w = \overline{K}_w, \tag{7}$$

$$K_w E_w = q^2 E_w K_w, \quad \overline{K}_w E_w = q^{-2} E_w \overline{K}_w, \tag{8}$$

$$K_w F_w = q^{-2} F_w K_w, \quad \overline{K}_w F_w = q^2 F_w \overline{K}_w, \tag{9}$$

$$[L_w, E_w] = q(E_w K_w + \overline{K}_w E_w), \quad [L_w, F_w] = -q^{-1}(F_w K_w + \overline{K}_w F_w), \tag{10}$$

$$E_w F_w - F_w E_w = L_w, \quad (q - q^{-1})L_w = (K_w - \overline{K}_w), \tag{11}$$

Then U_q^w is isomorphic with the algebra $U_q^{w\prime}$ with φ_w satisfying $\varphi_w(E_w) = E_w$, $\varphi_w(F_w) = F_w$, $\varphi_w(K_w) = K_w$, $\varphi_w(\overline{K}_w) = \overline{K}_w$. And, the relationship between $U_q^{w\prime}$ and $U(\mathfrak{sl}(2))$ is that for $q = 1$, (i) the algebra isomorphism $U(\mathfrak{sl}(2)) \cong U_1^{w\prime}/(K_w - 1)$ holds; (ii) there exists an injective algebra morphism π from U_1^w to $U(\mathfrak{sl}(2))[K_w]/(K_w^3 - K_w)$ satisfying $\pi(E_w) = XK_w$, $\pi(F_w) = Y$, $\pi(K_w) = K_w$, $\pi(L) = HK_w$.

For $w\mathfrak{sl}_q(2)$, define the maps $\Delta_w : w\mathfrak{sl}_q(2) \to w\mathfrak{sl}_q(2) \otimes w\mathfrak{sl}_q(2)$, $\varepsilon_w : w\mathfrak{sl}_q(2) \to k$ and $T_w : w\mathfrak{sl}_q(2) \to w\mathfrak{sl}_q(2)$ satisfying respectively $\Delta_w(E_w) = 1 \otimes E_w + E_w \otimes K_w$, $\Delta(F_w) = F_w \otimes 1 + \overline{K}_w \otimes F_w$, $\Delta_w(K_w) = K_w \otimes K_w$, $\Delta_w(\overline{K}_w) = \overline{K}_w \otimes \overline{K}_w$, $\varepsilon_w(E_w) = \varepsilon_w(F_w) = 0$, $\varepsilon_w(K_w) = \varepsilon_w(\overline{K}_w) = 1$, $T_w(E_w) = -E_w\overline{K}_w$, $T_w(F_w) = -K_w F_w$, $T(K_w) = \overline{K}_w$, $T_w(\overline{K}_w) = K_w$.

Proposition 5. *The relations above endow $w\mathfrak{sl}_q(2)$ with a bialgebra structure possessing a weak antipode T_w.*

Proposition 6. *T_w^2 is an inner endomorphism of the algebra $w\mathfrak{sl}_q(2)$ satisfying $T_w^2(X) = K_w X \overline{K}_w$ for any $X \in w\mathfrak{sl}_q(2)$.*

Using the Theorem 4, it can be shown that for the operations above, it is not possible that $w\mathfrak{sl}_q(2)$ would possess an antipode S so as to become a Hopf algebra. Hence, $w\mathfrak{sl}_q(2)$ is an example for a non-commutative and non-cocommutative weak Hopf algebra which is *not a Hopf algebra.*

Also, we can see easily that $U_q^{w\prime}$ comes into a weak Hopf algebra and φ_w is an isomorphism of weak Hopf algebras from $w\mathfrak{sl}_q(2)$ to $U_q^{w\prime}$.

For J-weak quantum algebra $v\mathfrak{sl}_q(2)$, a thorough analysis gives the following nontrivial definitions $\Delta_v(E_v) = J_v \otimes J_v E_v J_v + J_v E_v J_v \otimes K_v$, $\Delta_v(F_v) = J_v F_v J_v \otimes J_v + \overline{K}_v \otimes J_v F_v J_v$, $\Delta_v(K_v) = K_v \otimes K_v$, $\Delta_v(\overline{K}_v) = \overline{K}_v \otimes \overline{K}_v$, $\varepsilon_v(E_v) = \varepsilon_v(F_v) = 0$, $\varepsilon_v(K_v) = \varepsilon_v(\overline{K}_v) = 1$, $T_v(E_v) = -J_v E_v \overline{K}_v$, $T_v(F_v) = -K_v F_v J_v$, $T_v(K_v) = \overline{K}_v$, $T_v(\overline{K}_v) = K_v$.

These relations endow $v\mathfrak{sl}_q(2)$ with a bialgebra structure with a J-weak antipode T_v, i.e. satisfying the regularity conditions $(\mathbf{e}_v \star_v T_v \star_v \mathbf{e}_v)(X) = \mathbf{e}_v(X)$, $(T_v \star_v \mathbf{e}_v \star_v T_v)(X) = T_v(X)$, for any X in $v\mathfrak{sl}_q(2)$. From the difference between id and $\mathbf{e}_v$, $v\mathfrak{sl}_q(2)$ is not a weak Hopf algebra according to the definition of [2]. So we will call it *J-weak Hopf algebra* and T_v the *J-weak antipode.* Remark that the variable $\mathbf{e}_v$ can be treated as $n = 2$ example of the "tower identity" $e^{(n)}_{\alpha\beta}$ introduced for semisupermanifolds in [8, 6] or the "obstructor" $\mathbf{e}^{(n)}_X$ for general mappings, categories and Yang-Baxter equation in [7].

Now, we discuss the set $G(w\mathfrak{sl}_q(2))$ of all group-like elements of $w\mathfrak{sl}_q(2)$. The concept of *inverse monoid* can be found in [5].

Proposition 7. *The set of all group-like elements $G(w\mathfrak{sl}_q(2)) = \{J^{(ij)} = K^i_w \overline{K}^j_w :$ i, j run over all non-negative integers$\}$, which forms a regular monoid under the multiplication of $w\mathfrak{sl}_q(2)$.*

Proof. (Schema) Using of $\Delta_w(x) = x \otimes x$, we can conclude that only $x = \alpha_l K^l_w$, $\beta_m \overline{K}^m_w$ or J_w. It follows that $G(w\mathfrak{sl}_q(2)) = \{J^{(ij)}_w = K^i_w \overline{K}^j_w : i, j$ run over all non-negative integers$\}$ and $J^{(ij)}_w J^{(ji)}_w J^{(ij)}_w = J^{(ij)}_w$, which means that $G(w\mathfrak{sl}_q(2))$ forms a regular monoid under the multiplication of $w\mathfrak{sl}_q(2)$. □

For $v\mathfrak{sl}_q(2)$ we can get a similar statement.

Theorem 8. *$w\mathfrak{sl}_q(2)$ possesses an ideal W and a sub-algebra Y satisfying $w\mathfrak{sl}_q(2) = Y \oplus W$ and $W \cong \mathfrak{sl}_q(2)$ as Hopf algebras.*

Proof. (Schema) Let W be generated by $\{E^i_w F^j_w K^l_w, E^i_w F^j_w \overline{K}^m_w, E^i_w F^j_w J_w :$ for all $i \geq 0$, $j \geq 0$, $l > 0$ and $m > 0\}$, and Y is generated by $\{E^i_w F^j_w : i \geq 0, j \geq 0\}$. W is a Hopf algebra with the unit J_w, the comultiplication Δ^W_w satisfying $\Delta^W_w(E_w) = J_w \otimes E_w + E_w \otimes K_w$, $\Delta^W_w(F_w) = F_w \otimes J_w + \overline{K}_w \otimes F_w$, $\Delta^W_w(K_w) = K_w \otimes K_w$, $\Delta^W_w(\overline{K}_w) = \overline{K}_w \otimes \overline{K}_w$ and the antipode T_w. ρ is trivial. □

Let us assume here that q is a root of unity of order d in the field k where d is an odd integer and $d > 1$. Set $I = (E^d_w, F^d_w, K^d_w - J_w)$ the two-sided ideal of U^w_q and the algebra $\overline{U}^w_q = U^w_q / I$. I is also a coideal of U_q and $T_w(I) \subseteq I$. Then I is a weak Hopf ideal and $\overline{U}^w_q$ has a unique weak Hopf algebra structure with the same operations of U^w_q.

By Theorem 8, $\overline{U}^w_q = U^w_q / I = Y/I \oplus W/I \cong Y/(E^d_w, F^d_w) \oplus \widetilde{U}_q$ where $\widetilde{U}_q = \mathfrak{sl}_q(2)/(E^d_w, F^d_w, K^d - 1)$ is a finite dimensional Hopf algebra. As shown in [4], the sub-algebra $\widetilde{B}_q$ of $\widetilde{U}_q$ generated by $\{E^m_w K^n_w : 0 \leq m, n \leq d - 1\}$ is a finite dimensional Hopf sub-algebra and $\widetilde{U}_q$ is a braided Hopf algebra as a quotient of the quantum double of $\widetilde{B}_q$. The R-matrix of $\widetilde{U}_q$ is

$$\widetilde{R} = \frac{1}{d} \sum_{0 \le i,j,k \le d-1} \frac{(q-q^{-1})^k}{[k]!} q^{k(k-1)/2+2k(i-j)-2ij} E_w^k K_w^i \otimes F_w^k K_w^j.$$

Since $\mathfrak{sl}_q(2) \overset{\rho}{\cong} W$ and $(E^d, F^d, K^d - 1) \overset{\rho}{\cong} I$, we get $\widetilde{U}_q \cong W/I$ under the induced morphism of ρ. Then W/I possesses also an R-matrix

$$R^w = \frac{1}{d} \sum_{0 \le k \le d-1; 1 \le i,j \le d} \frac{(q-q^{-1})^k}{[k]!} q^{k(k-1)/2+2k(i-j)-2ij} E_w^k K_w^i \otimes F_w^k K_w^j.$$

In W/I, there exists the inverse $\hat{R}^w$ of R^w such that $\hat{R}^w R^w = R^w \hat{R}^w = J_w$(the identity). Then $R^w \hat{R}^w R^w = R^w$, $\hat{R}^w R^w \hat{R}^w = \hat{R}^w$, which means that the R-matrix is regular in $\overline{U}_q$. So, we get

Theorem 9. *$\overline{U}_q$ is a quasi-braided weak Hopf algebra with*

$$R^w = \frac{1}{d} \sum_{0 \le k \le d-1; 1 \le i,j \le d} \frac{(q-q^{-1})^k}{[k]!} q^{k(k-1)/2+2k(i-j)-2ij} E_w^k K_w^i \otimes F_w^k K_w^j$$

as its quasi-R-matrix, which is von Neumann's regular.

The quasi-R-matrix from J-weak Hopf algebra $v\mathfrak{sl}_q(2)$ has more complicated structure and will be considered elsewhere.

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References

[1] E. Abe: *Hopf Algebras*, Cambridge Univ. Press, Cambridge, 1980.

[2] F. Li: J. Algebra **208** (1998) 72.

[3] F. Li: Commun. Algebra **28** (2000) 2253.

[4] C. Kassel: *Quantum Groups*, Springer-Verlag, New York, 1995.

[5] M.V. Lawson: *Inverse Semigroups: The Theory of Partial Symmetries*, World Sci., Singapore, 1998.

[6] S. Duplij: *Semisupermanifolds and semigroups*, Krok, Kharkov, 2000.

[7] S. Duplij and W. Marcinek: in *Noncommutative Structures in Mathematics and Physics* (Eds. S. Duplij and J. Wess), Kluwer, Dordrecht, 2001, p. 125.

[8] S. Duplij: Pure Math. Appl. **9** (1998) 283.

[9] K. Goodearl: *Von Neumann Regular Rings*, Pitman, London, 1979.

[10] F. Li: J. Math. Research and Exposition **19** (1999) 325.

[11] J. Bernstein and T. Khovanova: *On quantum group $SL_q(2)$*, Preprint MIT, Cambridge, 1994; hep-th/9412056.