

On semigroup nature of superconformal symmetry

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A semigroup of $N = 1$ superconformal transformations is introduced and analyzed. Noninvertible ones can describe transitions from body to soul and form a proper ideal containing a set of nilpotent transformations. The projective superspace is also considered. Transformations twisting the parity of a tangent space are brought in. They can be a nonsuperconformal "square root" of noninvertible superconformal transformations and the analogs of the Poincaré metric and conformal invariance are suggested for them.

I. INTRODUCTION

The investigation of various transformations on a set having an extra structure plays an essential role in physical theory. As is well known,¹⁻³ all transformations on a set into itself generally form a full semigroup with respect to composition, while the transformations preserving structure form a semigroup of endomorphisms.¹ The invertible one-to-one transformations lie in a subgroup of the full semigroup and it is these transformations that mainly capture physicists' attention since any theory is usually defined over invertible elements. Another situation comes when working out theories with fermions in superspace approach⁴ that implies the existence of additional anticommuting coordinates. This brings into being nilpotent ingredients and zero divisors that forces one to deal with really noninvertible functions, but nevertheless only invertible transformations forming suitable (super) groups are commonly employed ("super-" will be omitted in evident cases). To save the information lost in this way it is natural and consistent from the algebraic viewpoint to treat noninvertible transformations on a par with invertible ones, i.e., to pass from groups to semigroups. Guided by these considerations, we suggest the "super-" and "semi-" generalizations of a physical theory to be carried out simultaneously.

On the other hand, at present the most promising unified models⁵ are based on the superstring theory.⁶ In it, as well as in some statistical models,⁷ the two-dimensional superconformal (SCf) symmetry⁸ is a vital attribute. So it would be exciting to apply the a. m. proposal to that one.

In this paper we study the semigroup extension of $N = 1$ SCf symmetry in two dimensions. In Sec. II the complex (1|1)-dimensional superspace is briefly given. Various transformations coming into existence due to the weakening of invertibility requirements are introduced in Sec. III. Both invertible and noninvertible transformations that satisfy the same SCf condition form an SCf semigroup. That is analyzed in Sec. IV where the projective superspace is also considered. In Sec. V we dwell on some exotic transformations that twist the parity of a tangent space and can be viewed as a non-SCf square root of SCf transformations. The appropriate analogs of conformal invariance and the Poincaré metric are introduced. Section VI contains the conclusions and some generalities.

II. PRELIMINARIES

Let $\mathbb{C}^{1,1}$ be a complex vector superspace with one even (z) and one odd (θ) coordinates (we consider the holomorphic sector only). In the spirit of, e.g., Rogers approach⁹ these coordinates are valued in a complex Grassmann algebra with L anticommuting generators $\wedge(\mathbb{C}_L)$ (L can be infinite¹⁰). The Cartesian product $\mathbb{C}_L^{1,0} \otimes \mathbb{C}_L^{0,1}$, where $\mathbb{C}_L^{1,0}$ and $\mathbb{C}_L^{0,1}$ are the even and odd sectors of \mathbb{C}_L having the natural gradation, is usually identified with $\mathbb{C}^{1,1}$.¹¹ If \mathbb{N}_L denotes the ideal of nilpotents of \mathbb{C}_L , then $\mathbb{C}_L = \mathbb{C} \oplus \mathbb{N}_L$, and the projection $\epsilon: \mathbb{C}_L \rightarrow \mathbb{C}$ is called a body map⁹ so that $\epsilon[z, \theta] = (z_0, 0)$, $z_0 \in \mathbb{C}$ (see Refs. 9-11 for more details). We only note that the elements from \mathbb{N}_L are evidently noninvertible, just this is important in the following but not the specific structure of the superspace. Then it is natural to treat the functions $\mathbb{C}^{1,1} \rightarrow \mathbb{C}_L$ that meet slackened invertibility requirements on the same ground with invertible ones and to bring in corresponding semigroups of transformations instead of groups. So the transformation $T: \mathbb{C}^{1,1} \rightarrow \mathbb{C}^{1,1}$ we deal with is polynomial in odd coordinates

$$\begin{aligned} \tilde{z} &= f(z) + \theta\chi(z), \\ \tilde{\theta} &= \psi(z) + \theta g(z), \end{aligned} \quad (2.1)$$

as is usually required by superanalyticity¹⁰ but the component functions can be noninvertible although they can satisfy partially supersmoothness conditions.¹¹ Hereafter, Roman (Greek) letters denote $\mathbb{C}^{1,0} \rightarrow \mathbb{C}^{1,0}$ ($\mathbb{C}^{1,0} \rightarrow \mathbb{C}^{0,1}$) functions and T is written on the left. A set of invertible and noninvertible T is a semigroup \mathbf{T} (with respect to composition) called a semigroup of superanalytic transformations. The set of constants on $\mathbb{C}^{1,1}$ is an ideal of \mathbf{T} . All transformations considered below are in various subsemigroups of \mathbf{T} . If $\epsilon[g(z)] \neq 0$, then the superanalog for the Jacobian, namely, the Berezinian¹² (*Ber* in the following), reads for (2.1) as

$$\text{Ber}(\tilde{Z}/Z) = \frac{f'(z)}{g(z)} + \frac{\chi(z)\psi'(z)}{g^2(z)} + \theta \left(\frac{\chi'(z)}{g(z)} \right)', \quad (2.2)$$

where $Z = (z, \theta)$ and prime denotes $\partial/\partial z$.

For the invertible transformations *Ber* has a body, i.e., $\epsilon[\text{Ber}(\tilde{Z}/Z)] \neq 0$. Then an element of the superanalytic semigroup \mathbf{S} can be defined as a set of four functions on $\mathbb{C}^{1,0} [\begin{smallmatrix} f & \chi \\ \psi & g \end{smallmatrix}] \in \mathbf{S}$ and the multiplication law is

$$\begin{bmatrix} f_1 & \chi_1 \\ \psi_1 & g_1 \end{bmatrix} \otimes \begin{bmatrix} f_2 & \chi_2 \\ \psi_2 & g_2 \end{bmatrix} = \begin{bmatrix} f_1 \circ f_2 + \chi_1 \circ f_2 \psi_2 & f_1 \circ f_2 \chi_2 + \chi_1 \circ f_2 g_2 + \chi_1' \circ f_2 \chi_2 \psi_2 \\ \psi_1 \circ f_2 + g_1 \circ f_2 \psi_2 & \psi_1 \circ f_2 \chi_2 + g_1 \circ f_2 g_2 + g_1' \circ f_2 \chi_2 \psi_2 \end{bmatrix}, \quad (2.3)$$

where $f \circ g \equiv f(g(z))$.

It is easily seen that the elements having $\psi(z) = 0$ and (or) $\chi(z) = 0$ form corresponding subsemigroups of S . Moreover, there is the homomorphism $\varphi: S \rightarrow T$ having $\ker \varphi = \begin{bmatrix} 0 & 0 \\ \delta_1 & 0 \end{bmatrix}$.

III. REDUCTIONS

The holomorphic tangent space on $\mathbb{C}^{1,1}$ is locally¹³ spanned by (D^2, D) , where $D = \partial/\partial\theta + \theta\partial/\partial z$ and $D^2 = \partial/\partial z$, while the dual cotangent space is spanned by $(dZ, d\theta)$, where $dZ = dz + \theta d\theta$. (Crane and Rabin's¹⁴ sign convention is used here.) Under T these transform as

$$\begin{pmatrix} D^2 \\ D \end{pmatrix} = U \begin{pmatrix} \bar{D}^2 \\ \bar{D} \end{pmatrix} \quad (3.1)$$

and

$$(d\bar{Z}, d\bar{\theta}) = (dZ, d\theta)U, \quad (3.2)$$

where

$$U = \begin{pmatrix} D^2 \bar{z} - D^2 \bar{\theta} \cdot \bar{\theta} & D^2 \bar{\theta} \\ D\bar{z} - D\bar{\theta} \cdot \bar{\theta} & D\bar{\theta} \end{pmatrix}. \quad (3.3)$$

So that the exterior differential $d = dZ D^2 + d\theta D$ is invariant. Using the relation

$$\begin{pmatrix} \frac{\partial \bar{z}}{\partial z} & \frac{\partial \bar{\theta}}{\partial z} \\ \frac{\partial \bar{z}}{\partial \theta} & \frac{\partial \bar{\theta}}{\partial \theta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\theta & 1 \end{pmatrix} U \begin{pmatrix} 1 & 0 \\ \bar{\theta} & 1 \end{pmatrix} \quad (3.4)$$

and the multiplicativity property of Ber^{12} one obtains

$$\text{Ber}(\bar{Z}/Z) = \text{Ber} U \quad (3.5)$$

in case the both sides are well defined. Usually this is achieved by choosing $U \in \text{GL}(1|1, C)$.¹⁵ Then the body of U odd-odd part is responsible for well definition of Ber , the body of U even-even part controlling its invertibility. But in general U can be in a semigroup and so three possibilities and respective types of T arise: (I) Ber exists and is invertible; (II) Ber exists but it is pure soul or vanishing; (III) Ber cannot be defined at all. (3.6)

The transformations of type I can be exploited as the transition functions between maps when constructing the supermanifolds for which $\text{GL}(1|1, C)$ is a structure group. Some reduction of the latter results in the supermanifold to become a super Riemann surface (SRS).¹⁴⁻¹⁷ In this connection an intriguing question comes into being: What analogous objects could be built using the transition functions of other types?

Let us consider all possible reductions of U via vanishing every of its entries in turn. In the trivial cases the entries from the right column of U vanish and the even sector is not affected. First, $D^2 \bar{\theta} = 0$ (but $D\bar{\theta} = a = \text{const} \neq 0$) determines a subsemigroup of transformations $T_0 \subset T$ and gives $d\bar{z}(z, \theta) = \partial\bar{z}/\partial z dz$ and $d\bar{\theta} = a d\theta$ so that $\text{Ber}(\bar{Z}/Z) = a^{-1} \partial\bar{z}/\partial z$ which means that the odd sector

disjoints being a global transformation $\bar{\theta} = \alpha + \theta a$. Hence, such $T_0 \in T$ can be called a "semiglobal" transformation for which (2.3) holds valid. Note that resemblance with the nonsupersymmetric case allows to define, in addition to d , the usual $d_0 = dz \partial/\partial z$ which is invariant under T_0 only. So this reduces the whole tangent space to its invariant even subspace. Secondly, if $D\bar{\theta} = a = 0$, then the odd sector of T_0 becomes degenerated and it is a left zero in the sense of (2.3).

Proceeding to nontrivial reductions of U we observe two remaining possible conditions

$$D\bar{z} = D\bar{\theta} \cdot \bar{\theta}, \quad (3.7)$$

$$D^2 \bar{z} = D^2 \bar{\theta} \cdot \bar{\theta}. \quad (3.8)$$

The former is called an SCf condition and the invertible transformations satisfying it are called SCf transformations.^{14,16} It is convenient to refer to all of them including noninvertible ones as "SCf" too and to denote a transformation satisfying (3.7) as T_{SCf} . Then using another form of (3.7)

$$\frac{\partial \bar{z}}{\partial z} + \bar{\theta} \frac{\partial \bar{\theta}}{\partial z} = (D\bar{\theta})^2, \quad (3.9)$$

we obtain

$$U_{\text{SCf}} = \begin{pmatrix} (D\bar{\theta})^2 & D^2 \bar{\theta} \\ 0 & D\bar{\theta} \end{pmatrix}. \quad (3.10)$$

It follows that Ber can be well defined if $\epsilon[D\bar{\theta}] \neq 0$:

$$\text{Ber}(\bar{Z}/Z) = D\bar{\theta}. \quad (3.11)$$

Therefore, a definition of the invertible $N = 1$ SCf transformation can be rewritten in the form

$$d\bar{Z} = (\text{Ber}(\bar{Z}/Z))^2 dZ, \quad (3.12)$$

which can be extended on general $N \neq 2$ in this way

$$d\bar{Z} = (\text{Ber}(\bar{Z}/Z))^{2/(2-N)} dZ \quad (3.13)$$

(cf. Cohn¹⁸ and Schoutens¹⁹).

An Abelian differential (or SCf superdifferential) dl ($dl^2 = dZ$) introduced by Friedan²⁰ can be used to determine line bundles²¹ and line integrals²² on an SRS. This dl transforms inversely to D by means of Ber as

$$d\bar{l} = \text{Ber}(\bar{Z}/Z) dl = (D\bar{\theta}) dl. \quad (3.14)$$

Then an exterior differential $d_{\text{SCf}} = dl D$ is invariant under T_{SCf} , which gives rise to the invariant odd subspace of the tangent space, and it can be utilized to construct fermionic string actions (see Refs. 16, 23).

Another condition (3.8) leads to

$$d\bar{Z} = \Delta d\theta, \quad (3.15)$$

where

$$\Delta = \frac{\partial \bar{z}}{\partial \theta} - \bar{\theta} \frac{\partial \bar{\theta}}{\partial \theta} \quad (3.16)$$

and so the transformation satisfying (3.8) can be called a

“twisting-parity-(of)-tangent-(space)” (TPt) transformation denoted by T_{TPt} . Now the reduced matrix U is

$$U_{\text{TPt}} = \begin{pmatrix} 0 & D^2\tilde{\theta} \\ \Delta & D\tilde{\theta} \end{pmatrix}. \quad (3.17)$$

In general a set of T_{TPt} do not form any semigroup, since the shape of U_{TPt} is not preserved without extra requirements. Nevertheless, T_{TPt} has another fascinating sense we shall find out later on.

In case $\epsilon[D\tilde{\theta}] \neq 0$ we express

$$D^2\tilde{\theta} = -\Delta'/2(D\tilde{\theta}) \quad (3.18)$$

and obtain Ber in the form

$$\text{Ber}(\tilde{Z}/Z) = \Delta'\Delta/2(D\tilde{\theta})^3. \quad (3.19)$$

Since it is pure soul, TPt transformations are of type II or III [see (3.6)] and hence noninvertible. By analogy with the SCf superdifferential dl we can introduce a TPt superdifferential dt as an object transforming inversely to

$$D^2 = (D^2\tilde{\theta})\tilde{D}, \quad (3.20)$$

in the following way

$$\tilde{d}t = (D^2\tilde{\theta})dt. \quad (3.21)$$

We observe that the parity of $\tilde{d}t$ is opposite to that of dt and so it is really “parity twisted.” Now a TPt invariant “exterior differential” is

$$d_{\text{TPt}} = dt D^2 = \tilde{d}t \tilde{D} = \tilde{d}_{\text{TPt}}, \quad (3.22)$$

which could be used to construct a TPt analog of a “line bundle.” It follows from (3.19) that

$$d\tilde{Z} \tilde{d}t = (D\tilde{\theta})^2 \text{Ber}(\tilde{Z}/Z) d\theta dt. \quad (3.23)$$

Consequently, if $(D\tilde{\theta})^2 = 1$, then (3.23) defines an object that transforms via Ber as in the previous SCf case. Now let us turn to the structure of SCf and TPt transformations.

IV. SCf SEMIGROUP

The SCf condition (3.7) results in

$$\chi(z) = g(z)\psi(z), \quad (4.1)$$

$$f'(z) = g^2(z) - \psi(z)\psi'(z), \quad (4.2)$$

which gives the following form of T_{SCf}

$$\begin{aligned} \tilde{z} &= f(z) + \theta\psi(z)g(z), \\ \tilde{\theta} &= \psi(z) + \theta g(z). \end{aligned} \quad (4.3)$$

We do not express here $g(z)$ from (4.2) as is usually done^{14,24} intentionally to recall that it is possible if $\epsilon[g(z)] \neq 0$ only which is one of the cases under consideration. Also (4.3) can be interestingly interpreted as follows: We rewrite it in such a way

$$\tilde{z} = \tilde{\theta}\psi(z) + f(z) \quad (4.4)$$

and observe the points satisfying $z = \text{const}$ to be mapped onto a “SCf straight line” with a pure soul “slope” on $(\tilde{z}, \tilde{\theta})$ plane, while the points of the “line” are labelled by θ .

When searching for a proper subsemigroup of \mathbf{S} (2.3) we see two functions be fixed by (4.1) and (4.2). Therefore, an element of $\mathbf{S}_{\text{SCf}} \subset \mathbf{S}$ is determined by one even and one odd function on $\mathbf{C}^{1,0}$ from the odd sector of T_{SCf} , that is $\mathbf{s}[g, \psi] \in \mathbf{S}_{\text{SCf}}$ and the multiplication law in \mathbf{S}_{SCf} is

$$\begin{aligned} \mathbf{s}[g_1, \psi_1] \otimes \mathbf{s}[g_2, \psi_2] \\ = \mathbf{s}[g_1 \circ f_2 g_2 + g_1' \circ f_2 \psi_1 \circ f_2 g_2 \psi_2, \psi_1 \circ f_2 + g_1 \circ f_2 \psi_2], \end{aligned} \quad (4.5)$$

where the constraint (4.2) holds valid for every one of the elements here. When the functions entered are fixed, \mathbf{S}_{SCf} is a finitely generated semigroup. It obviously follows from (4.5) that an identity of \mathbf{S}_{SCf} is $\mathbf{e} = \mathbf{s}[1, 0]$ and a zero is $\mathbf{z} = \mathbf{s}[0, 0]$.

A. Classification

To classify SCf transformations we need an analog of Ber for all of them including noninvertible ones. Let us assume that (3.11) can play the part of such an overdefined super Jacobian

$$J_{\text{SCf}} = D\tilde{\theta}. \quad (4.6)$$

Then the invertible T_{SCf} called a *body* transformation and denoted by T_{body} has $\epsilon[J_{\text{SCf}}] \neq 0$ and represents a maximal subgroup $\mathbf{G}_{\text{SCf}} \subset \mathbf{S}_{\text{SCf}}$ containing the identity $\mathbf{e} \in \mathbf{G}_{\text{SCf}}$. If $\epsilon[J_{\text{SCf}}] = 0$, then T_{SCf} is called a *soul* transformation T_{soul} representing a maximal proper ideal $\mathbf{I}_{\text{SCf}} \subset \mathbf{S}_{\text{SCf}}$ ($\mathbf{z} \in \mathbf{I}_{\text{SCf}}$). Hence, \mathbf{S}_{SCf} is not a simple semigroup and so the sequence of inclusions takes place

$$\mathbf{Z} \subset \mathbf{I}_{\text{SCf}}^{\mathbf{g}=0} \subset \mathbf{I}_{\text{SCf}}^{\mathbf{g}^2=0} \subset \mathbf{I}_{\text{SCf}}^{\epsilon(\mathbf{g})=0} \equiv \mathbf{I}_{\text{SCf}}, \quad (4.7)$$

where \mathbf{Z} is a minimal ideal and $\mathbf{z} \in \mathbf{Z}$. Note that \mathbf{G}_{SCf} is disjoint since $\mathbf{S}_{\text{SCf}} = \mathbf{G}_{\text{SCf}} \cup \mathbf{I}_{\text{SCf}}$ and $\mathbf{G}_{\text{SCf}} \cap \mathbf{I}_{\text{SCf}} = \emptyset$. Here, the Rees quotient $\mathbf{S}_{\text{SCf}}/\mathbf{I}_{\text{SCf}}$ can be defined if the equivalence relation on \mathbf{S}_{SCf} relative to \mathbf{I}_{SCf} or some other term of the sequence (4.7) is the Rees congruence.^{1,2} To examine the left and right cancellativity (or reductivity) of \mathbf{S}_{SCf} , which means that $\mathbf{s} \otimes \mathbf{s}_1 = \mathbf{s} \otimes \mathbf{s}_2$ and $\mathbf{s}_1 \otimes \mathbf{s} = \mathbf{s}_2 \otimes \mathbf{s}$ should imply $\mathbf{s}_1 = \mathbf{s}_2 \in \mathbf{S}_{\text{SCf}}$ for any (or all) $\mathbf{s} \in \mathbf{S}_{\text{SCf}}$, it is necessary to consider the following component relations

$$\begin{aligned} \psi \circ f_1 - \psi \circ f_2 &= g \circ f_2 \psi_2 - g \circ f_1 \psi_1, \\ g_1 g \circ f_1 - g_2 g \circ f_2 &= g' \circ f_1 \psi \circ f_1 g_1 \psi_1 - g' \circ f_2 \psi \circ f_2 g_2 \psi_2 \end{aligned} \quad (4.8)$$

and

$$\begin{aligned} g(g_1 \circ f - g_2 \circ f) &= g\psi(g_2' \circ f \psi_2 \circ f - g_1' \circ f \psi_1 \circ f), \\ \psi_1 \circ f - \psi_2 \circ f &= \psi(g_2 \circ f - g_1 \circ f), \end{aligned} \quad (4.9)$$

respectively, where (4.2) holds for every of the elements. It follows that in general \mathbf{S}_{SCf} is not right cancellative at any rate owing to possible nilpotency of the functions entered, nevertheless \mathbf{S}_{SCf} is reductive. Further algebraic and topological features of \mathbf{S}_{SCf} can be analyzed by using Refs. 1–2 and Refs. 3, 25, respectively.

Let us dwell on \mathbf{G}_{SCf} briefly. In this case for every $\mathbf{g}[g, \psi] \in \mathbf{G}_{\text{SCf}}$ an inverse element is

$$\mathbf{g}^{-1}[g, \psi] = \mathbf{g} \left[\frac{1}{g \circ f^{-1}}, \frac{-\psi \circ f^{-1}}{g \circ f^{-1}} \right], \quad (4.10)$$

where f^{-1} denotes the inverse function. Since $\epsilon[g(z)] \neq 0$ here one can solve (4.2) under $g(z)$ explicitly and obtain the body transformation in the form^{14,17,24}

$$\begin{aligned}\tilde{z} &= f(z) + \theta \sqrt{f'(z)} \psi(z), \\ \tilde{\theta} &= \psi(z) + \theta \sqrt{f'(z) + \psi(z)\psi'(z)}.\end{aligned}\quad (4.11)$$

Now Ber is well defined as

$$J_{\text{SCf}}^{\text{body}} = \text{Ber}(\tilde{Z}/Z) = \sqrt{f'(z) + \psi(z)\psi'(z)} + \theta\psi'(z) \quad (4.12)$$

and has a body because from (4.2) it follows that $\epsilon[f'(z)] = \epsilon[g^2(z)] \neq 0$. This transformation is of type I [see (3.6)]. The element of \mathbf{G}_{SCf} can be reexpressed in terms of the functions from (4.11) as $g^*[f, \psi]$ then the multiplication law will change accordingly.¹⁴

The soul transformations²⁶ can be further subdivided into three kinds in line with the sequence (4.7). The common property of them is impossible to solve (4.2) under $g(z)$ as in the previous case (4.11). Then the soul transformation has the form

$$\begin{aligned}\tilde{z} &= f_{\text{soul}}(z) + \theta\psi(z)g_{\text{soul}}(z), \\ \tilde{\theta} &= \psi(z) + \theta g_{\text{soul}}(z).\end{aligned}\quad (4.13)$$

If $g^2(z) = 0$ (4.2) becomes

$$f'(z) = \psi'(z)\psi(z), \quad (4.14)$$

which cannot be directly integrated because $\psi(z)$ is nilpotent. Nevertheless, the solution of (4.14) can be presented as the series

$$f_{\text{soul}}(z) = \sum_{n=0}^{\infty} \frac{z^{n+1}}{(n+1)!} \left(-\frac{\partial}{\partial z} \right)^n (\psi'(z)\psi(z)) + k, \quad (4.15)$$

being finite for a polynomial $\psi(z)$. We observe that T_{soul} is the transformation of type III [see (3.6)], therefore only the overdefined super Jacobian taking the form

$$J_{\text{SCf}}^{\text{soul}} = g_{\text{soul}}(z) + \theta\psi'(z), \quad (4.16)$$

may be considered now. In general it has the third degree of nilpotency and the second one if $g_{\text{soul}}(z) \subset \text{Ann}[\psi'(z)]$, where $\text{Ann}[a] = \{b \in B \mid ab = 0\}$ is a set B of the annihilators of a . For $g_{\text{soul}}(z)$ to have the second degree of nilpotency, which corresponds to the mean term in the sequence (4.7), the ansatz $g_{\text{soul}}(z) = \alpha(z)\beta(z)$ is suitable.²⁶ In case $\beta(z) = \psi(z)$ the even sector disjoints

$$\begin{aligned}\tilde{z} &= f_{\text{soul}}(z), \\ \tilde{\theta} &= \psi(z) + \theta\alpha(z)\psi(z).\end{aligned}\quad (4.17)$$

Let us introduce a set of left annihilators for a given element from \mathbf{S}_{SCf} as

$$\text{Ann}_L[\mathbf{s}] = \{\mathbf{s}_1 \in \mathbf{S}_{\text{SCf}} \mid \mathbf{s}_1 \otimes \mathbf{s} = \mathbf{z}, \mathbf{s} \in \mathbf{S}_{\text{SCf}}\}$$

and the same for right ones. Then $\text{Ann}[\mathbf{s}] = \text{Ann}_L[\mathbf{s}] \cap \text{Ann}_R[\mathbf{s}]$. Writing out $\mathbf{s}_1 \otimes \mathbf{s}_2 = \mathbf{z}$ in components

$$\begin{aligned}g_2(g_1 \circ f_2 + \psi_2 \psi_1 \circ f_2 g_1' \circ f_2) &= 0, \\ \psi_1 \circ f_2 + g_1 \circ f_2 \psi_2 &= 0,\end{aligned}\quad (4.18)$$

we observe that $\text{Ann}_L[\mathbf{g}] = \mathbf{z}$, $\mathbf{g} \in \mathbf{G}_{\text{SCf}}$, but the strong inclusions $\mathbf{z} \subset \text{Ann}_R[\mathbf{g}] \subset \mathbf{I}_{\text{SCf}}$ take place (a simple example: $\mathbf{g}[z, \gamma z] \otimes \mathbf{i}[\mu \delta z, \gamma z + \delta] = \mathbf{z}, \mathbf{i} \in \mathbf{I}_{\text{SCf}}$). For elements from \mathbf{I}_{SCf} we have

$$\begin{aligned}\text{Ann}_R[\mathbf{i}] \cap \mathbf{G}_{\text{SCf}} &\neq \emptyset, \\ \text{Ann}_R[\mathbf{i}] \cap \mathbf{I}_{\text{SCf}} &\neq \emptyset, \\ \text{Ann}_L[\mathbf{i}] &\subset \mathbf{I}_{\text{SCf}}.\end{aligned}\quad (4.19)$$

Thus an exciting peculiarity of \mathbf{S}_{SCf} is the existence of right annihilators other than \mathbf{z} for every $\mathbf{s} \in \mathbf{S}_{\text{SCf}}$ and left ones for every $\mathbf{i} \in \mathbf{I}_{\text{SCf}}$. For $\mathbf{i} \in \mathbf{I}_{\text{SCf}}$ we can also define a set of nilpotent elements by

$$\text{Nil}[\mathbf{i}] = \{\mathbf{i} \in \mathbf{I}_{\text{SCf}} \mid \mathbf{i} \otimes \mathbf{i} = \mathbf{z}\}. \quad (4.20)$$

So the strong inclusions $\mathbf{z} \subset \text{Nil}[\mathbf{i}] \subset \mathbf{I}_{\text{SCf}}$ hold (in this connection see Ref. 27).

B. Matrix representation in projective superspace

The consideration of the projective superspace $\mathbb{C}P^{1,1}$ is principal and indispensable from various points of view (e.g., see Refs. 19, 24, 28). Here, we touch the question of the direct SCf symmetry restriction for linear transformations on $\mathbb{C}P^{1,1}$.

Let $X^T = (x, y, \eta) \in \mathbb{C}^{2,1}$ be the homogeneous coordinates on $\mathbb{C}P^{1,1}$. A general linear transformation

$$\tilde{X} = MX, \quad (4.21)$$

where

$$M = \begin{pmatrix} a & b & \alpha \\ c & d & \beta \\ \gamma & \delta & e \end{pmatrix} \quad (4.22)$$

corresponds to the following fractional linear transformation on $\mathbb{C}^{1,1}$ ($z = x/y, \theta = \eta/y$)

$$\begin{aligned}\tilde{z} &= \frac{az + b}{cz + d} + \theta \frac{(\beta a - \alpha c)z + \beta b - \alpha d}{(cz + d)^2}, \\ \tilde{\theta} &= \frac{\gamma z + \delta}{cz + d} + \theta \frac{(\beta \gamma + e c)z + \beta \delta + e d}{(cz + d)^2}.\end{aligned}\quad (4.23)$$

Using the SCf condition (3.7) we immediately derive the system of equations for M entries

$$e\beta \det B = 0, \quad (4.24a)$$

$$e \det B = \beta \det A, \quad (4.24b)$$

$$\beta \text{ per } A - e \text{ per } B = 2\alpha cd, \quad (4.24c)$$

$$\det A = \beta e / 2cd \text{ per } B + \gamma \delta + e^2, \quad (4.24d)$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (4.25)$$

and $\text{per } A = ad + bc$ is a permanent,²⁹ and $\epsilon[cd] \neq 0$. For convenience we have formally introduced here a "semimatrix"

$$B = \begin{pmatrix} \gamma & \delta \\ c & d \end{pmatrix}. \quad (4.26)$$

In general, B 's form no linear semigroup under usual matrix multiplication. For B analogs of the ordinary permanent and determinant read as

$$\text{per } B = \gamma d + \delta c, \quad (4.27)$$

$$\det B = \gamma d - \delta c, \quad (4.28)$$

where the first letters are replaced by Greek ones to mark off

that the introduced functions are odd. The useful properties of them are

$$\det B \cdot \text{per } B = 2cd\gamma\delta, \quad (4.29)$$

$$\det \begin{pmatrix} \det B & \text{per } B \\ \det A & \text{per } A \end{pmatrix} = 2cd \cdot \det C, \quad (4.30)$$

where

$$C = \begin{pmatrix} \gamma & \delta \\ a & b \end{pmatrix}. \quad (4.31)$$

Now let us turn to possible solutions of (4.24). First we observe the key relation dividing them into three types to be (4.24a). These types are determined by the ways of vanishing the factors in (4.24a) as follows

$$\beta \cdot \det B = 0, \quad (4.32a)$$

$$e \cdot \det B = 0, \quad (4.32b)$$

$$\beta e = 0. \quad (4.32c)$$

The first equation has the solution $\beta = \det B$ that gives the conventional projective transformation matrix (see Refs. 16, 24, 28)

$$M_1 = \begin{pmatrix} a & b & \frac{\det C}{\sqrt{\det A}} \\ c & d & \frac{\det B}{\sqrt{\det A}} \\ \gamma & \delta & \sqrt{\det A} - \frac{2}{3}\gamma\delta \end{pmatrix}, \quad (4.33)$$

having

$$\text{Ber } M_1 = \sqrt{\det A} + \frac{2}{3}\gamma\delta - 2\gamma\delta/\sqrt{\det A}. \quad (4.34)$$

The corresponding fractional linear transformation on $\mathbb{C}^{1,1}$ reads as^{14,28}

$$\begin{aligned} \bar{z} &= \frac{az + b}{cz + d} + \theta \frac{\gamma z + \delta}{(cz + d)^2} \sqrt{\det A}, \\ \bar{\theta} &= \frac{\gamma z + \delta}{cz + d} + \theta \frac{\sqrt{\det A} - \gamma\delta}{cz + d}, \end{aligned} \quad (4.35)$$

Ber taking the form

$$\text{Ber} (\tilde{Z}/Z) = \frac{\sqrt{\det A} - \gamma\delta}{cz + d} + \theta \frac{\det B}{(cz + d)^2}. \quad (4.36)$$

This clears the meaning of $\det B$: that is $\det B$ which controls θ dependence of Ber, while $\sqrt{\det A}$ is responsible for its body.

The second equation from (4.32) can be solved in two ways: (a) $e = \mu \cdot \det B$ and (b) $e = k\gamma\delta$. For the case (a) using (4.29) and (4.22) we obtain

$$M_2^{(a)} = \begin{pmatrix} a & b & \mu\delta\gamma + (\beta/2cd) \text{ per } A \\ c & d & \beta \\ \gamma & \delta & \mu \cdot \det B \end{pmatrix}, \quad (4.37)$$

where

$$\det A = \gamma\delta \quad (4.38)$$

and

$$\beta \det A = 0, \quad (4.39)$$

which means that $\beta \in \text{Ann}[\det A]$ (see Sec. IV A). The

proper ansatz to solve (4.38) is $a = a_0\gamma\delta$ and $b = b_0\gamma\delta$, where $a_0d - b_0c = 1$. The constraint (4.39) requires β to be proportional to γ , δ , or $\gamma\delta$ or to vanish. Taking this into account we find the soul transformation

$$\begin{aligned} \bar{z} &= \frac{a_0z + b_0}{cz + d} \gamma\delta + \theta \frac{\mu\gamma\delta}{cz + d}, \\ \bar{\theta} &= \frac{\gamma z + \delta}{cz + d} + \theta \frac{\mu \cdot \det B}{cz + d}, \end{aligned} \quad (4.40)$$

for which (4.16) yields

$$J_{\text{scr}}^{\text{soul}} = \frac{\det B}{cz + d} \left(\mu + \frac{\theta}{cz + d} \right). \quad (4.41)$$

While comparing (4.36) with (4.41) we conclude that for the soul transformations $\det B$ plays the part analogous to that of $\sqrt{\det A}$ for T_{body} .

If another solution [(b) case] of (4.32b) is taken, then we obtain

$$M_2^{(b)} = \begin{pmatrix} a & b & (\beta/2cd) \text{ per } A \\ c & d & \beta \\ \gamma & \delta & k\gamma\delta \end{pmatrix}, \quad (4.42)$$

where (4.38) and (4.39) hold valid.

The third equation in (4.32) can be solved by $e = \beta\mu$. It follows from (4.24b) and (4.24d) that $\det A = \mu \cdot \det B = \gamma\delta$ which yields

$$M_3 = \begin{pmatrix} a & b & (\beta/2cd) \text{ per } A \\ c & d & \beta \\ \gamma & \delta & -(\beta/2cd) \text{ per } B \end{pmatrix}. \quad (4.43)$$

Using the ansatz analogous to that of (4.40) we find another type of the soul transformation on $\mathbb{C}^{1,1}$

$$\begin{aligned} \bar{z} &= \frac{a_0z + b_0}{cz + d} \gamma\delta (1 + \theta\beta), \\ \bar{\theta} &= \frac{\gamma z + \delta}{cz + d} + \theta\beta \cdot \det B \cdot \frac{cz - d}{(cz + d)^2}, \end{aligned} \quad (4.44)$$

having (4.16) in the form

$$J_{\text{scr}}^{\text{soul}} = - \frac{\det B}{(cz + d)^2} (\beta(cz - d) + \theta), \quad (4.45)$$

which also confirm the above remark after (4.41).

C. Superdistance

A superdistance between two points on $\mathbb{C}^{1,1}$ is usually defined as (Refs. 8 and 14) $|Z_{12}|$, where $Z_{12} = z_1 - z_2 - \theta_1\theta_2$ is a covariant quantity under the projective (body) transformations (4.35). At first sight the conventional relation¹⁹

$$\tilde{Z}_{12} = J_{\text{scr}}(Z_1)J_{\text{scr}}(Z_2)Z_{12} \quad (4.46)$$

could be extended for all T_{scr} including both body and soul transformations by means of (4.6). But let us dwell on this point in detail. Using (2.1) we can present the lhs of (4.46) in such a way

$$\begin{aligned} \tilde{Z}_{12} = & f(z_1) - f(z_2) - \psi(z_1)\psi(z_2) \\ & + (\theta_1 g(z_1) + \theta_2 g(z_2))(\psi(z_1) - \psi(z_2)) \\ & - \theta_1 \theta_2 g(z_1)g(z_2). \end{aligned} \quad (4.47)$$

For one thing we notice that any fractional linear $f(z) = (az + b)/(cz + d)$ and $\psi(z) = (\gamma z + \delta)/(cz + d)$ being responsible for the first two columns of (4.22) obey the relations

$$f(z_1) - f(z_2) = R \cdot \det A \cdot (z_1 - z_2), \quad (4.48a)$$

$$\psi(z_1) - \psi(z_2) = R \cdot \delta \det B \cdot (z_1 - z_2), \quad (4.48b)$$

$$\psi(z_1)f(z_2) - \psi(z_2)f(z_1) = R \cdot \delta \det C \cdot (z_1 - z_2), \quad (4.48c)$$

$$\psi(z_1) \cdot \psi(z_2) = R\gamma\delta(z_1 - z_2), \quad (4.48d)$$

where $R^{-1} = (cz_1 + d)(cz_2 + d)$ [see also (4.28) and (4.31)]. Then for (4.47) we have

$$\begin{aligned} \tilde{Z}_{12} = & R (\det A - \gamma\delta)(z_1 - z_2) \\ & + (\theta_1 g(z_1) + \theta_2 g(z_2))R \cdot \delta \det B \cdot (z_1 - z_2) \\ & - \theta_1 \theta_2 g(z_1)g(z_2), \end{aligned} \quad (4.49)$$

and so $g(z)$ distinguishes between further particular cases, e.g., for the body transformations (4.35) we choose $g(z) = \sqrt{\det A - \gamma\delta}/(cz + d)$. Turning to the soul transformations we first observe that (4.38) implies the vanishing of θ -independent terms in (4.49), while the nilpotency of $g(z)$ gives linearity of \tilde{Z}_{12} on the odd coordinates. Furthermore, for the first type of the projective soul transformations (4.40) $g(z)$ is proportional to $\delta \det B$ and for the others $g(z) \subset \text{Ann}[\delta \det B]$. Therefore, for the projective soul transformations we obtain

$$\tilde{Z}_{12} = 0, \quad (4.50)$$

which can be viewed as the definition of them: The superdistance between any soul-transformed points vanishes. Instead we see the relation

$$\delta \det \tilde{Q} = R \cdot \delta \det C \cdot (z_1 - z_2), \quad (4.51)$$

where

$$Q = \begin{pmatrix} \theta_1 & \theta_2 \\ z_1 & z_2 \end{pmatrix},$$

hold for all projective soul transformations. Moreover, for the transformation (4.40) the differences of the transformed even and odd coordinates for the same points are proportional

$$\begin{aligned} \tilde{z}_1 - \tilde{z}_2 = & R \cdot \det A \cdot Z_{12}^{\mu}, \\ \tilde{\theta}_1 - \tilde{\theta}_2 = & R \cdot \delta \det B \cdot Z_{12}^{\mu}, \end{aligned} \quad (4.52)$$

where

$$Z_{12}^{\mu} = z_1 - z_2 + c \cdot \delta \det Q \cdot \mu + (\theta_1 - \theta_2)\mu d, \quad (4.53)$$

contains three types of distances appearing in the lhs of (4.51) and (4.52). The analogous relations can be derived for the transformation (4.44) as follows

$$\begin{aligned} \tilde{z}_1 - \tilde{z}_2 = & R \cdot \det A \cdot Z_{12}^{\beta}, \\ \tilde{\theta}_1 - \tilde{\theta}_2 = & R \cdot \delta \det B \cdot Z_{12}^{\beta}, \end{aligned} \quad (4.54)$$

where

$$\begin{aligned} Z_{12}^{\beta} = & z_1 - z_2 + (\theta_1 (cz_1 - d)(cz_2 + d) \\ & - \theta_2 (cz_2 - d)(cz_1 + d))(\beta/2cd). \end{aligned} \quad (4.55)$$

Thus we conclude that the introduced odd analogs of permanent and determinant [see (4.27) and (4.28)] formally play the parity-dual part for some soul-transformed quantities. Meanwhile, the ‘‘semimatrix’’ (4.26) in itself can represent the following ‘‘parity-twisting’’ linear mappings: $\mathbb{C}^{2,0} \rightarrow \mathbb{C}^{1,1}$ with reversed parity and $\mathbb{C}^{0,2} \rightarrow \mathbb{C}^{1,1}$ with usual one.

V. TPt TRANSFORMATIONS

A TPt transformation is also determined by two functions on $\mathbb{C}^{1,0}$ from the odd sector while the others can be found from the TPt condition (3.8) written as

$$f'(z) = \psi'(z)\psi(z) \equiv f'_{\text{soul}}(z), \quad (5.1)$$

$$\chi'(z) = g'(z)\psi(z) - g(z)\psi'(z) \quad (5.2)$$

[cf. (4.1) and (4.14)]. First we note that all TPt transformations are noninvertible and become degenerated after the body mapping just as T_{soul} (4.13). Their classification can also be provided in terms of $g(z)$. If $\epsilon[g(z)] \neq 0$, then Ber is a well defined although pure soul, which gives the transformation of type II [see (3.6)], while $\epsilon[g(z)] = 0$ defines that of type III. The special case of the latter is $g(z) = \psi'(z)\psi(z)$ which turns the rhs of (5.2) to zero and yields

$$\begin{aligned} \tilde{z} = & f_{\text{soul}}(z) + \theta\alpha, \\ \tilde{\theta} = & \psi(z) + \theta\psi'(z)\psi(z) \end{aligned} \quad (5.3)$$

[cf. (4.17)]. The fractional linear variant of (5.3) is

$$\begin{aligned} \tilde{z} = & \delta\gamma/c(cz + d) + k + \theta\alpha, \\ \tilde{\theta} = & (\gamma z + \delta)/(cz + d) + \theta[\gamma\delta/(cz + d)^2]. \end{aligned} \quad (5.4)$$

As was already pointed out TPt transformations do not form a semigroup since the composition law for U_{TPt} (3.17) is not closed in general

$$U_{\text{TPt}} \tilde{U}_{\text{TPt}} = \begin{pmatrix} D^2 \tilde{\theta} \tilde{\Delta} & D^2 \tilde{\theta} \tilde{D} \tilde{\theta} \\ D \tilde{\theta} \tilde{\Delta} & D \tilde{\theta} \tilde{D} \tilde{\theta} + \Delta \tilde{D}^2 \tilde{\theta} \end{pmatrix}, \quad (5.5)$$

where $\tilde{\Delta}$ is given by (3.17) but for $\tilde{Z} \rightarrow \tilde{\tilde{Z}}$ transition. It should be noted that the additional condition $D^2 \tilde{\theta} \subset \text{Ann}[\tilde{\Delta}]$ can determine a TPt semigroup. But in the case $D \tilde{\theta} \subset \text{Ann}[\tilde{\Delta}]$ we obtain the SCf matrix (3.10). This allows us to conclude that properly restricted TPt transformations could be interpreted as the non-SCf ‘‘square root’’ of the soul SCf transformations. Indeed, if we relate a TPt transformation to $\mathfrak{t}[g, \psi]$, then

$$\mathfrak{t}[g, \psi] \otimes \mathfrak{t}[g, \psi] = \mathfrak{i}[gg \circ f + g' \circ f \chi \psi + \chi \psi' \circ f, \psi \circ f + \psi g \circ f], \quad (5.6)$$

while the extra conditions are

$$\begin{aligned} g(\chi \circ f - \psi \circ f g \circ f - (g \circ f)^2 \psi) = & 0, \\ \psi'(\chi \circ f - \psi \circ f g \circ f - (g \circ f)^2 \psi) = & 0, \\ -g(g \circ f)(gg \circ f + 2\chi \psi g' \circ f) = & 0. \end{aligned} \quad (5.7)$$

In the projective superspace $\mathbb{C}P^{1,1}$ the TPt condition (3.8) formally gives the relations for M entries [see (4.21) and (4.22)]

$$\det A = \gamma\delta, \quad (5.8)$$

$$\det D = \gamma e, \quad (5.9)$$

where

$$D = \begin{pmatrix} \alpha & \beta \\ a & c \end{pmatrix} \quad (5.10)$$

[cf. (4.38)]. Lend attention to the facts that now there are two equations for M entries in contrast with the SCf case (4.24) and (5.8) coincides with the SCf condition (4.38), which underlines their similarity proving (5.6) value.

Now the superdistance between two points on $\mathbb{C}^{1,1}$ is the TPt image of the odd distance

$$\tilde{Z}_{12} = R (\det E - e\delta) (\theta_1 - \theta_2), \quad (5.11)$$

where

$$E = \begin{pmatrix} \alpha & \beta \\ b & d \end{pmatrix} \quad (5.12)$$

Here, we assume that the parameters are chosen to vanish θ -squared term in (4.47).

In searching for the analogies with the supersymmetrization of the upper half-plane and the Poincaré metric^{14,30} we take M entries to be real and (5.11) to be infinitesimal, which yields

$$d\tilde{Z} = \frac{\delta \det E - e\delta}{|cz + d|^2} d\theta \quad (5.13)$$

[cf. (3.16)]. Further, from (5.11) it follows that

$$\text{Im } \tilde{z} + \frac{1}{2} \tilde{\theta}\tilde{\theta} = \frac{\delta \det E - e\delta}{|cz + d|^2} \text{Im } \theta \quad (5.14)$$

and hence the key relation takes place (obviously we cannot use division)

$$|d\tilde{Z}| \text{Im } \theta = |d\theta| (\text{Im } \tilde{z} + \frac{1}{2}\tilde{\theta}\tilde{\theta}), \quad (5.15)$$

which leads to the following TPt analog of the Poincaré metric ds satisfying simultaneously

$$|ds| \text{Im } \theta = |d\theta| \quad (5.16)$$

and

$$|ds| (\text{Im } \tilde{z} + \frac{1}{2}\tilde{\theta}\tilde{\theta}) = |d\tilde{Z}|. \quad (5.17)$$

Very likely the relations (5.13)–(5.15) could be viewed as the definition of the “TPt invariance” of the introduced “metric.”

VI. CONCLUSION

In this paper we have introduced and studied the semigroup unifying both invertible (body) and noninvertible (soul) transformations on $\mathbb{C}^{1,1}$ which satisfy the same SCf condition. The body transformations form the disjoint subgroup of the semigroup, while the soul transformations are in the maximal ideal of it which also contains a set of nilpotent transformations differing from zero. The soul transformations are partial ones having degenerated second projection and nonvanishing defect after the body mapping. So they are not “body preserving”³¹ and do not admit an infinitesimal form, therefore superderivation algebras³² cannot be defined for them. Furthermore, the soul transformations describe transitions from pure body into pure soul superdo-

main and hence could be interpreted as a “one-way bridge” between the body and soul worlds. In this respect, using the procedure similar to the usual gluing of superdomains when constructing an SRS locally^{14,20} one could try to build a suggesting itself analogous object by means of body and soul transformations, which may be worthwhile due to, e.g., Crane and Rabin’s “a general SRS need not have a body at all.”¹⁴

We have also brought in the transformations twisting the parity of a tangent space. Those can be viewed as a non-SCf square root of the soul SCf transformations. Some analogies with the Poincaré metric and conformal invariance have been outlined for them.

Almost all results obtained can be extended on $N > 1$. Some steps in this direction have already been done.³³ Finally, it should be concluded that a further thorough study is desirable to understand the role of semigroups in supersymmetric theories.

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