THE SEARCH FOR A HOLOGRAPHIC DUAL TO $AdS_3 \times S^3 \times S^3 \times S^1$

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The problem of finding a holographic CFT dual to string theory on $AdS_3 \times \mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^1$ is examined in depth. This background supports a large $\mathcal{N} = 4$ superconformal symmetry. While in some respects similar to the familiar small $\mathcal{N} = 4$ systems on $AdS_3 \times \mathbf{S}^3 \times K3$ and $AdS_3 \times \mathbf{S}^3 \times T^4$, there are important qualitative differences. Using an analog of the elliptic genus for large $\mathcal{N} = 4$ theories we rule out all extant proposals – in their simplest form – for a holographic duality to supergravity at generic values of the background fluxes. Modifications of these extant proposals and other possible duals are discussed.

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1. Introduction and Summary

The AdS/CFT correspondence has been a powerful tool in understanding nonperturbative string theory (for a review, see [1]). This is especially true in two dimensions, due to the infinite dimensional structure of the conformal group. The examples most studied are the conformal field theories dual to type II string theory on geometries of the form $AdS_3 \times \mathbf{S}^3 \times \mathcal{M}$, with $\mathcal{M} = K3$ or T^4. These geometries arise from the near-horizon limit of Q_1 onebranes coincident with Q_5 fivebranes, with the fivebranes wrapping $\mathcal{M}$ and the onebranes transverse to $\mathcal{M}$. The dual CFT's obtained in this way are sigma models on the moduli space of Q_1 instantons in $U(Q_5)$ gauge theory on $\mathcal{M}$. They possess *small* $\mathcal{N} = (4,4)$ superconformal symmetry, in which the four (anti)holomorphic supercurrents are charged under a single $SU(2)$ R-symmetry current. They also form a doublet under a global, custodial $SU(2)$ R-symmetry. U-duality implies [2,3] that the CFT's for different Q_1, Q_5 having the same product $N = Q_1 Q_5$, are different descriptions of the same theory appropriate to different asymptotic regimes of its moduli space. These CFT's are all deformations of the much-studied symmetric product orbifold $Sym^N(\mathcal{M})$ [4].

Type II string theory also has a solution with the geometry $AdS_3 \times \mathbf{S}^3_+ \times \mathbf{S}^3_- \times \mathbf{S}^1$, where the three-spheres $\mathbf{S}^3_\pm$ are threaded by integral fivebrane flux $Q_5^\pm$, and there is also a onebrane charge Q_1 [5-9]. This solution is distinguished in having 16 Killing spinors and a corresponding *large* $\mathcal{N} = (4,4)$ superconformal symmetry. Large $\mathcal{N} = 4$ supersymmetry is distinguished from its small counterpart in that both $SU(2)$ R-symmetries under which the supercharges transform give rise to current algebras (at levels $k^\pm$ related to the background fluxes). Despite this enhanced symmetry, this example is much less well understood than that of its $AdS_3 \times \mathbf{S}^3 \times K3$ or $AdS_3 \times \mathbf{S}^3 \times T^4$ cousins. In particular, the holographic dual has not been established.

For the special case $Q_5^+ = Q_5^- \equiv Q_5$, a seemingly obvious candidate dual is obtained by replacing $K3$ or T^4 with $\mathbf{S}^3 \times \mathbf{S}^1$ in the symmetric product CFT. This was first suggested in [8], and further studied and elaborated in [9]. More specifically one takes (deformations of) the symmetric product $Sym^{Q_1 Q_5}(\mathcal{S})$, where $\mathcal{S} \sim \mathbf{S}^3 \times \mathbf{S}^1$ is the supersymmetric $U(2)$ WZW model with central charge $c = 3$. $\mathcal{S}$ can be described by a free boson and four free fermions and is the smallest large $\mathcal{N} = 4$ CFT. Many aspects of this construction appear promising. First, it carries the large $\mathcal{N} = (4,4)$ superconformal symmetry and has a central charge $c = 6Q_1Q_5$ which agrees with the Brown-Henneaux formula [10] as applied to $AdS_3 \times \mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^1$. It has the small RR-sector gap (of order $\frac{1}{Q_1 Q_5}$) required for agreement with black hole thermodynamics [11], and the low-lying states of the Hilbert space have the structure of a Fock space, much like supergravity/string theory quanta in the AdS background. Indeed it is hard to see how one could satisfy these requirements in any way other than with a Q_1Q_5-fold symmetric product. Given the assumption of a Q_1Q_5-fold symmetric product, $\mathcal{S}$ is the only game in town with the required central charge $c = 3$. On top of this, we match the CFT and supergravity moduli as well as the indices (as far as they can be compared) in the sector of the theory with zero $\mathbf{S}^1$ charge.

Despite these promising features, this proposed duality has a fatal flaw (for generic Q_5) in its simplest form. The basic problem is that $Sym^{Q_1Q_5}(\mathcal{S})$ depends only on the product Q_1Q_5, while the natural formulation of string theory on $AdS_3 \times \mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^1$ depends on Q_1, Q_5 separately, as we will deduce from by comparing a certain index of the conformal field theory with a partition function of the supergravity theory. [a] The $K3$ and T^4 cases are rescued from such a contradiction by a large U-duality group which relates all theories with the same value for the product Q_1Q_5. In striking contrast, we find in Section 2 below that the U-duality group is extremely limited for $\mathbf{S}^3 \times \mathbf{S}^1$ and does not relate theories of the same central charge and different Q_1, Q_5. It is possible that this difficulty may be overcome by some kind of modification or twisting of the symmetric product but we do not have a concrete suggestion.

The considerations of the preceding paragraph do not rule out the possibility of a 'duality' to $Sym^{Q_1Q_5}(\mathcal{S})$ when $Q_5 = 1$.[b] One can generalize this proposal to the case where only one of $Q_5^{\pm}$ equals one; then the symmetric product $Sym^{Q_1}(\mathcal{S})$ is still a viable candidate (one of the $SU(2)$ R-symmetries of the component $\mathbf{S}^3 \times \mathbf{S}^1$ CFT is then a current algebra of level $Q_5' > 1$). For general $Q_5^+ \neq Q_5^-$ there is not even a full conjecture for a dual. (An interesting and tentative partial proposal was made in [9].)

For general values of Q_5 alternatives should be considered. One possibility is the low-energy dynamics of fivebranes wrapped on $\mathbf{S}^3 \times \mathbf{S}^1$. The gauge theory and related supergravity solutions for Q_5 fivebranes wrapped on a special Lagrangian $\mathbf{S}^3$ threaded by Q_5' units of three-form flux, were

[a] More precisely, the index of $Sym^N(\mathcal{S})$ depends on all the prime factors of N "democratically" but the supergravity depends on the particular factorization $N = Q_1Q_5$.

[b] Such a duality may well ultimately make sense, but at present it is not so well-defined because supergravity is strongly coupled when $Q_5 = 1$. There may be a duality to a bulk string theory when $Q_5 = 1$, but at our current level of string technology this is not well-understood – even in the NS case there are singularities [2]. Nevertheless in this paper we shall continue to speak of a $Q_5 = 1$ duality with the idea that the difficulties on the bulk side may eventually be overcome.

considered in [12-15]. The worldvolume of the fivebranes is $\mathbb{R}^{1,2} \times \mathbf{S}^3$ with a warp factor for the $\mathbf{S}^3$. The solution in Section 3.1.1 of [12] has 1/16 supersymmetry, and $SU(2)^3$ symmetry. We conjecture that, with $\mathbb{R}^{1,2}$ compactified to $\mathbb{R}^{1,1} \times \mathbf{S}^1$ and Q_1 instantons on $\mathbf{S}^3 \times \mathbf{S}^1$, the theory will flow in the IR to a sigma model with large $\mathcal{N} = (4,4)$ superconformal symmetry, i.e. 1/2 supersymmetry and its associated $SU(2)^4$ global symmetry. This sigma model should be closely related to the sigma model on the moduli space of Q_1 instantons in $U(Q_5)$ gauge theory on $\mathbf{S}^3 \times \mathbf{S}^1$. This sigma model has not been studied (some relevant mathematical results can be found in [16]); indeed, it is not known if this model has large $\mathcal{N} = 4$ supersymmetry.

The difficulties in establishing a holographic duality might seem surprising. One might have expected that the enhanced large $\mathcal{N} = 4$ supersymmetry would give greater control for this case. While that may ultimately prove correct, there are substantial qualitative differences between large and small $\mathcal{N} = 4$ which prevent us from drawing on the familiar bag of tricks. To name a few:

1. The BPS bound is nonlinear in the charges and implies that some BPS states must get mass corrections at every order in perturbation theory.
2. The large $\mathcal{N} = 4$ algebra has a finite dimensional $\mathcal{N} = 4$ superconformal subalgebra $D(2,1|\alpha)$. However, BPS states of the global $D(2,1|\alpha)$ subalgebra are not in general BPS states of the large $\mathcal{N} = 4$ super Virasoro algebra.
3. There can be any number – odd or even – of moduli, and there are few known constraints on the moduli space geometry.[c]

Even so, we will report on progress in understanding both sides of the correspondence.

On the supergravity side, we revisit in Section 2 the solution of the supergravity equations of motion on this background, for both NS and R background fluxes. We determine the massless moduli, which can be parametrized by the string coupling g_s and (in the IIB theory) a linear combination of RR axion C_0 and four form C_4 . The radius of the $\mathbf{S}^1$ is determined in terms of g_s and the charges.[d] We discuss the global structure of the moduli space, the low-energy descriptions appropriate to various regimes, and the locus in moduli space where the CFT becomes singular. In

[c] We will demonstrate one constraint in Section 4.4 – that the moduli space is a real slice of a self-mirror $\mathcal{N} = 2$ theory, which is also fixed under the mirror map.

[d] This formula differs from the one in [9].

section 3 we discuss the relation of the solution to the near-horizon geometry of intersecting branes.

On the CFT side, we review in Section 4 (following [17-22]) the large $\mathcal{N} = 4$ superconformal algebra and its representation theory. We demonstrate the fact mentioned above, that the BPS bound of large $\mathcal{N} = 4$ superconformal symmetry in general differs from that of its global subalgebra $D(2,1)$ which comprises the super-isometries of $AdS_3 \times \mathbf{S}^3_+ \times \mathbf{S}^3_-$. We exhibit the general structure of marginal deformations, and we examine the question of whether an $h = 1/2$ chiral primary field generates a modulus that preserves large $\mathcal{N} = 4$. In yet another surprise, Dixon's proof of this fact for $\mathcal{N} = (2,2)$ [23] does not immediately apply to the case of large $\mathcal{N} = (4,4)$ supersymmetry. We will nevertheless find an appropriate generalization of Dixon's proof which does apply to large $\mathcal{N} = 4$.

We also introduce an index for theories with large $\mathcal{N} = 4$ supersymmetry, with rather remarkable properties: The index is not a number, rather it is a nontrivial modular form. Consequently, the analogue of the elliptic genus is not holomorphic. We introduce the index and the analogue of the elliptic genus in Section 4 and evaluate the index in Section 6 for the symmetric product $Sym^N(\mathbf{S}^3 \times \mathbf{S}^1)$. Detailed derivations and further discussion of these indices will be the subject of a companion paper [24]. The related BPS spectrum and the moduli of the symmetric product are exhibited in Section 5.

Sections 7 and 8 analyze the BPS and near-BPS spectra of supergravity and compare them to the symmetric product. We find that the BPS spectra do not match, in that the one-particle states of the classical supergravity limit with different spins $\ell^+ \neq \ell^-$ on $\mathbf{S}^3_\pm$ do not have a BPS counterpart in the symmetric product (this was already noted in [8] for a special case). This might indicate that such states are not protected by large $\mathcal{N} = 4$ supersymmetry (assuming that the correct dual has been identified). Indeed, as mentioned above, the BPS bound already requires that the masses receive perturbative corrections; furthermore, we show that the (BPS) short multiplets of supergravity occur in combinations that can naturally pair up into (non-BPS) long multiplets, so there is no reason *a priori* that they should survive across moduli space. The near-BPS spectrum is of course also not protected, but in recent studies [25] has been seen to be remarkably robust. In our case, the spectrum provides an indication that the symmetric product orbifolds indeed only describe the situation where one of the fivebrane charges is one.

Finally, in Section 9 we discuss aspects of the $U(1) \times U(1)$ Chern–Simons gauge theory which appears in low-energy supergravity. A study of the associated topological field theory yields further constraints on the structure of the holographic dual, and provides further strong evidence that the symmetric product has $Q_5^+ = 1$ or $Q_5^- = 1$. Again, details and generalizations are deferred to another companion paper [26].

While our results should help guide the search for holographic duals for supergravity backgrounds with large $\mathcal{N} = 4$ supersymmetry, many open questions remain. To list a few:

i. What are the geometrical conditions on a sigma model target space in order that it admit large $\mathcal{N} = 4$ supersymmetry? The examples discussed to date are based on current algebra cosets [18]. Are all models with large $\mathcal{N} = 4$ automatically conformally invariant, as is the case for small $\mathcal{N} = 4$?

ii. What is the geometrical interpretation of the large $\mathcal{N} = 4$ index?

iii. Does the sigma model on the moduli space of Q_1 instantons in $U(Q_5)$ gauge theory on $\mathbf{S}^3 \times \mathbf{S}^1$ have large $\mathcal{N} = 4$ supersymmetry? Is it a viable dual for $Q_5^+ = Q_5^-$?

iv. Are there possible alternatives to the naive orbifold $Sym^N(\mathbf{S}^3 \times \mathbf{S}^1)$ (Making use, for example, of discrete torsion, extensions of the orbifold group S_N, asymmetric shifts on the $\mathbb{R}$ factor, etc.), which could serve as candidate duals? For $Q_5^+ = Q_5^-$, are such orbifold theories on the moduli space of the sigma model proposed in *iii.*

v. What can we say about the (Zamolodchikov) metric and the corresponding geometry of moduli space as a consequence of large $\mathcal{N} = 4$ superconformal symmetry?

vi. The new large $\mathcal{N} = 4$ index predicts "long string" BPS states. What are the corresponding geometries/bulk states? (A natural conjecture is that they are generalizations of the supertube solutions found in [27].)

vii. The intersecting D-brane configurations that naively give rise to large $\mathcal{N} = 4$ supersymmetry have chiral fermions bound to the intersection. What is their role, and does their presence imply any constraint on the CFT dual? Do they decouple, as we will assume below? (See the discussion near equation (3.5) below.)

These and many other questions remain for future research.

2. Supergravity solutions

2.1. *Type II conventions*

The IIB Lagrangian is [a]

$$\frac{2\pi}{g_B^2}\int\sqrt{-g}e^{-2\phi}\big(\mathcal{R}+4(\nabla\phi)^2\big)-\frac{\pi}{g_B^2}\int e^{-2\phi}H\wedge *H$$
$$-\pi\int R_1\wedge *R_1-\pi\int R_3\wedge *R_3-\frac{1}{2}\pi\int R_5\wedge *R_5+\pi\int C_4\wedge H\wedge F_3\ . \tag{2.1}$$

Here R_1 has integral periods; locally $R_1 = dC_0$. R_3 satisfies the Bianchi identity

$$dR_3 + R_1\wedge H = 0\ . \tag{2.2}$$

When $R_1 = dC_0$ can be trivialized then

$$F_3 = R_3 + C_0 H \tag{2.3}$$

is closed and has integral periods. R_5 has integral periods when $H = 0$, is self-dual and obeys

$$dR_5 = H\wedge F_3\ . \tag{2.4}$$

The IIA Lagrangian is similarly

$$\begin{aligned}&\frac{2\pi}{g_A^2}\int\sqrt{-g}\,e^{-2\phi}\big(\mathcal{R}+4(\nabla\phi)^2\big)-\frac{\pi}{g_A^2}\int e^{-2\phi}H\wedge *H\\&\qquad-\pi\int R_2\wedge *R_2-\pi\int R_4\wedge *R_4+\pi\int C_3\wedge dC_3\wedge H\,,\end{aligned} \tag{2.5}$$

where $R_4 = dC_3 - H\wedge C_1$.

We now look for AdS solutions to the equations of motion following from (2.1), (2.5) on $AdS_3\times\mathbf{S}^3\times\mathbf{S}^3\times\mathbf{S}^1$ with either NS or RR three-form background fluxes.

2.2. *Pure NS solutions*

We take $R_5 = 0, F_3 = 0$, $\phi = 0$ and

$$H = \lambda_0\omega_0 + \lambda_+\omega_+ + \lambda_-\omega_-\,, \tag{2.6}$$

[a] We set $\alpha' = \frac{1}{(2\pi)^2}$. In the notation of [28], we have $\kappa_{10}^2 = \frac{1}{4\pi}$, $\widetilde{F}_k = R_k$ and $\mu^p = 2\pi$.

where the volume forms

$$\begin{aligned} \omega_0 &= \mathrm{vol}(AdS_3) = (\ell/x_2)^3 dt \wedge dx^1 \wedge dx^2 \, , \\ \omega_\pm &= \mathrm{vol}(\mathbf{S}^3_\pm) \end{aligned} \tag{2.7}$$

are normalized so that $\int_{\mathbf{S}^3_\pm} \omega_\pm = 2\pi^2 R^3_\pm$. We take the metric

$$ds^2 = \frac{\ell^2}{{x_2}^2}\Big(-dt^2 + (dx_1)^2 + (dx_2)^2\Big) + R_+^2 ds^2(\mathbf{S}^3_+) + R_-^2 ds^2(\mathbf{S}^3_-) + L^2 (d\theta)^2 \tag{2.8}$$

with $\theta \sim \theta + 1$. The curvatures are

$$\begin{aligned} R_{\mu\nu\lambda\rho} &= -\ell^{-2}(g_{\mu\lambda} g_{\nu\rho} - g_{\mu\rho} g_{\nu\lambda}) \, , \\ \mathcal{R}_{\mu\nu} &= -2\,\ell^{-2} g_{\mu\nu} \, , \\ \mathcal{R} &= -6\,\ell^{-2} \ . \end{aligned} \tag{2.9}$$

Similarly, $ds^2(\mathbf{S}^3)$ is the round metric of $\mathbf{S}^3$ normalized as in the unit sphere in Euclidean $\mathbb{R}^4$. With this normalization we have curvatures:

$$\begin{aligned} R_{\mu\nu\lambda\rho} &= R^{-2}(g_{\mu\lambda} g_{\nu\rho} - g_{\mu\rho} g_{\nu\lambda}) \, , \\ \mathcal{R}_{\mu\nu} &= 2R^{-2} g_{\mu\nu} \, , \\ \mathcal{R} &= 6R^{-2} \ . \end{aligned} \tag{2.10}$$

We look for solutions with constant dilaton. The ϕ equation of motion then forces $H_{MNP}H^{MNP} = 0$,[b]

$$\frac{1}{6} H_{MNP} H^{MNP} = -\lambda_0^2 + \lambda_+^2 + \lambda_-^2 = 0 \ . \tag{2.11}$$

The stress-energy simplifies and $\mathcal{R} = 0$. The Einstein equations then give

$$\begin{aligned} \ell^{-2} &= \frac{1}{4}\,\lambda_0^2 \, , \\ R_+^{-2} &= \frac{1}{4}\,\lambda_+^2 \, , \\ R_-^{-2} &= \frac{1}{4}\,\lambda_-^2 \ . \end{aligned} \tag{2.12}$$

The fivebrane charges on the two $\mathbf{S}^3$s are

$$\int_{\mathbf{S}^3_\pm} H = Q_5^\pm = 4\pi^2 R_\pm^2 \tag{2.13}$$

[b] We absorb the constant mode of the dilaton in g_s, and set $\phi = 0$ at infinity.

with $Q_5^\pm$ integers. The fundamental string charge – also an integer – is

$$Q_1 = \frac{1}{g_B^2}\int *H = \frac{8\pi^4 R_+^3 R_-^3 L}{\ell g_B^2} \ . \tag{2.14}$$

In summary we have

$$\begin{aligned} \ell &= \frac{1}{2\pi}\sqrt{\frac{Q_5^+ Q_5^-}{Q_5^+ + Q_5^-}} \ , \\ R_\pm &= \frac{1}{2\pi}\sqrt{Q_5^\pm} \ , \\ L &= \frac{4\pi g_B^2 Q_1}{Q_5^+ Q_5^- \sqrt{Q_5^+ + Q_5^-}} \ . \end{aligned} \tag{2.15}$$

Note that the radius L and the string coupling g_B^2 are not separate moduli, rather their ratio is fixed by this relation in terms of the charge quanta.

The pure NS solution considered here can be constructed as an exact worldsheet conformal field theory, using products of $SU(2)$ level $Q_5^\pm$, $SL(2,R)$ level $\frac{Q_5^+ Q_5^-}{Q_5^+ + Q_5^-}$ and $U(1)$ WZW models [8]. This conformal field theory provides solutions of all the $d = 10$ superstring theories.

For the case of the IIA string almost the same equations apply. The result is exactly (2.15) with $g_B \to g_A$. Note that this makes good sense since under T-duality

$$L_B/g_B^2 = L_A/g_A^2 \tag{2.16}$$

and this is the quantity which is fixed when we have purely NS sector fluxes.

2.3. *Pure RR solutions*

The case of purely RR fluxes, which is related to the near-horizon geometry of the intersecting D1-D5-D5' system in the next section, is also of interest. The spacetime solution is easily obtained using the S-duality of the supergravity equations of motion, under which $g_B \to 1/g_B$, lengths are rescaled[c] by a factor of $\sqrt{g_B}$ and the integer NS charges (Q_1, Q_5^+, Q_5^-) become integer RR charges which we continue to denote (Q_1, Q_5^+, Q_5^-). The relations (2.15)

[c] Since they are referred to the string tension. The fundamental and D-string tensions differ by a factor of g_B, so a factor of $\sqrt{g_B}$ takes into account the change in conventions.

become

$$\ell = \frac{1}{2\pi}\sqrt{\frac{g_B Q_5^+ g_B Q_5^-}{g_B Q_5^+ + g_B Q_5^-}} \ ,$$
$$R_\pm = \frac{1}{2\pi}\sqrt{g_B Q_5^\pm} \ , \tag{2.17}$$
$$L = \frac{4\pi g_B Q_1}{g_B Q_5^+ g_B Q_5^- \sqrt{g_B Q_5^+ + g_B Q_5^-}} \ .$$

We have written the expression in a manner which emphasizes the fact that $R_\pm$, L and ℓ are finite in the $Q \to \infty$ limit with $g_B Q$ held fixed.

2.4. *Chern–Simons terms and central charges*

The central charge of the spacetime conformal field theory can be computed from an analysis of the algebra of diffeomorphisms near the conformal boundary of AdS_3 [10]; the result is

$$c = \frac{3\ell}{2G_N^{(3)}} \ . \tag{2.18}$$

By dimensional reduction (in the NS background) we have

$$\frac{1}{16\pi G_N^{(3)}} = \frac{8\pi^5 R_+^3 R_-^3 L}{g_B^2} = \frac{1}{2} Q_1 \sqrt{\frac{Q_5^+ Q_5^-}{Q_5^+ + Q_5^-}} \ . \tag{2.19}$$

and so

$$c = 6Q_1 \frac{Q_5^+ Q_5^-}{Q_5^+ + Q_5^-} \ . \tag{2.20}$$

Similarly, the left and right $SU(2) \times SU(2) \times U(1)$ isometries of $\mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^1$ yields a set of corresponding gauge fields from the Kaluza-Klein reduction of the metric and NS B-field. The action for these fields on AdS_3 contains Chern–Simons terms. (The abelian Chern–Simons term is discussed in more depth in in Section 9). The result is [29,30,31,32,33,34]

$$S = \frac{1}{16\pi G_N^{(3)}} \int d^3x \sqrt{-g}\Big(\mathcal{R}^{(3)} + \frac{2}{\ell^2}\Big) + \Big[\frac{Q_1 Q_5^+}{8\pi}\int \mathrm{Tr}\Big(\mathcal{A}_L^+ d\mathcal{A}_L^+ + \tfrac{2}{3}\mathcal{A}_L^{+\,3}\Big)$$
$$+ \frac{Q_1 Q_5^-}{8\pi}\int \mathrm{Tr}\Big(\mathcal{A}_L^- d\mathcal{A}_L^- + \tfrac{2}{3}\mathcal{A}_L^{-\,3}\Big) + \frac{Q_1}{8\pi}\int A_L dA_L\Big] - (L \leftrightarrow R) \tag{2.21}$$

where $\mathcal{A}^{\pm}_{L,R}$ are the gauge fields in AdS_3 that transform under left- and right-handed $SU(2)$ isometries of $\mathbf{S}^3_{\pm}$, and $A_{L,R}$ are the corresponding $U(1)$ gauge fields for $\mathbf{S}^1$. We have also included the 3d Einstein term, which can be written as a Chern–Simons form [35]. These Chern–Simons terms enforce the integer quantization of the background charges Q_1, $Q_5^{\pm}$.

While we have isolated this apparently three-dimensional action for the bosonic modes of an AdS_3 supergravity, it is important to note that the radii of $\mathbf{S}^3_{\pm}$ are typically of the same order as the curvature radius of AdS_3, and set the scale of the masses of KK modes. There is no sense in which the bulk theory is effectively 2+1 dimensional; the reason for exhibiting the Chern–Simons forms (2.21) is to manifest the central extensions of the various current algebras in the spacetime CFT. Indeed, the dual CFT contains left and right $SU(2)\times SU(2)\times U(1)$ current algebras; the SU(2) current algebras are at levels $k_{\pm} = Q_1 Q_5^{\pm}$, possibly up to $O(1)$ corrections that are invisible in the classical supergravity limit.[d] In the gauge field equations of motion (see for example [36]), the Chern–Simons term gives mass to half of the components, such that their lowest modes have conformal weight $(h_L, h_R) = (1,2)$ or $(2,1)$; the lowest modes of the other components are the 'singleton' modes of weight $(1,0)$ or $(0,1)$, dual to the respective $(0,1)$ and $(1,0)$ $SU(2)\times SU(2)\times U(1)$ currents $j_{L,R}$ of the dual CFT via the usual boundary coupling

$$\int_{\partial AdS_3} \Big(\mathcal{A}_L j_R + \mathcal{A}_R j_L \Big) \ . \tag{2.22}$$

The spacetime supersymmetry of the background requires a supersymmetric completion of this $SU(2)\times SU(2)\times U(1)$ current algebra, and an action of two-dimensional conformal symmetry. The supersymmetry currents must transform as $(\frac{1}{2},\frac{1}{2})$ under $SU(2)\times SU(2)$. The only known algebra with these properties is the large $\mathcal{N}=4$ superconformal algebra of [17] with generators

$$T\ ;\quad G^a\ ;\quad A^i_+\ ,\quad A^i_-\ ,\quad U\ ;\quad Q^a \tag{2.23}$$

where $a = 0,1,2,3$ and $i = 1,2,3$. The currents A_+, A_-, and U are dual to the gauge fields $\mathcal{A}^+$, $\mathcal{A}^-$, and A, respectively. The supersymmetry generated by G^a relates the $U(1)$ current U to a set of four free fermions Q^a, which are thus required for completion of the algebra.

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[d] The D-brane analysis of the next section provides evidence that there are no such $O(1)$ corrections.

We will describe this large $\mathcal{N}=4$ algebra in more detail below in Section 4. However, at this point we wish to point out a surprise: In contrast to other supergravity backgrounds based on AdS_3, the large $\mathcal{N}=4$ superalgebra does *not* have a realization as an Chern–Simons-type AdS_3 supergravity — at least not an obvious one.[e] Extended AdS_3 supergravities can be written as Chern–Simons theories [37] with gauged supergroup containing $SL(2,R)_{L,R}$ factors for the isometries of AdS_3, as well as factors for the gauged R-symmetry (in this case $(SU(2)_+ \times SU(2)_-)_{L,R}$). The unique supergroup with this bosonic subalgebra and fermionic generators transforming as $(\frac{1}{2},\frac{1}{2})$ is the supergroup $D(2,1|\alpha)$. In AdS_3 supergravities with $\mathcal{N}=0,1,2,3$ or small $\mathcal{N}=4$ supersymmetry, there is a Chern–Simons action using the super-isometry group $\mathcal{I}$; the superconformal algebra of the spacetime CFT is a Hamiltonian reduction of the affinization $\widehat{\mathcal{I}}$ imprinted on the boundary by Chern–Simons gauge transformations. However, the symmetry generators (2.23) are *not* a Hamiltonian reduction of the currents of affine $D(2,1|\alpha)$ (although (2.23) contains $D(2,1|\alpha)$ as a subalgebra). [f]

In fact, it is easy to see that there is no finite dimensional superalgebra that could serve as the basis for an AdS_3 supergravity corresponding to the large $\mathcal{N}=4$ algebra. Such an algebra would have to contain the $D(2,1|\alpha)$ subalgebra generated by $L_{\pm1}$, L_0, $G^a_{\pm1/2}$, and $A^{\pm,i}_0$. Adding the zero mode of the $U(1)$ current, U_0, then requires us to add the fermion modes $Q^a_{-1/2}$ by supersymmetry. But then the anticommutator $\{Q^a_{-1/2}, G^b_{-1/2}\}$ includes $A^{\pm,i}_{-1}$, and so on – we end up generating the entire large $\mathcal{N}=4$ algebra.

2.5. *Moduli*

In this subsection we analyze the moduli of the solution. It turns out to be simplest to analyze the pure NS form of the solution. Since the action is even in RR fields, the linearized equations of motion which determine the number of massless moduli do not mix RR and NS fluctuations. Hence the two possible types of moduli can be analyzed separately.

We begin with the NS fluctuations.

1. *The metric.* The only possible scalar fluctuations of the metric are

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[e] Thus providing another reason why (2.21) is not the whole story when we wish to compare the spacetime CFT with the effective supergravity theory.

[f] We should note, however, that ref. [38] shows that the Hamiltonian reduction of the affinization of $D(2,1|\alpha)$ leads to $\widetilde{\mathcal{A}}_\gamma$.

parameterized as

$$ds^2 = \frac{\ell^2}{{x_2}^2}\Big(-dt^2 + (dx_1)^2 + (dx_2)^2\Big) + \frac{Q_5^+}{4\pi^2}\, ds^2\,(\mathbf{S}^3_+) + \frac{Q_5^-}{4\pi^2}\, ds^2\,(\mathbf{S}^3_-) + (L + \delta L(x))^2 (d\theta)^2\,, \tag{2.24}$$

where $\delta L(x)$ is a scalar depending only on the AdS_3 coordinates. It is clear from the construction of the solution as a worldsheet CFT that the $\mathbf{S}^3$ and AdS_3 radii cannot be moduli because they appear as levels of WZW models.

2. *The dilaton.* We also get a scalar $\phi(x)$, whose zero mode we have absorbed into the string coupling g_B. However we have already seen in equation (2.15) or (2.17) that this is not a separate modulus, but rather is fixed in terms of the $\mathbf{S}^1$ radius and the charges. A direct Kaluza-Klein reduction reveals a mass for fluctuations which change the sizes of the $\mathbf{S}^1$ and $\mathbf{S}^3$ radii (2.15) or (2.17).
3. *The NS B-field.* Again there are no possible moduli here as the H fluxes are quantized.

Now we consider possible RR moduli in the IIA context in order to avoid subtleties related to the self-duality of the RR four-form C_4 in the IIB description. The RR equations of motion following from (2.5) are

$$d * dC_1 + H \wedge *(dC_3 - H \wedge C_1) = 0\,, \tag{2.25}$$

$$d * (dC_3 - H \wedge C_1) - dC_3 \wedge H = 0\ . \tag{2.26}$$

Our ansatz is

$$C_1 = c_1 + \sigma d\theta\,, \tag{2.27}$$

where c_1 is a 1-form on AdS_3 and σ is a scalar on AdS_3. Even though it is a gauge field and not a scalar we include c_1 at this point because it eats one of the scalars. We also take

$$C_3 = \alpha_+(x)\omega_+ + \alpha_-(x)\omega_-\,, \tag{2.28}$$

where $\alpha_\pm$ are scalars. Because of large C_3 field gauge transformations, they are periodic scalars (more on this below).

Choosing the orientation to be $\omega_0 \wedge \omega_+ \wedge \omega_- \wedge d\theta$ we obtain

$$\begin{aligned} &\nabla^2\sigma = 0\,, \\ &d *_3 dc_1 + \lambda_+ *_3 (d\alpha_+ + \lambda_+ c_1) + \lambda_- *_3 (d\alpha_- + \lambda_- c_1) = 0 \end{aligned} \tag{2.29}$$

from (2.25). Here $\nabla^2\sigma = *d * d\sigma/\omega_0$. We also get

$$\begin{aligned} &d *_3 (d\alpha_+ + \lambda_+ c_1) = 0\,, \\ &d *_3 (d\alpha_- + \lambda_- c_1) = 0\,, \\ &d(\lambda_0\sigma) + L(\lambda_- d\alpha_+ - \lambda_+ d\alpha_-) = 0 \end{aligned} \tag{2.30}$$

from (2.26). The third equation of (2.30) freezes one linear combination of $\alpha_\pm$ to equal σ. We also recognize the other linear combination as the Goldstone boson eaten by c_1. The remaining scalar σ has mass

$$m_\sigma^2 = 0\ . \tag{2.31}$$

Hence there is one massless modulus in the RR sector.

2.6. *Effects of the second modulus*

Now, let us consider effects associated with the second modulus. In type IIB setup with NS background, it corresponds to a combination of the axion field C_0 and the 4-form field. Expanding the 4-form field C_4 in terms of (2.7),

$$C_4 = (\alpha_+\omega_+ + \alpha_-\omega_-) \wedge d\theta\ , \tag{2.32}$$

the charge quantization conditions in the RR sector

$$\begin{aligned} &\int_{\mathbf{S}^3_+ \times \mathbf{S}^3_- \times \mathbf{S}^1} \left[*(C_0 H - F_3) - H \wedge C_4 \right] = 0\,, \\ &\int_{\mathbf{S}^3_\pm} F_3 = 0 \end{aligned} \tag{2.33}$$

are solved by

$$\lambda_+\alpha_- - \lambda_-\alpha_+ = \lambda_0 L C_0\ . \tag{2.34}$$

From (2.29), we learn that the linear combination $\lambda_+\alpha_+ + \lambda_-\alpha_-$ is proportional to a Goldstone mode for the gauge field c_1 for the IIA theory, (and $\int_{S^1} C_2$ for the IIB theory). The orthogonal combination $-\lambda_-\alpha_+ + \lambda_+\alpha_-$ is a modulus. From (2.34) when C_0 is turned on we must also turn on the RR potential C_4. If we set the Goldstone mode to zero then we may write

$$C_4 = \frac{C_0 L}{\lambda_0}\left(-\lambda_-\omega_+ \wedge d\theta + \lambda_+\omega_- \wedge d\theta\right)\ . \tag{2.35}$$

The only equation of motion that gets modified in the background (2.34) is the Einstein equation. Now, instead of (2.12), it gives

$$\begin{aligned} \ell^{-2} &= \frac{1}{4}\,\lambda_0^2\left(1+(g_B C_0)^2\right)\ , \\ R_\pm^{-2} &= \frac{1}{4}\,\lambda_\pm^2\left(1+(g_B C_0)^2\right)\ . \end{aligned} \tag{2.36}$$

Evaluating the NS5-brane charges, *cf.* (2.13),

$$\int_{\mathbf{S}^3_\pm} H = Q_5^\pm \tag{2.37}$$

we find a relation between $Q_5^\pm$ and $R_\pm$

$$Q_5^\pm = 2\pi^2 \lambda_\pm R_\pm^3$$

which together with (2.36) yields

$$R_\pm = \frac{1}{2\pi}\sqrt{Q_5^\pm}\left(1+(g_B C_0)^2\right)^{1/4}\ . \tag{2.38}$$

Since the background H-flux (2.6) is still defined so that (2.11) holds, we can use this equation to find the AdS radius,

$$\ell = \frac{1}{2\pi}\sqrt{\frac{Q_5^+ Q_5^-}{Q_5^+ + Q_5^-}}\left(1+(g_B C_0)^2\right)^{1/4}\ . \tag{2.39}$$

Finally, from the fundamental string charge quantization condition

$$\int *(|\tau_B|^2 H - C_0 F_3) = Q_1 \in \mathbb{Z} \tag{2.40}$$

we find

$$4\pi^4 R_+^3 R_-^3 L \lambda_0 |\tau_B|^2 = Q_1\,. \tag{2.41}$$

Here τ_B is the complexified type IIB coupling

$$\tau_B = C_0 + \frac{ie^{-\phi}}{g_B} \tag{2.42}$$

evaluated at $\phi = 0$. This relation can be used to solve for the size of the $\mathbf{S}^1$. Thus, substituting (2.36), (2.38), and (2.39), we find

$$L = Q_1\,\frac{4\pi g_B^2}{Q_5^+ Q_5^- \sqrt{Q_5^+ + Q_5^-}}\left(1+(g_B C_0)^2\right)^{-7/4} \tag{2.43}$$

which is similar to the previous expression, except for the last factor.

To summarize, turning on the second (axion) modulus in our NS background modifies the expressions (2.15) for the radii in the following way

$$\begin{aligned} \ell &= \frac{1}{2\pi}\sqrt{\frac{Q_5^+ Q_5^-}{Q_5^+ + Q_5^-}}\left(1+(g_B C_0)^2\right)^{1/4} , \\ R_\pm &= \frac{1}{2\pi}\sqrt{Q_5^\pm}\left(1+(g_B C_0)^2\right)^{1/4} , \\ L &= \frac{4\pi g_B^2 Q_1}{Q_5^+ Q_5^- \sqrt{Q_5^+ + Q_5^-}}\left(1+(g_B C_0)^2\right)^{-7/4} . \end{aligned} \tag{2.44}$$

These modifications leave the Brown-Henneaux central charge (2.18) unchanged. It will be useful for later purposes to note that

$$1+(g_B C_0)^2 = \left(\frac{|\tau_B|}{\Im\tau_B}\right)^2 . \tag{2.45}$$

2.7. *Moduli space metric*

The metric on the moduli space is most easily computed for the case of RR charges, which may be obtained from (2.44) by S-duality. Quite generally, under $SL(2,\mathbb{Z})$ transformations we have

$$\begin{aligned} \ell' &= \ell\left(\frac{\Im\tau_B}{\Im\tau'_B}\right)^{1/4} , \\ R'_\pm &= R_\pm\left(\frac{\Im\tau_B}{\Im\tau'_B}\right)^{1/4} , \\ L' &= L\left(\frac{\Im\tau_B}{\Im\tau'_B}\right)^{1/4} . \end{aligned} \tag{2.46}$$

It is convenient to write the answer in terms of $g_B Q$ which is held fixed in the classical limit of the RR background. One finds the simple C_0-independent expressions

$$\begin{aligned} \ell &= \frac{1}{2\pi}\sqrt{\frac{g_B Q_5^+ g_B Q_5^-}{g_B Q_5^+ + g_B Q_5^-}} , \\ R_\pm &= \frac{1}{2\pi}\sqrt{g_B Q_5^\pm} , \\ L &= \frac{4\pi g_B Q_1}{g_B Q_5^+ g_B Q_5^- \sqrt{g_B Q_5^+ + g_B Q_5^-}} . \end{aligned} \tag{2.47}$$

The moduli space metric follows from the kinetic terms of the three-dimensional low-energy effective action. These in turn descend from the ten-dimensional kinetic terms in (2.1)[g]

$$\int d^{10}x\sqrt{-g}\Big(\frac{2\pi e^{-2\phi}}{g_B^2}\big(\mathcal{R}+4(\nabla\phi)^2\big)-\pi(dC_0)^2-\frac{\pi}{5!}(dC_4)^2\Big)\ , \tag{2.48}$$

with the metric ansatz

$$ds^2=e^{\phi(x)}g^{(3)}_{\mu\nu}dx^\mu dx^\nu+e^{\phi(x)}R_+^2ds^2(\mathbf{S}^3_+)+e^{\phi(x)}R_-^2ds^2(\mathbf{S}^3_-)+e^{-3\phi(x)}L^2(d\theta)^2 \tag{2.49}$$

corresponding to the modulus generated by taking the coupling $g_B \to g_B e^{\phi(x)}$ in (2.47) (together with a Weyl rescaling of $g^{(3)}$). Using the S-dual of relation (2.34) to express C_4 in terms of C_0 one finds, after some computation, the three-dimensional effective action

$$\frac{2\pi V}{g_B^2}\int d^3x\sqrt{-g^{(3)}}\big(\mathcal{R}^3-4(\nabla\phi)^2-g_B^2e^{2\phi}(\nabla C_0)^2\big)\ , \tag{2.50}$$

where $V=4\pi^4R_+^3R_-^3L$ is the internal volume. From this we can read off the moduli space metric

$$ds^2=\frac{d\tau d\overline{\tau}}{(\Im\tau)^2},\qquad \tau=C_0+\frac{2ie^{-\phi}}{g_B} \tag{2.51}$$

which is the hyperbolic metric on the upper half plane. Note that the τ in (2.51) is not the same as the ten-dimensional coupling τ_B.

2.8. *Speculations on the global structure of the moduli space*

We now consider the RR gauge transformations which preserve the IIB solution with NS-sector fluxes described in Section 2.6, in equations (2.32)-(2.45). The unbroken gauge group is generated by three types of transformations. First, there are $SL(2,\mathbb{Z})$ transformations

$$\tau_B'=\frac{a\tau_B+b}{c\tau_B+d}\ ,\qquad\qquad \begin{pmatrix}F_3'\\ H_3'\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}F_3\\ H_3\end{pmatrix} \tag{2.52}$$

[g] In the ten-dimensional Einstein frame, the first three terms in this action can be written as $\frac{2\pi}{g_B^2}\int d^{10}x\sqrt{-g}\big(\mathcal{R}-\frac{d\tau_B d\overline{\tau}_B}{2(\Im\tau_B)^2}\big)$. Also, the contribution of the 4-form field C_4 is best described in the T-dual type IIA theory, which automatically avoids subtleties related to self-duality.

leaving C_4 invariant. The background 3-fluxes break the S-duality group down to the group of transformations

$$\begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix} \tag{2.53}$$

with $m \in \mathbb{Z}$.

Next there are small RR gauge transformations. Of these the only significant ones are the shifts of C_2 by exact forms. Only the combination $R_5 = dC_4 - C_2 H$ is gauge invariant, so in the presence of H-flux small C_2 gauge transformations acting by

$$C_2 \to C_2 + d(\chi d\theta) \tag{2.54}$$

must be accompanied by

$$\begin{aligned} \alpha_+ &\to \alpha_+ - \lambda_+ \chi \,, \\ \alpha_- &\to \alpha_- - \lambda_- \chi \,. \end{aligned} \tag{2.55}$$

Note that $\chi \in \mathbb{R}$ is an arbitrary *real number*. We define the Goldstone mode to be

$$\phi_{GB} = \frac{\lambda_+ \alpha_+ + \lambda_- \alpha_-}{\lambda_0^2} \ . \tag{2.56}$$

Then (2.55) shifts $\phi_{GB} \to \phi_{GB} - \chi$, but leaves $\lambda_+ \alpha_- - \lambda_- \alpha_+$ and C_0 invariant. Finally there are large C field gauge transformations. These act by

$$\begin{aligned} \alpha_+ &\to \alpha_+ + \frac{\lambda_+}{Q_5^+} n_+ \,, \\ \alpha_- &\to \alpha_- + \frac{\lambda_-}{Q_5^-} n_- \,, \\ C_0 &\to C_0 \,, \end{aligned} \tag{2.57}$$

where $n_\pm \in \mathbb{Z}$ are independent integers. Defining

$$C_4^\pm := \int_{S^3_\pm \times S^1} C_4 \tag{2.58}$$

they act by

$$C_4^\pm \to C_4^\pm + n_\pm \ . \tag{2.59}$$

Note that these transformations are not all independent. For example, some transformations of the type (2.57) are in fact of the form (2.55).

The transformations (2.53)(2.55)(2.57) generate a commutative group of unbroken gauge transformations. However, we must consider the subgroup of transformations which preserve the condition (2.34). The transformation of lengths under $SL(2,\mathbb{Z})$ (2.46) shows that

$$\begin{aligned} \lambda'_\pm &= \lambda_\pm \left(\frac{\Im\tau'_B}{\Im\tau_B}\right)^{3/4} , \\ \alpha'_\pm &= \alpha_\pm \left(\frac{\Im\tau'_B}{\Im\tau_B}\right)^{3/4} \end{aligned} \tag{2.60}$$

and hence the subgroup of transformations preserving (2.34) is determined from

$$\lambda_+(\alpha_- + \frac{\lambda_-}{Q_5^-}n_-) - \lambda_-(\alpha_+ + \frac{\lambda_+}{Q_5^+}n_+) = \lambda_0 L \left(\frac{\Im\tau_B}{\Im\tau'_B}\right) C'_0 \ . \tag{2.61}$$

A little bit of algebra reveals that this is true if

$$Q_5^+ n_- - Q_5^- n_+ = m Q_1 \ . \tag{2.62}$$

Let us now introduce $d := gcd(Q_5^+, Q_5^-)$ and $Q_5^\pm := d\widehat{Q}_5^\pm$. Moreover, we make the 1-1 invertible change of variables:

$$\begin{pmatrix} n_+ \\ n_- \end{pmatrix} = \begin{pmatrix} S_+ & \widehat{Q}_5^+ \\ S_- & \widehat{Q}_5^- \end{pmatrix} \begin{pmatrix} \widetilde{n}_+ \\ \widetilde{n}_- \end{pmatrix} \tag{2.63}$$

where $S_\pm$ are integers with $S_- Q_5^+ - S_+ Q_5^- = d$. The parameter $\widetilde{n}^-$ is equivalent to a small gauge transformation (2.55) and hence can be dropped. If we fix the gauge by setting $\phi_{GB} = 0$ then the unbroken symmetry group is $\mathbb{Z}$, generated by ($\widetilde{n}_+ = 1, \widetilde{n}_- = 0$) (which must be accompanied by a small gauge transformation χ to preserve the gauge condition $\phi_{GB} = 0$).

The resulting unbroken gauge transformations are *much* more simply expressed in the S-dual background related by $\tau_B \to -1/\tau_B$. In this background (2.53) is mapped to the usual RR shift symmetry $C_0 \to C_0 - m$. Henceforth we shall work in this S-dual picture. In this picture the unbroken gauge group has a generator acting by

$$\begin{aligned} C_0 &\to C_0 - d \, , \\ C_4^+ &\to C_4^+ + S_+ Q_1 \, , \\ C_4^- &\to C_4^- + S_- Q_1 \ . \end{aligned} \tag{2.64}$$

The equivalence relation (2.64) is very much analogous to an identification of the moduli space in the D1-D5 system [3]. Since we parametrize the moduli space by τ, equation (2.51), we say that the moduli space is identified under shifts $\Re\tau \to \Re\tau + d$. Note that d depends on the arithmetic of $Q_5^\pm$. If Q_5^+, Q_5^- are relatively prime, or have small common divisors, then the identification is by a distance of order g_B. On the other hand, if $Q_5^+ = Q_5^-$ then $d = Q_5^+$. In the scaling required for the supergravity limit this shift is very large, of order $1/g_B$, and the distance on moduli space is order 1. Such shifts mix up all orders of string perturbation theory, and our supergravity analysis cannot reliably conclude that (2.64) is a symmetry of the exact theory.[h] In principle it could be spoiled by D3 instantons, for example. Nevertheless we proceed in the rest of this subsection under the assumption that supergravity is indeed a reliable guide in this case.

Apart from this RR shift symmetry, the U-duality group is generated by various 'inversion' transformations:

- 1) T-duality, which sends $L \to 1/L$, $g \to g/L$, and interchanges IIA/B;
- 2) S-duality in IIB, which sends $g \to 1/g$, $L \to L/\sqrt{g}$, and interchanges NS and RR backgrounds;
- 3) '9/11 flip', which sends $L \to \sqrt{Lg}$, $g \to L^{3/2}/g^{1/2}$ in type IIA.

One easily checks that $TFT = S$, so essentially there is just T-duality and S-duality – the flip is just the image of S-duality in type IIA. Clearly T-duality interchanges momentum and fundamental string winding on the circle in the NS background, but this interchanges IIA/B. In the IIB D-brane background, the equivalent operation is STF, which again interchanges IIA/B. Hence there is no inversion automorphism of the theory leaving the background charges fixed – the only candidate is S-duality, and that interchanges the RR and NS descriptions of the background.

The only identification of the moduli space is thus the RR shift symmetry (2.64). If we parametrize the modulus by τ, equation (2.51), we find the fundamental domain of the moduli space is the strip in the upper half plane with $\Re\,\tau \in (-\frac{1}{2}d, \frac{1}{2}d)$.

The restricted scope of the U-duality group is quite different from the situation in the D1-D5 system on $\mathcal{M} = K3$ or T^4. There it was found that all supergravities with the same value of the product $N = Q_1 Q_5$ of background charges were located in different cusps of the moduli space [2,3]. In particular, this allowed the symmetric product orbifold $Sym^N(\mathcal{M})$, which was naturally associated to the background with $Q_5 = 1$, to be continuously connected to all other backgrounds with the same central charge. In the present case, all distinct sets of charges (Q_1, Q_5^+, Q_5^-) lead to distinct, disconnected moduli spaces of theories. This leads to the possibility that

[h] This is why the title of our section contains the word "speculations."

symmetric product orbifolds will only lie on a subset of these moduli spaces of theories; for instance, they might not describe both (pQ_1, Q_5, Q_5') and (Q_1, pQ_5, pQ_5'), which are backgrounds having the same central charge in the spacetime CFT but lying on disconnected moduli spaces.

2.9. *Regions of the moduli space*

We now turn to a discussion of the various regions of the moduli space. Different low-energy descriptions are appropriate in different regions. To fix notation, let us refer all quantities to the IIB RR background, via the appropriate dualities. In that frame, the moduli space is the strip in the UHP $|\Re\tau| < \frac{1}{2}d$. Weakly coupled IIB string theory is appropriate as we move up into the cusp of the moduli space at large $\Im\tau$. The cycle sizes and curvature radii are not too small provided $\ell, R_\pm, L > 1$ in string units; for instance we want $g_B Q_5^\pm > 1$. If this is not true, then we are in a regime of weak coupling of the dual CFT (just like $g_B Q_3 < 1$ is weak coupling for $\mathcal{N} = 4$ super Yang–Mills), and the geometrical interpretation of the target breaks down. Thus, the region far up in the cusp is the perturbative regime of the spacetime CFT.

From equations (2.47) and their various duals, e.g. (2.44), we have the following criteria to impose:

1) If $g_B > 1$ we should S-dualize to the NSB description.
2) If $L < 1$ we should T-dualize the $\mathbf{S}^1$ to type IIA. This will be the F1-NS5A-NS5A' background if we arrive from the NSB description, otherwise we arrive from RRB and get D2-D4-D4'. Referred to the RRB background, the condition to T-dualize is

$$g_B^{3/2} > \frac{Q_1}{Q_5^+ Q_5^- \sqrt{Q_5^+ + Q_5^-}} \,. \qquad (2.65)$$

3) If the IIA coupling becomes strong, we go to M-theory with the charges M2-M5-M5'. Referred back to the RRB frame, the condition is

$$g_B > \frac{Q_1}{Q_5^+ Q_5^- \sqrt{Q_5^+ + Q_5^-}} \qquad (2.66)$$

(note that the RHS is the same as in (2.65)).

Note that the natural boundary at $\Im\tau = 0$ is arrived at from an effective M-theory description. In the M-theory description, the RR axion has transformed into the shear of the T^2 comprised of the $\mathbf{S}^1$ and the M-theory circle.

To summarize: The cusp region is the weakly coupled dual CFT. Coming down from the cusp, we encounter RRB sugra. Then, depending on whether (1) or (2) is satisfied first, we go to either (a) NSB supergravity, then NSA after T-duality, then M-theory; or (b) RRA supergravity by T-duality, then M-theory. A sketch of the first possibility is given in figure 1.

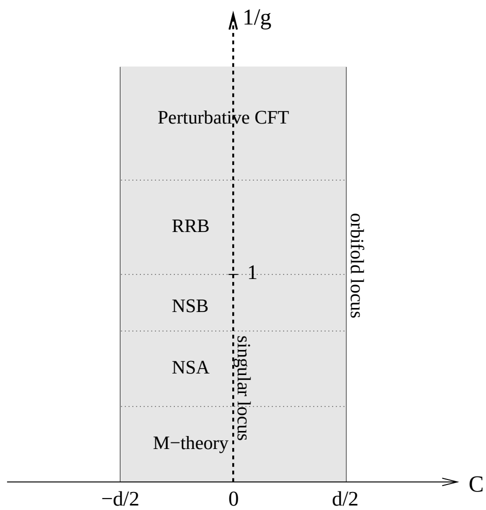

Figure 1. Sketch of the moduli space, and the regimes in which various low energy descriptions are valid. The dashed line is the singular locus, where a long string continuum appears (see Section 2.10). The line $C = d/2$ is argued to be the location of the symmetric product orbifolds.

The region of RRB supergravity can be vanishingly small if e.g. $Q_5^{\pm}$ are small. However, there will be a regime described by the perturbative string formalism of [39,8] for $g_B > 1$ (recall we are referring everything to the RR duality frame), although it may only involve type IIA.

The dual CFT is typically perturbative up in the cusp. In the D1-D5 and related systems, the dual CFT had a description as a symmetric orbifold along a line $\Re\tau = \frac{1}{2}$ in a duality frame where the background charges were RR and $Q_5 = 1$ [3]. Below we will argue that similarly, there is a symmetric product orbifold when one of the fivebrane charges is one, say $Q_5^+ = 1$; and then the orbifold line is $\Re\tau = \frac{1}{2}$.

The perturbative string description of [39] is only equipped to handle backgrounds with vanishing RR potentials, i.e. $\Re\tau = 0$. The spacetime CFT is actually singular on this subspace of the moduli space. We now turn to a discussion of this phenomenon.

2.10. *Long strings and singular CFT's*

A common feature of conformal field theories dual to AdS_3 string backgrounds is that, in certain regions of the moduli space, they exhibit a continuum of states above a gap Δ_0. The continuum is associated to the appearance of a new branch of the configuration space where the onebrane-fivebrane ensemble can fragment into separate pieces [40,39,2]. The new branches of the configuration space describe separating clusters of onebranes and fivebranes, often called 'long strings', since they are codimension one objects in AdS_3 whose proper length grows to infinity as they approach the AdS_3 boundary.

Typically one thinks of the spacetime CFT dual to AdS_3 as the Higgs branch of the onebrane-fivebrane system, where the onebranes are dissolved in the fivebranes as finite-size instantons. For instance, in the D1-D5 system, the CFT is the sigma model on the moduli space of instantons on T^4 or $K3$ (see e.g. [41,2,3,42,43] for reviews and further references). In this dual CFT, the new branches of the configuration space are Coulomb branches, where some number of instantons shrink to zero size [2]. Whereas the instanton of non-zero size gives a potential for the coordinates of the dissolved string in the directions transverse to the fivebrane, the zero-size instanton string allows these fields to turn on, so that the string moves away from the fivebrane background (out to the boundary of AdS_3).

In either description, long strings or zero-size instantons, the appearance of the continuum results in singularities in correlation functions. The singularity arises only when strings can become infinitely long at finite energy cost (or correspondingly instantons can shrink to exactly zero size). This is not allowed at generic points on the moduli space, typically only when the RR potentials vanish.

In the NS duality frame, the tension of a long string receives compensating contributions from the tension of a fundmental string (determined e.g. from the Nambu-Goto action) and from the background B field (the Wess-Zumino term in the $AdS_3 = SL(2,R)$ sigma model). We choose global coordinates for AdS_3,

$$ds^2 = \ell^2(-\cosh^2\!\rho\, dt^2 + \sinh^2\!\rho\, d\phi^2 + d\rho^2) \qquad (2.67)$$

and we choose the duality frame with NS fluxes turned on so that

$$H = \lambda_0\omega_0 = \frac{1}{2}\lambda_0\ell^3 \sinh(2\rho) dt \wedge d\phi \wedge d\rho\ . \qquad (2.68)$$

Consider a string at fixed ρ, with a worldsheet that spans a time Δt. The action consists of two pieces

$$S = S_{\rm NG} - S_{WZ} = 2\pi \int \sqrt{-h} - 2\pi \int B \ . \tag{2.69}$$

We define q (following [40]) to be the ratio $S_{\rm WZ}/S_{\rm NG}$ as $\rho \to \infty$. Thus, $(1-q)$ measures the coefficient of a "cosmological term" $\sim e^{2\rho}$ giving the energy cost per unit proper length of the string. When $q \neq 1$, it costs infinite energy to take the string to the boundary of AdS_3 and so it is effectively bound to the system. There is no continuum in the spectrum.

In our conventions,

$$S_{\rm NG} = 2\pi^2 \ell^2 \sinh(2\rho)\ \Delta t \ . \tag{2.70}$$

Next we choose a gauge $B = \frac{1}{4}\lambda_0 \ell^3 \cosh 2\rho dt d\phi$, so that

$$S_{\rm WZ} = \frac{1}{4}\lambda_0 \ell^3 (2\pi)^2 \cosh(2\rho)\ \Delta t \ . \tag{2.71}$$

Using the Einstein equation (2.36) we compute

$$q = \lim \frac{S_{\rm WZ}}{S_{\rm NG}} = \frac{1}{2}\lambda_0 \ell = \frac{1}{g_B|\tau_B|} = \frac{1}{\sqrt{1+(g_B C_0)^2}} \ . \tag{2.72}$$

We thus conclude that the singular locus on moduli space is at $C_0 = 0$, the positive imaginary axis for τ.

The energy cost of a long string is related to the change of the central charge (2.20) resulting from pulling it completely out of the background. The difference in the ground state (Casimir) energies is given by the change in the central charge as $\delta h = -\delta c/24$. One finds the gap Δ_0 to the continuum of long string states

$$\Delta_0 = -\delta c/24 = \frac{Q_5^+ Q_5^-}{4(Q_5^+ + Q_5^-)} \tag{2.73}$$

associated to pulling out a onebrane.

Long strings can also carry non-zero fivebrane charge – strings in AdS_3 can be obtained by wrapping fivebranes over $\mathbf{S}^3_+ \times \mathbf{S}^1$ or $\mathbf{S}^3_- \times \mathbf{S}^1$. We have computed q for these strings. This is a more difficult computation, but it does indicate that sometimes these strings can produce singularities, again on the locus of vanishing C_0. As the details would take us somewhat far afield we do not include them here.

3. Relation to intersecting D-branes

Anti-de Sitter backgrounds are often realized as near-horizon limits of the geometry surrounding intersecting brane sources. In this section we study the geometries surrounding intersecting brane configurations that are expected to exhibit large $\mathcal{N} = 4$ supersymmetry in their infrared dynamics. While we have not found a brane configuration with all the desired properties, we discuss three different ones which exhibit different aspects of the dynamics:

1. The collection of branes in $\mathbb{R}^{1,9}$

$$\begin{array}{lll} Q_1 & \text{D1 branes along} & x^0,\ x^5\ , \\ Q_5^+ & \text{D5 branes along} & x^0,\ x^5,\ x^6, \ldots,\ x^9\ , \\ Q_5^- & \text{D5}' \text{ branes along} & x^0,\ x^5,\ x^1, \ldots,\ x^4 \end{array} \tag{3.1}$$

   preserves 1/8 supersymmetry, and has near-horizon geometry

$$AdS_3 \times \mathbf{S}^3 \times \mathbf{S}^3 \times \mathbb{R}\ . \tag{3.2}$$

   There are thus strong reasons to believe that the IR theory has large $\mathcal{N} = 4$ supersymmetry. However, what sort of dynamics describes the intersecting fivebranes is not understood, and it is not clear how to implement a compactification of $\mathbb{R}$ to $\mathbf{S}^1$.
2. For $Q_5^+ = Q_5^-$, the locus of fivebrane sources above can be deformed to a special Lagrangian 4-manifold $\mathcal{M} \subset \mathbb{R}^8$; the two sets of intersecting branes deforms to a single set of branes, much as in [44]. The $SO(4) \times SO(4)$ symmetry of the branes in (1) is broken to the diagonal $SO(4)$. The near-horizon geometry is still (3.2), but the fivebrane dynamics in the IR appears to be a more conventional $U(Q_5)$ gauge theory; the addition of onebranes dissolved in the fivebranes should be described as the Higgs branch of the corresponding D1-D5 system.
3. While this second configuration points toward the appropriate IR dynamics (for $Q_5^+ = Q_5^-$), in the near-horizon geometry the fivebranes are wrapping $\mathbf{S}^3 \times \mathbb{R}$ and not $\mathbf{S}^3 \times \mathbf{S}^1$. To find the latter, we can change the setup somewhat and consider Q_5^+ fivebranes wrapping a special Lagrangian $\mathbf{S}^3$ supported by Q_5^- units of three-form flux. The remaining directions on these branes can be taken to be $\mathbb{R}^{1,1} \times \mathbf{S}^1$. As we will see below, there are good reasons to expect that with onebranes along $\mathbb{R}^{1,1}$ included, the geometry in the infrared flows to a theory with large $\mathcal{N} = 4$ supersymmetry; and at the same time, the dynamics is that of the onebranes dissolved in the fivebranes – a sigma model on the moduli space of instantons on $\mathbf{S}^3 \times \mathbf{S}^1$.

We will now describe each of these brane configurations in more detail.

First consider the configuration (3.1) of flat branes intersecting in flat spacetime. Each of these D-branes is invariant under half of the supersymmetries, and altogether the D-branes preserve only $\frac{1}{8} \times 32 = 4$ supersymmetries. Hence, the two-dimensional field theory on the D1-branes has $\mathcal{N} = (0, 4)$ supersymmetry. The D-brane configuration (3.1) breaks the Lorentz group $SO(1, 9)$ to the subgroup

$$SO(1,1)_{05} \times \left[SU(2)_L \times SU(2)_R\right]_{1234} \times \left[SU(2)_L \times SU(2)_R\right]_{6789} . \tag{3.3}$$

The $SU(2)$ factors in this symmetry group play the role of the R-symmetry in the effective two-dimensional field theory on the intersection.

The 05 field theory on the world-volume of intersecting D-branes is composed of 11, 15, $15'$ and $55'$ string states, which form complete representations under the unbroken symmetry group (3.3). Among various states, the 15 and $15'$ strings are in a chiral representation of the rotational R-symmetry and contribute to an R-charge anomaly. Since the 15 ($15'$) fermions are invariant under $[SU(2)_L \times SU(2)_R]_{1234}$ (respectively $[SU(2)_L \times SU(2)_R]_{6789}$), the computation of this contribution to the anomaly is exactly as in the standard D1-D5 system. Specifically, one has

$$k_L^+ = Q_1 Q_5^+ \ , \quad k_R^+ = -Q_1 Q_5^+ \ , \quad k_L^- = Q_1 Q_5^- \ , \quad k_R^- = -Q_1 Q_5^- \ . \tag{3.4}$$

At the IR fixed point, the theory must have $\mathcal{N} = (0, 4)$ supersymmetry; $(0, 4)$ supersymmetry implies at least one of the $SU(2)$ R-symmetries must become an $SU(2)$ current algebra. However both right-moving $SU(2)$ R-symmetries are on the same footing. In other words, we have at least the large $\mathcal{N} = 4$ supersymmetry algebra on the right. The large $\mathcal{N} = 4$ supersymmetry algebra has the four supercharges transforming as $(\frac{1}{2}, \frac{1}{2})$ under two $SU(2)$ R-symmetry currents. We will describe this algebra in the next section. Here we simply note that the $SU(2) \times SU(2)$ currents have central extensions $k_R^+ = -Q_1 Q_5^+$ and $k_R^- = -Q_1 Q_5^-$, and the large $\mathcal{N} = 4$ superalgebra with these two R-currents indeed has conformal central charge (2.20).[a]

In the $55'$ spectrum, since there are 8 DN directions the ground state energy in the NS sector is $+\frac{1}{2}$ and there are no massless bosons. In the R sector, the fermions are periodic in the two NN directions so there are fermion zero modes ψ^0 and ψ^1. The ground state is then in the $(-\frac{1}{2}; 0, 0; 0, 0)$ representation of these zero modes, which is a trivial (left-moving) representation of the right moving superalgebra. There is a non-trivial contribution to the

[a] In particular, we see that there are no $O(\hbar)$ corrections to the central extensions.

central charge

$$c_L - c_R = \tfrac{1}{2} Q_5^+ Q_5^- \ . \tag{3.5}$$

On the other hand, the supergravity background seems to respect $\mathcal{N} = 4$ supersymmetry of both chiralities on the AdS_3 boundary. One way to accomodate these facts is to suppose that these R-invariant $55'$ fields decouple, becoming free fermions in the IR, and that the remaining theory has its symmetry enhanced to a large $\mathcal{N} = (4,4)$ superconformal algebra (the $55'$ fields cannot be fit into a representation of the large $\mathcal{N} = (4,4)$ algebra). It would certainly be helpful to understand this issue better, but for now we are going to ignore these fermionic 'singleton' modes, and assume that the infrared theory has large $\mathcal{N} = (4,4)$ superconformal symmetry.

To describe the supergravity solution corresponding to this configuration of branes, we begin with the geometry of fivebranes intersecting over a string [45]:

$$\begin{aligned} ds^2 &= (\mathrm{det} U)^{-1/2} \left[(-dt^2 + dx_5^2) + U_{ij} d\vec{x}_i \cdot d\vec{x}_j \right] , \\ F_3 &= *_x dU_{11} + *_y dU_{22} , \\ e^{\phi} &= g_B \, (\mathrm{det} U)^{-1/2} \ . \end{aligned} \tag{3.6}$$

Here U is a 2×2 symmetric matrix, whose entries are harmonic functions of the coordinates $\vec{x}_i = (\vec{x}, \vec{y})$ on the $\mathbb{R}^8 = \mathbb{R}^4 \times \mathbb{R}^4$, and $*_x$ denotes the Hodge dual on $\mathbb{R}^4$ parametrized by the 4-vectors $\vec{x} = (x^1, x^2, x^3, x^4)$ (similarly, $*_y$ is the Hodge dual in $\vec{y} = (x^6, x^7, x^8, x^9)$).

We will in fact consider a slight generalization the geometry, in which the branes intersect at angles; this will be useful below when we describe the deformation to fivebranes wrapping a special Lagrangian submanifold. Thus, we rotate the D5'-branes by an angle ϑ in every two-plane $x^k - x^{k+5}$:

$$\begin{array}{lll} Q_5^+ & \text{D5 branes :} & 056789 \ , \\ Q_5^- & \text{D5}' \text{ branes :} & 05[16]_\vartheta [27]_\vartheta [38]_\vartheta [49]_\vartheta \ . \end{array} \tag{3.7}$$

This configuration of intersecting fivebranes preserves 3/16 of the original supersymmetry and, as will be shown below, leads to the same near-horizon geometry (3.2) for *any* value of ϑ. Therefore, all such theories are expected to have the same IR physics, described by a CFT with large $\mathcal{N} = (4,4)$ superconformal symmetry.[b]

[b] One can check, using the representation theory that we will introduce in the next section, that the symmetries preserved by the deformation guarantee that it represents an irrelevant deformation of the infrared physics.

For fivebranes at the special angles (3.7), the IIB supergravity solution has the form

$$U = U^{(\infty)} + \begin{pmatrix} \frac{g_B Q_5^+}{x^2} & 0 \\ 0 & \frac{g_B Q_5^-}{y^2} \end{pmatrix} . \tag{3.8}$$

Since constant terms are omitted in the near-horizon limit, this is the first indication that the near-horizon geometry of this intersecting D-brane configuration is the same for any value of $\vartheta \neq 0$. Explicitly, the rotation angle ϑ is given in terms of $U_{ij}^{(\infty)}$ by

$$\cos\vartheta = -\frac{U_{12}^{(\infty)}}{\sqrt{U_{11}^{(\infty)} U_{22}^{(\infty)}}} . \tag{3.9}$$

We may restrict $U^{(\infty)}$ to be such that $\mathrm{det}U^{(\infty)} = 1$. Specifically, we choose $U^{(\infty)}$ to be

$$U^{(\infty)} = \begin{pmatrix} \mathrm{cosh}\alpha \ \mathrm{sinh}\alpha \\ \mathrm{sinh}\alpha \ \mathrm{cosh}\alpha \end{pmatrix} . \tag{3.10}$$

Then, from (3.9) one finds a relation between α and the rotation angle ϑ:

$$\cos\vartheta = -\mathrm{tanh}\alpha . \tag{3.11}$$

Now let us include D1-branes smeared in the directions $x^{1,2,3,4}$ and $x^{6,7,8,9}$. This will further break the supersymmetry from 3/16 to 1/16, unless $\vartheta = 0$ where we preserve 1/8. The complete supergravity solution looks like (in string frame):

$$\begin{aligned} ds^2 = & \left(H_1^{(+)} H_1^{(-)} \mathrm{det}U\right)^{-1/2} (-dt^2 + dx_5^2) + \sqrt{H_1^{(+)} H_1^{(-)}}\, \frac{U_{11}}{\sqrt{\mathrm{det}U}}\, (d\vec{x})^2 \\ & + \sqrt{H_1^{(+)} H_1^{(-)}} \frac{U_{22}}{\sqrt{\mathrm{det}U}}\, (d\vec{y})^2 + \frac{2U_{12}}{\sqrt{\mathrm{det}U}}\, d\vec{x} \cdot d\vec{y} \ , \\ F_3 = & dt \wedge dx_5 \wedge d \left(H_1^{(+)} H_1^{(-)}\right)^{-1} + *_x dU_{11} + *_y dU_{22} \ , \\ e^{-2\phi} = & \frac{1}{g_B^2} \frac{\mathrm{det}U}{H_1^{(+)} H_1^{(-)}} \ , \end{aligned} \tag{3.12}$$

where

$$H_1^{(+)} = 1 + \frac{g_B q_1}{x^2} \ , \qquad H_1^{(-)} = 1 + \frac{g_B q_1}{y^2} \ . \tag{3.13}$$

Notice, that since D1-branes are smeared along four spatial directions, these harmonic functions exhibit the same radial dependence as the fivebrane harmonic functions (3.8). The parameter q_1 is the density of onebrane charge along the fivebrane.

Now we are in a position to take the near-horizon limit of the solution (3.12) with the matrix U given by (3.8), (3.10). Omitting constant terms in the harmonic functions, we find the near-horizon limit of the metric (3.12)

$$
\begin{aligned}
ds^2 = &\frac{x^2 y^2}{g_B^2 q_1 \sqrt{Q_5^+ Q_5^-}}(-dt^2 + dx_5^2) + g_B q_1 \sqrt{\frac{Q_5^+}{Q_5^-}} \Big(\frac{dx^2}{x^2} + \Omega_+^2 \Big) \\
&+ g_B q_1 \sqrt{\frac{Q_5^-}{Q_5^+}} \Big(\frac{dy^2}{y^2} + \Omega_-^2 \Big) \ .
\end{aligned}
\tag{3.14}
$$

By a change of variables,

$$
\begin{aligned}
u &= xy \left(q_1^2 g_B^3 \, \frac{Q_5^+ Q_5^-}{Q_5^+ + Q_5^-} \right)^{-1/2} , \\
\widehat{\theta} &= \frac{(Q_5^+ Q_5^-)^{-1/4}}{\sqrt{Q_5^+ + Q_5^-}} \left[-Q_5^+ \log x + Q_5^- \log y \right]
\end{aligned}
\tag{3.15}
$$

we can write the near-horizon metric in the form (2.8)

$$
ds^2 = \ell^2 ds^2(AdS_3) + R_+^2 ds^2(\mathbf{S}_+^3) + R_-^2 ds^2(\mathbf{S}_-^3) + \widehat{L}^2 (d\widehat{\theta})^2 \, , \tag{3.16}
$$

where

$$
\begin{aligned}
\ell^2 &= g_B q_1 \, \frac{\sqrt{Q_5^+ Q_5^-}}{Q_5^+ + Q_5^-} \ , \\
R_+^2 &= g_B q_1 \sqrt{\frac{Q_5^+}{Q_5^-}} \ , \\
R_-^2 &= g_B q_1 \sqrt{\frac{Q_5^-}{Q_5^+}} \ , \\
\widehat{L}^2 &= q_1 g_B \ .
\end{aligned}
\tag{3.17}
$$

Note especially that in the near-horizon geometry obtained from the intersecting branes, the coordinate $\widehat{\theta}$ is non-compact; the near-horizon geometry is $AdS_3 \times \mathbf{S}^3 \times \mathbf{S}^3 \times \mathbb{R}$.

This near-horizon geometry can formally be further compactified using a new isometry that only appears after taking the near-horizon limit. Namely, following [9], we observe that $x \to e^{-h}x, y \to e^{h}y$, is a symmetry of (3.14) for any real number h. If we take a quotient by $\mathbb{Z}$ with the generator acting as $x \to e^{-h_*}x, y \to e^{h_*}y$ with

$$h_* = \frac{(2\pi L)^2}{g_B} \frac{1}{Q_5^+ Q_5^-} \tag{3.18}$$

and make a Weyl rescaling of (2.8) by $(2\pi)^2 q_1/\sqrt{Q_5^+ Q_5^-}$ then we obtain $AdS_3 \times \mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^1$. Note, however, that this orbifold action relates points nonperturbatively far apart. Thus, the relevance of this orbifold action is open to question. It is certainly not a symmetry of the full string theory of the intersecting branes (3.1).

The $AdS_3 \times \mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^1$ supergravity solution of the previous section is more conventionally related to the brane construction above, if in the former we take the limit $Q_1, L \to \infty$, with g_B, $Q_5^{\pm}$ and $q_1 \sim Q_1/L$ fixed; then we obtain the near-horizon geometry (3.16). In order to find the relation between various parameters, one can e.g. compute the Brown-Henneaux central charge (2.18) - (2.19) using (3.17). This gives

$$q_1 = \frac{1}{4\pi^2}\sqrt{Q_5^+ Q_5^-} \tag{3.19}$$

which, after substituting it back into (3.17), leads to the expressions (2.17) found in the previous section.

While useful for illustrating geometrically the appearance of large $\mathcal{N} = 4$ supersymmetry in the near-horizon limit of branes, the above intersecting brane configuration is somewhat less useful for illuminating the nature of the dual CFT, since the dynamics of fivebranes intersecting over a string is poorly understood. To shed some light on this side of the duality, we can deform the above brane configuration, simplifying the brane dynamics at the cost of breaking some of the symmetry. The deformation involved is expected to be irrelevant, so that we should still recover large $\mathcal{N} = 4$ supersymmetry in the infrared.

The deformation we wish to consider is only allowed for $Q_5^+ = Q_5^- \equiv Q_5$; we will also take the angle ϑ to have the value $\vartheta = \pi/4$. Then, the D5-branes and D5'-branes can join together to form a single set of Q_5 D5-branes along a smooth 4-manifold $\mathcal{M} \subset \mathbb{R}^8$. In order to preserve supersymmetry, $\mathcal{M}$ must be a calibrated submanifold inside $\mathbb{R}^8$. Namely, the 4-manifold $\mathcal{M}$ must be a special Lagrangian submanifold inside $\mathbb{C}^4 \cong \mathbb{R}^8$. Fortunately, the explicit geometry of a special Lagrangian submanifold in $\mathbb{C}^4$ with the right properties was found by Harvey and Lawson [46].

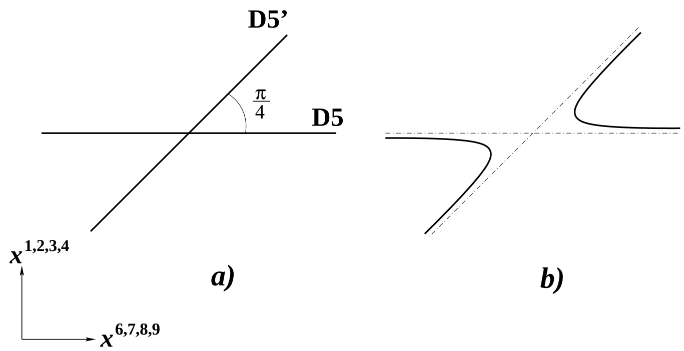

Figure 2. Intersection of special Lagrangian D5-branes (a) and its non-singular deformation (b).

As before, let us represent

$$\mathbb{C}^4 \cong \mathbb{R}^4 \times \mathbb{R}^4 \tag{3.20}$$

where each copy of $\mathbb{R}^4$ is parametrized by the 4-vectors $\vec{x}$ and $\vec{y}$. Then, the explicit form of the special Lagrangian 4-manifold $\mathcal{M}$ is given by a set of points in $\mathbb{C}^4 = \mathbb{R}^4 \times \mathbb{R}^4$, which satisfy the conditions [46]

$$\mathcal{M} = \{(\vec{x}, \vec{y}) \in \mathbb{C}^4 \ \mid \ \frac{\vec{x}}{|\vec{x}|} = \frac{\vec{y}}{|\vec{y}|}\,, \quad xy(x^2 - y^2) = \rho\} \tag{3.21}$$

where $x \equiv |\vec{x}|$, $y \equiv |\vec{y}|$. In other words, $\mathcal{M}$ is a cohomogeneity one submanifold in $\mathbb{C}^4 = \mathbb{R}^4 \times \mathbb{R}^4$, represented by a graph of the function

$$xy(x^2 - y^2) = \rho\ . \tag{3.22}$$

Note that the Lagrangian submanifold $\mathcal{M}$ has topology

$$\mathcal{M} \cong \mathbb{R} \times \mathbf{S}^3 \tag{3.23}$$

where the radius of the 3-sphere is determined by the (real) deformation parameter ρ, *cf.* (3.21).

In the limit $\rho \to 0$ the Lagrangian submanifold (3.21) degenerates into a union of two 4-planes

$$\mathcal{M}_{\rm sing} = \mathbb{R}^4 \cup \mathbb{R}^4 \tag{3.24}$$

and we recover the geometry of intersecting fivebranes (3.7), see Figure 2a. On the other hand, when $\rho \neq 0$, $\mathcal{M}$ is a smooth 4-manifold with topology (3.23), which is asymptotic to the union of planes (3.24).

For generic values of ρ, the Lagrangian submanifold $\mathcal{M}$ is invariant under the symmetry group,

$$\big[SU(2)_L \times SU(2)_R\big]_D \subset \big[SU(2)_L \times SU(2)_R\big]_{1234} \times \big[SU(2)_L \times SU(2)_R\big]_{6789} \tag{3.25}$$

which is a diagonal subgroup of the R-symmetry group (3.3). The undeformed rotated brane source (3.7) has the same symmetry. Nevertheless, the near-horizon geometry (3.16) of the latter is clearly invariant under the full symmetry group on the RHS of (3.25); the symmetry breaking to the diagonal is an irrelevant perturbation in the infrared limit. We expect the curved geometry of fivebranes located along $\mathcal{M}$ also to flow to one with the $SU(2)^4$ isometry of (3.2).

Such a configuration, with D5 and D5' branes joined in a single smooth manifold $\mathcal{M}$, admits a Higgs branch where D1-branes are realized as instantons in the D5-brane. In fact, this branch is very similar to the Higgs branch in the ordinary D1-D5 system, where D5-branes are wrapped on a 4-manifold $\mathcal{M} = T^4$ or $K3$. In the present case, on a single fivebrane the vevs of the scalar fields in the 1-5 string sector parametrize a 4-manifold $\mathcal{M}$ with the topology of $\mathbf{S}^3 \times \mathbb{R}$, so that the geometry of the Higgs branch is given by the symmetric product of this space,

$$Sym^N(\mathcal{M}) \tag{3.26}$$

where, roughly speaking, one can interpret the coordinates on this moduli space as parameters of the D1-brane instantons on the D5-brane. Unfortunately, because the space wrapped by the fivebrane is non-compact, a duality between this Higgs branch sigma model and supergravity can only take place at $N = \infty$.

As mentioned in the beginning of this section, a rather different approach using intersecting branes considers the onebrane-fivebrane system wrapping a special Lagrangian $\mathbf{S}^3$. This approach does allow us to compactify $\mathbb{R}$ to $\mathbf{S}^1$. The low-energy gauge theory of $N = Q_5$ fivebranes wrapping a special Lagrangian $\mathbf{S}^3$ was considered in [12-15,47]. In this theory, the $k = Q_5'$ units of three-form flux through the $\mathbf{S}^3$ wrapped by the fivebranes appears in the effective gauge dynamics through a Chern–Simons term, which is most easily seen using the RR background frame

$$\begin{aligned} \frac{1}{16\pi^3}\int_{\mathbb{R}^{1,2}\times\mathbf{S}^3_\parallel} C_2 \wedge \mathrm{Tr}[F\wedge F] &= -\frac{1}{16\pi^3}\int_{\mathbb{R}^{1,2}\times\mathbf{S}^3_\parallel} F_3 \wedge \mathrm{Tr}[AdA + \tfrac{2}{3}A^3] \\ &= -\frac{k}{4\pi}\int_{\mathbb{R}^{1,2}} \mathrm{Tr}[AdA + \tfrac{2}{3}A^3] \ . \end{aligned} \tag{3.27}$$

Dual supergravity solutions [12,13,14,15,47] have been considered in the NS background frame (appropriate to the strong coupling gauge theory that appears in the IR of the D-brane gauge theory). A solution with $Q_5 = Q_5'$, which preserves 1/16 supersymmetry and $SU(2)^3$ symmetry, was found in [12]:

$$\begin{aligned} ds^2 &= ds^2_{\mathbb{R}^{1,2}} + dr^2 + \tfrac{1}{2} r\, d\Omega^2_{3,\parallel} + \tfrac{1}{4}\, d\Omega^2_{3,\perp}\,, \\ e^{2\phi} &= g_s^2\, e^{-2r} r^{3/4}\,, \\ H &= \tfrac{1}{32}[\sigma_2 \wedge \sigma_3 \wedge \nu_1 + \sigma_3 \wedge \sigma_1 \wedge \nu_2 + \sigma_1 \wedge \sigma_2 \wedge \nu_3] + \tfrac{1}{8}\nu_1 \wedge \nu_2 \wedge \nu_3\ . \end{aligned} \tag{3.28}$$

Here σ_a (ω_a) are the left-invariant one-forms on $\mathbf{S}^3_\parallel$ ($\mathbf{S}^3_\perp$), and $\nu_a \equiv \omega_a - \frac{1}{2}\sigma_a$.

It was demonstrated in [48] that the effective 2+1d gauge theory obtained by KK reduction on $\mathbf{S}^3_\parallel$ spontaneously breaks supersymmetry unless $|k| \geq N$. [c] In [14] it was argued that for $k \neq N$ (i.e. $Q_5 \neq Q_5'$), one needs to introduce explicit sources for the three-form field strength corresponding to the fivebranes wrapping $\mathbf{S}^3_\perp$, and their effects are crucial for determining the IR dynamics of the theory. Once again, we suffer from our lack of understanding of the dynamics of intersecting fivebranes. However, for equal fivebrane charges it appears that the effects of one set of fivebranes is taken into account through the background three-form flux, and the Chern–Simons term (3.27) it induces on the other set of fivebranes.

The geometry (3.28) is quite similar to the throat geometry of NS5-branes in flat space; an $\mathbb{R}^3$ parallel to the brane has been replaced by an $\mathbf{S}^3$ whose warp factor is power law in r. By the UV/IR relation of fivebrane holography, this variation in the size of the $\mathbf{S}^3_\parallel$ is logarithmic in the energy scale. Thus, we expect the addition of onebranes to the background to be essentially the same as adding them to fivebranes in flat space, up to additional logarithmic warping. Let us compactify $\mathbb{R}^{1,2}$ to $\mathbb{R}^{1,1} \times \mathbf{S}^1$, and put onebrane sources along $\mathbb{R}^{1,1}$, parametrized by (t, x_5) in keeping with previous notation. The expected form of the metric is then

$$ds^2 = h_1(r)e^{2r}(-dt^2 + dx_5^2) + h_2(r)d\theta^2 + h_3(r)dr^2 + h_4(r)d\Omega^2_{3,\parallel} + h_5(r)d\Omega^2_{3,\perp} \tag{3.29}$$

with $h_i(r)$ having at most polynomial growth at large r (similarly, one expects the dilaton ϕ to vary logarithmically in r). Constant h_i and dilaton ϕ corresponds to $AdS_3 \times \mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^1$, and logarithmic (in energy) dependence

[c] This phenomenon is familiar in the context of supersymetric gauge theories, such as $\mathcal{N} = 1$ super-Yang–Mills in four dimensions, where a similar argument can be used to show that the number of BPS domain walls in a $U(N)$ gauge theory is conserved modulo N [49].

of the geometry would correspond to the RG flow of the dual sigma model toward an infrared CFT.

One might then look for instanton solutions to the gauge theory compactified on $\mathbf{S}^3 \times \mathbf{S}^1$, and propose that the CFT we are interested in is a sigma model on the instanton moduli space, that represents the small fluctuations around these configurations.[d] This is the standard logic by which one motivates the sigma models in the hyperkahler cases of fivebranes wrapping T^4 or $K3$ (see for examples [41,2,3,42,43] for a discussion), and we propose that it can be adapted to the case of $\mathbf{S}^3 \times \mathbf{S}^1$.

4. The superconformal algebra $\mathcal{A}_\gamma$ and its representations

The superconformal symmetry of the spacetime CFT for $AdS_3 \times \mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^1$ consists of left and right copies of the two-dimensional large $\mathcal{N} = 4$ supersymmetry algebra, denoted $\mathcal{A}_\gamma$ [17]. In this section we review some of the properties of this algebra and its representations.

4.1. *The superconformal algebra* $\mathcal{A}_\gamma$

Apart from the usual Virasoro algebra, the large $\mathcal{N} = 4$ superconformal algebra $\mathcal{A}_\gamma$ contains two copies of the affine $\widehat{SU(2)}$ Lie algebras, at the levels k^+ and k^-, respectively. The relation between $k^\pm$ and the parameter γ is

$$\gamma = \frac{k^-}{k^+ + k^-} \ . \tag{4.1}$$

Unitarity implies that the Virasoro central charge is:

$$c = \frac{6k^+k^-}{k^+ + k^-} \ . \tag{4.2}$$

The superconformal algebra $\mathcal{A}_\gamma$ is generated by six affine $\widehat{SU(2)}$ generators $A^{\pm,i}(z)$, four dimension 3/2 supersymmetry generators $G^a(z)$, four dimension 1/2 fields $Q^a(z)$, a dimension 1 field $U(z)$, and the Virasoro current $T(z)$. The OPEs with the Virasoro generators, T_m, have the usual form.

[d] The Chern–Simons term will not affect the solution of the Yang–Mills equations on $\mathbf{S}^3 \times \mathbf{S}^1$; it will, however, generate additional couplings in the sigma model obtained by expanding around the instanton solutions.

The remaining OPEs are [17,50]:

$$
\begin{aligned}
G^a(z)G^b(w) &= \frac{2c}{3}\frac{\delta^{ab}}{(z-w)^3} - \frac{8\gamma\alpha^{+,i}_{ab}A^{+,i}(w) + 8(1-\gamma)\alpha^{-,i}_{ab}A^{-,i}(w)}{(z-w)^2} - \\
&\quad - \frac{4\gamma\alpha^{+,i}_{ab}\partial A^{+,i}(w) + 4(1-\gamma)\alpha^{-,i}_{ab}\partial A^{-,i}(w)}{z-w} + \frac{2\delta^{ab}L(w)}{z-w} + \dots, \\
A^{\pm,i}(z)A^{\pm,j}(w) &= -\frac{k^\pm\delta^{ij}}{2(z-w)^2} + \frac{\epsilon^{ijk}A^{\pm,k}(w)}{z-w} + \dots, \\
Q^a(z)Q^b(w) &= -\frac{(k^+ + k^-)\delta^{ab}}{2(z-w)} + \dots, \\
U(z)U(w) &= -\frac{k^+ + k^-}{2(z-w)^2} + \dots, \\
A^{\pm,i}(z)G^a(w) &= \mp\frac{2k^\pm\alpha^{\pm,i}_{ab}Q^b(w)}{(k^+ + k^-)(z-w)^2} + \frac{\alpha^{\pm,i}_{ab}G^b(w)}{z-w} + \dots, \\
A^{\pm,i}(z)Q^a(w) &= \frac{\alpha^{\pm,i}_{ab}Q^b(w)}{z-w} + \dots, \\
Q^a(z)G^b(w) &= \frac{2\alpha^{+,i}_{ab}A^{+,i}(w) - 2\alpha^{-,i}_{ab}A^{-,i}(w)}{z-w} + \frac{\delta^{ab}U(w)}{z-w} + \dots, \\
U(z)G^a(w) &= \frac{Q^a(w)}{(z-w)^2} + \dots\ .
\end{aligned} \tag{4.3}
$$

Here $\alpha^{\pm,i}_{ab}$ are 4×4 matrices, which project onto (anti)self-dual tensors. Explicitly,

$$\alpha^{\pm,i}_{ab} = \frac{1}{2}\Big(\pm\delta_{ia}\delta_{b0} \mp \delta_{ib}\delta_{a0} + \epsilon_{iab}\Big)\ . \tag{4.4}$$

They obey $SO(4)$ commutation relations,

$$[\alpha^{\pm,i},\alpha^{\pm,j}] = -\epsilon^{ijk}\alpha^{\pm k}\ , \qquad [\alpha^{+,i},\alpha^{-,j}] = 0\ , \qquad \{\alpha^{\pm,i},\alpha^{\pm,j}\} = -\frac{1}{2}\,\delta^{ij}\ . \tag{4.5}$$

It is sometimes useful to employ spinor notation, where for instance $G^a \to G^{\alpha\dot\alpha} = \gamma^{\alpha\dot\alpha}_a G^a$ (and $\gamma^{\alpha\dot\alpha}_a$ are Dirac matrices); $A^{+,i} \to A^{\alpha\beta} = \tau^{\alpha\beta}_i A^{+,i}$ (where τ^i are Pauli matrices); $A^{-,i} \to A^{\dot\alpha\dot\beta} = \tau^{\dot\alpha\dot\beta}_i A^{-,i}$; and so on. Our conventions are spelled out in appendix B.

An important subalgebra of $\mathcal{A}_\gamma$ is denoted $D(2,1|\alpha)$; here $\alpha = k^-/k^+ = \frac{\gamma}{1-\gamma}$. It is generated (in the NS sector) by L_0, $L_{\pm1}$, $G^a_{\pm1/2}$, and $A^{\pm,i}_0$. The superalgebra $D(2,1|\alpha)\times D(2,1|\alpha)$ constitutes the super-isometries of $AdS_3\times \mathbf{S}^3\times\mathbf{S}^3$.

Yet another useful subalgebra of $\mathcal{A}_\gamma$ is the $\mathcal{N}=2$ subalgebra generated by

$$T\ , \quad \mathcal{G}^+ = i\sqrt{2}\, G^{+\dot{+}}\ , \quad \mathcal{G}^- = i\sqrt{2}\, G^{-\dot{-}}\ , \quad J = 2i\,[\gamma A^{+,3} - (1-\gamma)A^{-,3}]\,, \tag{4.6}$$

where the supercurrents are written in spinor notation. For instance, it will be useful to consider the states that are chiral with respect to this $\mathcal{N}=2$.

4.2. *Examples of large $\mathcal{N}=4$ SCFT's*

The simplest example of a large $\mathcal{N}=4$ theory can be realized as a theory of a free boson, ϕ, and four Majorana fermions, ψ_a, $a = 0, \ldots, 3$. Specifically, we have [51]:

$$\begin{aligned} T &= -\frac{1}{2}\,(\partial\phi)^2 - \frac{1}{2}\,\psi^a\partial\psi^a\,, \\ G^a &= -\frac{1}{6}\,i\epsilon^{abcd}\psi^b\psi^c\psi^d - i\psi^a\partial\phi\,, \\ A^{\pm,i} &= \frac{i}{2}\,\alpha^{\pm,i}_{ab}\psi^a\psi^b\,, \\ Q^a &= \psi^a\,, \\ U &= i\partial\phi\ . \end{aligned} \tag{4.7}$$

This theory was called the $\mathcal{T}_3$ theory in [8], but we shall herein use the notation $\mathcal{S}$ for simple. In [8] it was conjectured that in the case $k^+ = k^-$ the boundary SCFT dual to type IIB string theory on $AdS_3 \times \mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^1$ is a sigma-model based on the symmetric product orbifold of this $c = 3$ theory.

The CFT $\mathcal{S}$ belongs to a family of large $\mathcal{N}=4$ theories, labeled by a non-negative integer number κ [17]:

$$\begin{aligned} T &= -J^0J^0 - \frac{J^aJ^a}{\kappa+2} - \partial\psi^a\psi^a\,, \\ G^a &= 2J^0\psi^a + \frac{4}{\sqrt{\kappa+2}}\,\alpha^{+,i}_{ab}J^i\psi^b - \frac{2}{3\sqrt{\kappa+2}}\,\epsilon_{abcd}\psi^b\psi^c\psi^d\,, \\ A^{-,i} &= \alpha^{-,i}_{ab}\psi^a\psi^b\,, \\ A^{+,i} &= \alpha^{+,i}_{ab}\psi^a\psi^b + J^i\,, \\ U &= -\sqrt{\kappa+2}\,J^0\,, \\ Q^a &= \sqrt{\kappa+2}\,\psi^a \end{aligned} \tag{4.8}$$

where J^i denote $SU(2)$ currents at level κ and $J^0(z)J^0(w) \sim -\frac{1}{2}(z-w)^{-2}$. We shall denote these theories $\mathcal{S}_\kappa$. It is easy to check that (4.8) indeed generate the large $\mathcal{N}=4$ algebra with $k^+ = \kappa+1$ and $k^- = 1$. In fact, the $U(2)$ level κ theory of [17] admits *two* distinct large $\mathcal{N} = 4$ algebras. The second algebra is obtained by the outer automorphism and has ($k^+ = 1, k^- = \kappa + 1$):

$$\begin{aligned} T &= -J^0 J^0 - \frac{J^a J^a}{\kappa + 2} - \partial\psi^a \psi^a \,, \\ G^a &= 2J^0 \psi^a + \frac{4}{\sqrt{\kappa+2}}\, \alpha^{+,i}_{ab} J^i \psi^b - \frac{2}{3\sqrt{\kappa+2}}\, \epsilon_{abcd} \psi^b \psi^c \psi^d \,, \\ A^{-,i} &= \alpha^{-,i}_{ab} \psi^a \psi^b + J^i \,, \\ A^{+,i} &= \alpha^{+,i}_{ab} \psi^a \psi^b \,, \\ U &= +\sqrt{\kappa+2}\, J^0 \,, \\ Q^a &= -\sqrt{\kappa+2}\, \psi^a \ . \end{aligned} \tag{4.9}$$

The $c = 3$ CFT $\mathcal{S} = \mathcal{S}_0$ appears as a special case, $\kappa = 0$. We will consider these simple large $\mathcal{N} = 4$ theories below in the context of symmetric product orbifolds as candidates for the spacetime CFT.

Additional examples of large $\mathcal{N} = 4$ are provided by WZW coset models $\mathcal{W} \times U(1)$, where $\mathcal{W}$ is a gauged WZW model associated to a quaternionic (Wolf) space. Examples based on classical groups are $\mathcal{W} = G/H = \frac{SU(n)}{SU(n-2)\times U(1)}$, $\frac{SO(n)}{SO(n-4)\times SU(2)}$, and $\frac{Sp(2n)}{Sp(2n-2)}$. These theories carry large $\mathcal{N} = 4$ supersymmetry, with $k^+ = \kappa + 1$ and $k^- = \check{c}_G$; here κ is the level of the bosonic current algebra for the group G and $\check{c}_G$ its dual Coxeter number. However, they are unsuitable as building blocks for a symmetric product orbifold dual to supergravity. For example, any modulus associated to the RR axion would generically come from the component theory and not the twisted sector of the symmetric product, and would thus not deform the spectrum in the appropriate way as one moves from the orbifold locus to the supergravity regime.[a]

4.3. *Unitary representations*

The unitary representations of the superconformal algebra $\mathcal{A}_\gamma$ are labeled by the conformal dimension h, by the $SU(2)$ spins $\ell^\pm$, and by the $U(1)$ charge u. The generic *long* or *massive* representation has no null vectors under

[a] Also, the BPS spectrum of these theories does not seem to have the requisite properties. The BPS states are associated to the cohomology of $\mathcal{W}$, which is in turn related to the elements of the Weyl group of affine G (related to the symmetric group). Instead, in order to match the structure of supergravity, one typically would want the cohomology to be associated to the conjugacy classes of the symmetric group, as in the orbifold cohomology of the symmetric product, whose cohomology matches supergravity in for example the D1-D5 system.

the raising operators of the algebra. On the other hand, the highest weight states $|\Omega\rangle_{\mathcal{A}_\gamma}$ of *short* or *massless* representations have the null vector [19]

$$\Big(G^{++\dot{+}}_{-1/2} - \frac{2u}{k^+ + k^-}\, Q^{++\dot{+}}_{-1/2} - \frac{2i(\ell^+ - \ell^-)}{k^+ + k^-}\, Q^{++\dot{+}}_{-1/2}\Big)|\Omega\rangle_{\mathcal{A}_\gamma} = 0 \ . \tag{4.10}$$

(We have used the property that $|\Omega\rangle_{\mathcal{A}_\gamma}$ is a highest weight state for the $SU(2)$ current algebras.) Squaring this null vector leads to a relation among the spins $\ell^\pm$ and the conformal dimension h [19,20,21,9]

$$h_{\rm short} = \frac{1}{k^+ + k^-}\left(k^-\ell^+ + k^+\ell^- + (\ell^+ - \ell^-)^2 + u^2\right) \ . \tag{4.11}$$

Unitarity demands that all representations, short or long, lie at or above this bound: $h \geq h_{\rm short}$; and that the spins lie in the range $\ell^\pm = 0, \frac{1}{2}, ..., \frac{1}{2}(k^\pm - 1)$. When we consider $U(1)$ singlets, we shall denote representations by their labels (h, ℓ^+, ℓ^-); for short representations with $u = 0$ it is sufficient to specify them simply by (ℓ^+, ℓ^-). The representations of the spacetime SCFT can be obtained by combining left and right sectors. Following [9], we shall label such (short) representations by $(\ell^+, \ell^-; \overline{\ell}^+, \overline{\ell}^-)$.

The conformal dimension of short representations is protected, as long as they do not combine into long ones. The highest weight components of operators in short representations with $\ell^+ = \ell^-$ form a ring. Their dimensions are additive, since $h = \ell^+ = \ell^-$. This ring is the chiral ring of the $\mathcal{N} = 2$ subalgebra of $\mathcal{A}_\gamma$ introduced in subsection 4.1.

We will also be interested in the representations of the super-isometry group $D(2,1|\alpha)$; for example, the normalizable wavefunctions on $AdS^3 \times \mathbf{S}^3 \times \mathbf{S}^3$ lie in representations of $D(2,1|\alpha) \times D(2,1|\alpha)$. A *short* $D(2,1|\alpha)$ representation $(\ell^+, \ell^-)_{\mathbf{s}}$ of $D(2,1|\alpha)$ has a highest weight vector $|\Omega\rangle_D$ which obeys the condition

$$G^{++\dot{+}}_{-1/2}|\Omega\rangle_D = 0 \ . \tag{4.12}$$

Long representations $(\ell^+, \ell^-)_{\mathbf{l}}$ have no such null vector in the action of $G^a_{-1/2}$.

The base of a short representation $(\ell^+, \ell^-)_{\mathbf{s}}$ of $D(2,1|\alpha)$ can be obtained by acting with $G^a_{-1/2}$:

$$\begin{array}{lccc}
h & & (\ell^+, \ell^-) & \\
h + \frac{1}{2} & (\ell^+ - \frac{1}{2}, \ell^- - \frac{1}{2}) & (\ell^+ + \frac{1}{2}, \ell^- - \frac{1}{2}) & (\ell^+ - \frac{1}{2}, \ell^- + \frac{1}{2}) \\
h + 1 & (\ell^+, \ell^- - 1) & (\ell^+ - 1, \ell^-) & (\ell^+, \ell^-) \\
h + \frac{3}{2} & & (\ell^+ - \frac{1}{2}, \ell^- - \frac{1}{2}) &
\end{array} \tag{4.13}$$

with the rest of the representation filled out by the action of L_{-1}. We denote these short representations of $D(2,1|\alpha)$ by $(\ell^+,\ell^-)_{\mathbf{s}}$ with lower case subscript $\mathbf{s}$ to distinguish them from short representations $(\ell^+,\ell^-)_S$ of $\mathcal{A}_\gamma$. These representations are smaller than the generic large $D(2,1|\alpha)$ representation, due to the absence of a spin $(\ell^+ + \frac{1}{2}, \ell^- + \frac{1}{2})$ component in $G^a_{-1/2}|h,\ell^+,\ell^-\rangle$. Note that since $\ell^\pm$ take only non-negative values, short representations with $\ell^\pm < 1$ have some of the components missing, see e.g. [9] and below.

Squaring the null vector (4.12) and using the algebra gives the BPS bound $h = \frac{k^+\ell^- + k^-\ell^+}{k^+ + k^-}$. Note that *unless* $u = 0$ *and* $\ell^+ = \ell^-$ *the* $\mathcal{A}_\gamma$ *and* $D(2,1|\alpha)$ *BPS conditions are different.* In particular, if $u \neq 0$ or $\ell^+ \neq \ell^-$, then a BPS state in the $\mathcal{A}_\gamma$ sense is *not* a BPS state in the $D(2,1|\alpha)$ sense. In fact, by unitarity, if $\ell^+ \neq \ell^-$ then the representation (ℓ^+,ℓ^-) of $\mathcal{A}_\gamma$ cannot contain any BPS representations of $D(2,1|\alpha)$!

The representations of $\mathcal{A}_\gamma$ can however be decomposed into representations of $D(2,1|\alpha)$. Let $\rho(\ell^+,\ell^-,u)$ be a short representation of $\mathcal{A}_\gamma$. Then, as a representation of $D(2,1;\alpha)$ ρ contains:

a.) All long $D(2,1|\alpha)$ representations for $u \neq 0$ or for $\ell^+ \neq \ell^-$.
b.) Exactly two short $D(2,1|\alpha)$ representations for $\ell^+ = \ell^-$ and $u = 0$. That is,

$$\rho(\ell,\ell,0) = (\ell,\ell)_{\mathbf{s}} + (\ell + \tfrac{1}{2}, \ell + \tfrac{1}{2})_{\mathbf{s}} + \cdots \tag{4.14}$$

where all representations in $\cdots$ satisfy $h > \frac{k^+\ell^- + k^-\ell^+}{k^+ + k^-}$.

Part *(a)* is trivial. From the $\mathcal{A}_\gamma$ bound we get the inequality:

$$(k^+ + k^-)h = k^+\ell^- + k^-\ell^+ + (\ell^+ - \ell^-)^2 + u^2 > k^+\ell^- + k^-\ell^+ \ . \tag{4.15}$$

We have also explained this in detail in comparing the highest weight conditions above. For part *(b)* we take the BPS highest weight state $|\Omega\rangle_{\mathcal{A}_\gamma}$ for $\mathcal{A}_\gamma$. Under the conditions of part *(b)* this is also a BPS highest weight state for $D(2,1|\alpha)$. We also have the state

$$Q^{+\dot{+}}_{-1/2}|\Omega\rangle_{\mathcal{A}_\gamma} \ . \tag{4.16}$$

This is a descendent in the $\mathcal{A}_\gamma$ representation, but since

$$\begin{aligned} [A_0^{\pm,+}, Q^{+\dot{+}}_{-1/2}] &= 0\,, \\ \{G^{+\dot{+}}_{-1/2}, Q^{+\dot{+}}_{-1/2}\} &= 0 \end{aligned} \tag{4.17}$$

the state (4.16) is a BPS highest weight state for the $D(2,1;\alpha)$ subalgebra. It generates the representation $(\ell+1/2, \ell+1/2)_{\mathbf{s}}$. Finally, we must show there are no other short $D(2,1|\alpha)$ highest weight vectors. The BPS bound is linear in h, ℓ and must be obtained from the $\mathcal{A}_\gamma$ highest weight state by applying $G_{+,-1/2}$ and $Q_{+,-1/2}$. Using (4.17) above we see that the only state we can generate is the second one we have already accounted for. The two short representations in (4.14) are distinct from a long representation of $D(2,1|\alpha)$, even though they have the same spin content.

4.4. *The general structure of marginal deformations*

Our goal in this subsection is to identify the states in the spacetime CFT which correspond to moduli of the type IIB string theory on $AdS_3 \times \mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^1$. Such states should have conformal dimensions $(h, \overline{h}) = (1,1)$ and must be $SU(2)\times SU(2)$ singlet components of short multiplets. Inspection of (4.13) shows that marginal deformations in the large $\mathcal{N}=4$ superconformal field theory come from upper components of the $u=0$ short multiplets $(\frac{1}{2},\frac{1}{2})_S$. These representations have the $D(2,1|\alpha)$ structure

$$\begin{array}{lccc} h=\frac{1}{2} & & (\frac{1}{2},\frac{1}{2}) & \\ h=1 & (0,0) & (1,0) & (0,1) \\ h=\frac{3}{2} & & (\frac{1}{2},\frac{1}{2}) & \\ h=2 & & (0,0) & \end{array} \tag{4.18}$$

and so are even more truncated than the generic short representation. The spin $(0,0)$ state on the second level is dimension one and invariant under the $SU(2)\times SU(2)$ R-symmetry; acting by the raising operators $G^{\alpha\dot{\alpha}}_{-1/2}$ gives only L_{-1} descendants, as we will now show momentarily.

First we note a very interesting consequence of the structure (4.18): there are no constraints on the number of moduli. This result should be compared with a similar situation in superconformal theories based on the small $\mathcal{N}=4$ algebra, where marginal deformations are also upper components of chiral primary states $\Phi_{\alpha\overline{\beta}}$ with $(j,\overline{j}) = (\frac{1}{2},\frac{1}{2})$. However, in that case every short multiplet contains four singlet states with $(h,\overline{h}) = (1,1)$, namely, $\mathcal{T}^{a\overline{b}} = G^{a\alpha}_{-\frac{1}{2}}\overline{G^{b\beta}_{-\frac{1}{2}}}\Phi_{\alpha\overline{\beta}}$, where $a, \overline{b}$ are custodial $SU(2)$ indices. In particular, the number of massless moduli has to be a multiple of 4.

A large $\mathcal{N}=4$ chiral primary with $\ell^+ = \ell^-$ has the null vector (4.10), which may be written more invariantly,

$$G_{-\frac{1}{2}(\dot{\alpha}}^{\;(\alpha}\,\Phi^{\alpha_1\ldots\alpha_n)}_{\dot{\alpha}_1\ldots\dot{\alpha}_n)} = 0\ . \tag{4.19}$$

The candidate modulus operator is

$$\mathcal{T} = G^{\beta\dot{\beta}}_{-\frac{1}{2}}\overline{G}^{\overline{\alpha\dot{\alpha}}}_{-\frac{1}{2}}\Phi_{\beta\dot{\beta};\overline{\alpha\dot{\alpha}}}\ , \tag{4.20}$$

where $\Phi_{\beta\dot{\beta};\overline{\alpha\dot{\alpha}}}$ has $(\ell^+,\ell^-;\overline{\ell}^+,\overline{\ell}^-) = (\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2})$. We will for the remainder of the discussion suppress the anti-holomorphic structure, which will not be needed explicitly. Expanding in components,

$$G^{\beta\dot{\beta}}_{-\frac{1}{2}}\Phi_{\beta\dot{\beta}} = G^{+\dot{+}}_{-\frac{1}{2}}\Phi^{-\dot{-}} + G^{-\dot{-}}_{-\frac{1}{2}}\Phi^{+\dot{+}} - G^{+\dot{-}}_{-\frac{1}{2}}\Phi^{-\dot{+}} - G^{-\dot{+}}_{-\frac{1}{2}}\Phi^{+\dot{-}} \ . \tag{4.21}$$

The supercharge anticommutation relations and the nullvector condition (4.19) then imply

$$G^{+\dot{+}}_{-\frac{1}{2}}(G^{+\dot{-}}_{-\frac{1}{2}}\Phi^{-\dot{+}} + G^{-\dot{+}}_{-\frac{1}{2}}\Phi^{+\dot{-}}) = \{G^{+\dot{-}}_{-\frac{1}{2}}, G^{-\dot{+}}_{-\frac{1}{2}}\}\Phi^{+\dot{+}} = -L_{-1}\Phi^{+\dot{+}} \ . \tag{4.22}$$

Similarly, we have

$$G^{+\dot{+}}_{-\frac{1}{2}}(G^{+\dot{+}}_{-\frac{1}{2}}\Phi^{-\dot{-}} + G^{-\dot{-}}_{-\frac{1}{2}}\Phi^{+\dot{+}}) = \{G^{+\dot{+}}_{-\frac{1}{2}}, G^{-\dot{-}}_{-\frac{1}{2}}\}\Phi^{+\dot{+}} = L_{-1}\Phi^{+\dot{+}} \ ; \tag{4.23}$$

putting it all together, we have [b]

$$G^{\alpha\dot{\alpha}}_{-\frac{1}{2}}(G^{\beta\dot{\beta}}_{-\frac{1}{2}}\Phi_{\beta\dot{\beta}}) = 2\partial\Phi^{\alpha\dot{\alpha}} \ . \tag{4.24}$$

Thus, while the candidate modulus is not the highest component of the supermultiplet based on $\Phi^{\alpha\dot{\alpha}}$, it nevertheless varies into a total derivative under the action of the supercharges and so its integral preserves all the supersymmetries. All that remains to be checked is that it preserves conformal invariance. A proof of conformal invariance to all orders in conformal perturbation theory, following [23], is given in Appendix A.

As an aside, it is curious that it appears not to be possible to write this candidate modulus operator as an integral over even $\mathcal{N} = 1$ superspace! In particular, we cannot directly use the results of Dixon [23] on the marginality of $h = 1$, $\mathcal{N} = 2$ chiral operators, even though the lowest component $\Phi_{\alpha\dot{\alpha}}$ of the multiplet is a chiral operator under the canonical $\mathcal{N} = 2$ subalgebra of large $\mathcal{N} = 4$. The argument of [23] uses the structure of $\mathcal{N} = 2$ chiral superspace integrals in an essential way. Fortunately, it is possible to adapt the analysis to fit the structure of large $\mathcal{N} = 4$.

A key ingredient of the analysis of Appendix A is the demonstration that, in the partition function, one can replace the operator (4.20) by the operator

$$\begin{aligned}\widetilde{\mathcal{T}} =& \Big(G^{+\dot{+}}_{-\frac{1}{2}}\overline{G}^{+\dot{+}}_{-\frac{1}{2}}\Phi_{+\dot{+},+\dot{+}} + G^{-\dot{-}}_{-\frac{1}{2}}\overline{G}^{-\dot{-}}_{-\frac{1}{2}}\Phi_{-\dot{-},-\dot{-}}\Big) \\ &+ \Big(G^{-\dot{-}}_{-\frac{1}{2}}\overline{G}^{+\dot{+}}_{-\frac{1}{2}}\Phi_{-\dot{-},+\dot{+}} + G^{+\dot{+}}_{-\frac{1}{2}}\overline{G}^{-\dot{-}}_{-\frac{1}{2}}\Phi_{+\dot{+},-\dot{-}}\Big)\end{aligned} \tag{4.25}$$

which is a sum, in equal proportion, of a chiral and a twisted chiral modulus under the canonical $\mathcal{N} = 2$ algebra (4.6); moreover, the chiral and twisted

[b] Similarly, one can show that $G^{\alpha\dot{\alpha}}_{+\frac{1}{2}}(G^{\beta\dot{\beta}}_{-\frac{1}{2}}\Phi_{\beta\dot{\beta}}) = 2\Phi^{\alpha\dot{\alpha}}$.

chiral moduli are real. If we were to give each term in (4.25) a different coefficient (compatible with hermiticity), we would explore the moduli space of an $\mathcal{N}=2$ superconformal theory. This theory is manifestly self-mirror. The large $\mathcal{N}=4$ locus on this moduli space is thus the fixed point set under both the mirror map, and also the antiholomorphic involution of the $\mathcal{N}=2$ algebra.[c] This rather constrains the geometry of the moduli space; it would be interesting if the structure of large $\mathcal{N}=4$ could yield further information about this geometry.

There is a universal $(\frac{1}{2},\frac{1}{2})_S$ representation that canonically appears in the theory – the singleton bilinear $U\overline{U}$. In the application to $AdS_3\times\mathbf{S}^3\times\mathbf{S}^3\times\mathbf{S}^1$, it implements (among other things) a change in the boundary condition on the corresponding bulk gauge field [52]. There is also a second modulus associated to the $\mathbf{S}^1$, the mode which corresponds to changing the $\mathbf{S}^1$ radius in supergravity, the combination of the metric and dilaton found in equation (2.47). In general the structure is rather complicated, since these two moduli mix non-trivially. Conventionally, the singleton bilinear modulus is turned off, and only the supergraviton mode is considered. We will make the same restriction here.

We are also expecting that the spacetime CFT contains another modulus, corresponding to the RR axion, as discussed in Section 2. In symmetric products of the $U(2)$ WZW model, we will find the corresponding marginal deformation in twisted sectors.

4.5. *Spectral flow*

Since the superconformal algebra (4.3) contains two copies of $SU(2)$, there are several types of spectral flow one can consider.[d] Following [53], let us call the corresponding parameters ρ and η. Then, the relation between the generators looks like [53],

$$\begin{aligned}
L_m^{\rho;\eta} &= L_m - i(\rho A_m^{+3} + \eta A^{-3}) + \frac{1}{4}(k^+\rho^2 + k^-\eta^2)\delta_{0,m}\,,\\
A_m^{\rho;\eta;+3} &= A_m^{+3} + \frac{i}{2}\rho k^+\delta_{m,0}\,,\\
A_m^{\rho;\eta;-3} &= A_m^{-3} + \frac{i}{2}\eta k^-\delta_{m,0}\,,\\
U_m^{\rho;\eta} &= U_m\ .
\end{aligned} \tag{4.26}$$

[c] Note that this reduces a $4n$ dimensional moduli space to an n dimensional one. Again there is no constraint on n.

[d] In the case $k^+ = k^-$, the superconformal algebra $\mathcal{A}_\gamma$ has additional automorphisms, which we are not going to discuss here.

The Neveu-Schwarz sector corresponds to $(\rho,\eta) = (0,0)$, whereas the Ramond sector can be obtained by a spectral flow with $\rho = 1$, $\eta = 0$ or $\rho = 0$, $\eta = 1$. From (4.26) one finds the following relation between the conformal dimensions and other quantum numbers in the Ramond and Neveu-Schwarz sectors (note, that our transformations of $\ell^{\pm}$ differ from those given in [19]),

$$\begin{aligned} h_R &= h_{NS} - \ell^+_{NS} + \frac{1}{4}\,k^+ \,, \\ \ell^+_R &= \ell^+_{NS} - \frac{1}{2}\,k^+ \,, \\ \ell^-_R &= \ell^-_{NS} \,, \\ u_R &= u_{NS} \end{aligned} \tag{4.27}$$

for the spectral flow in the $SU(2)^+$. Similarly, for the spectral flow in the $SU(2)^-$, we have

$$\begin{aligned} h_R &= h_{NS} - \ell^-_{NS} + \frac{1}{4}\,k^- \,, \\ \ell^+_R &= \ell^+_{NS} \,, \\ \ell^-_R &= \ell^-_{NS} - \frac{1}{2}\,k^- \,, \\ u_R &= u_{NS} \;. \end{aligned} \tag{4.28}$$

In particular, NS states saturating the BPS bound (4.11) flow to R states with

$$h_R - \frac{c}{24} = \frac{(\ell^+ + \ell^-)^2 + u^2}{k^+ + k^-} \;. \tag{4.29}$$

Note the rather peculiar fact that the right-hand side is nonzero. Here again we see an important qualitative difference between the large $\mathcal{N} = 4$ algebra and other superconformal algebras.

The $\mathcal{N} = 2$ subalgebra (4.6) leads to yet another version of spectral flow to a Ramond sector, with $\rho = \eta = 1/2$. This leads to Ramond boundary conditions for the $\mathcal{N} = 2$ currents $\mathcal{G}^{\pm} = i\sqrt{2}G^{\pm\pm}$, but the boundary conditions on $G^{\pm\mp}$ become fractionally moded (as do the raising and lowering operators $A^{\pm,i}$, $i = \pm$, of the two $SU(2)$'s).

4.6. *An index for theories with $\mathcal{A}_\gamma$ symmetry*

When one is working with families of theories with $\mathcal{A}_\gamma$ symmetry, as we are in the present paper, it is useful to know quantities which are invariant under deformations. The traditional elliptic genus does not provide useful information in the present context, but one can nevertheless define an index which summarizes some important information about the BPS spectrum of the theory and which remains invariant under deformations. In this section we briefly define such an index.[e] Further details and comments can be found in a companion paper [24] where we investigate this large $\mathcal{N}=4$ index in some detail.

The representation content of a theory with $\mathcal{A}_\gamma$ symmetry is summarized by the RR sector supercharacter,

$$Z(\tau,\omega_+,\omega_-;\overline{\tau},\widetilde{\omega}_+,\widetilde{\omega}_-) := \mathrm{Tr}_{\mathcal{H}_{RR}} q^{L_0-\frac{c}{24}} \widetilde{q}^{\widetilde{L}_0-\frac{c}{24}} z_+^{2T_0^{+,3}} (-z_-)^{2T_0^{-,3}} \widetilde{z}_+^{2\widetilde{T}_0^{+,3}} (-\widetilde{z}_-)^{2\widetilde{T}_0^{-,3}}. \tag{4.30}$$

Here and hereafter we denote $z_\pm = e^{2\pi i\omega_\pm}$ for left-movers and $\widetilde{z}_\pm = e^{2\pi i\widetilde{\omega}_\pm}$ for right-movers. The spectrum in other sectors can be obtained from (4.30) by spectral flow.

Now (4.30) can be expanded in the supercharacters of the irreducible representations, defined by

$$\mathrm{SCh}(\rho)(\tau,\omega_+,\omega_-) = \mathrm{Tr}_\rho q^{L_0-c/24} z_+^{2T_0^{+,3}} z_-^{2T_0^{-,3}} (-1)^{2T_0^{-,3}} \tag{4.31}$$

we just write $\mathrm{SCh}(\rho)$ when the arguments are understood. Explicit formulae for these characters have been derived by Peterson and Taormina. Using the formulae of [21] one finds that short representations have a character with a first order zero at $z_+ = z_-$, while long representations have a character with a second order zero at $z_+ = z_-$.[f]

Thanks to the second order vanishing of the characters of long representations we can define the *left-index* of the CFT $\mathcal{C}$ by

$$I_1(\mathcal{C}) := -z_+ \frac{d}{dz_-}\bigg|_{z_-=z_+} Z\ . \tag{4.32}$$

Only short representations can contribute on the left. On the right, long representations might contribute. However, due to the constraint $h-\overline{h} = 0 \mathrm{mod} 1$ the right-moving conformal weights which do contribute are rigid, and hence I_1 is a deformation invariant.

Of course, one could also define a right-index. Since we will consider left-right symmetric theories here this is redundant information. Nevertheless,

[e] For a related discussion see also [22].

[f] The fact that all characters vanish at $z_+ = z_-$ is a reflection of the fact that one can always make a GKO coset construction factoring out the free $\mathcal{S}$-theory defined by $U, Q^{A\dot{A}}$. The character of this theory has a first order zero. The characters of the quotient $\widetilde{\mathcal{A}_\gamma}$ W-algebra are nonvanishing for short representations, and have a first order zero for long representations.

it is often useful to define the left-right index

$$I_2(\mathcal{C}) := z_+ \widetilde{z}_+ \frac{d}{dz_-} \frac{d}{d\widetilde{z}_-} Z\,, \tag{4.33}$$

where one evaluates at $z_- = z_+$, $\widetilde{z}_- = \widetilde{z}_+$.

4.7. *Digression: Taking the tensor product of two large $\mathcal{N} = 4$ algebras*

Although it is not directly used in the present paper, we would like to mention in this section on $\mathcal{A}_\gamma$ symmetry a curious behavior of these theories under the tensor product operation. Since the Virasoro central charge (4.2) is nonlinear it is therefore not obvious how to take a tensor product of algebras $\mathcal{A}(k_1^+, k_1^-)$ with $\mathcal{A}(k_2^+, k_2^-)$.

The tensor product formula is given as follows. Denote the generators of the two *commuting* $\mathcal{N} = 4$ algebras by G_1^a, G_2^a, etc. Then we form:

$$\begin{aligned} T &= T_1 + T_2 + \frac{1}{2}\,\partial(pU_1 + qU_2)\,, \\ G^a &= G_1^a + G_2^a + \partial(pQ_1^a + qQ_2^a)\,, \\ A^{\pm,i} &= A_1^{\pm,i} + A_2^{\pm,i}\,, \\ Q^a &= Q_1^a + Q_2^a\,, \\ U &= U_1 + U_2\,, \end{aligned} \tag{4.34}$$

with

$$p = 2\,\frac{k_1^+ k_2^- - k_1^- k_2^+}{k_1(k_1 + k_2)}\,, \tag{4.35}$$

$$q = 2\frac{k_2^+ k_1^- - k_2^- k_1^+}{k_2(k_1 + k_2)}\,, \tag{4.36}$$

where $k_i = k_i^+ + k_i^-$, $i = 1, 2$. Moreover, this is the unique way of combining the generators to form a large $\mathcal{N} = 4$ algebra.

Remarks

1. The computation of the AG commutator shows that one *cannot* give a Feigin-Fuks deformation of a single copy of the large $\mathcal{N} = 4$ algebra, it is too rigid. This is actually a special case of (4.35)(4.36) with $k_2^\pm = 0$.
2. Note that $p = q = 0$ when $k_1^\pm = \lambda k_2^\pm$. Thus, for example, in symmetric products the generators are simply made by direct sum.

3. Given a large $\mathcal{N}=4$ algebra one can form [17] a small $\mathcal{N}=4$ algebra $\widehat{\mathcal{A}} = (\widehat{L}, \widehat{G}^a, A^+)$ with $c = 6k^+$ and

$$\begin{aligned} \widehat{T} &= T + \frac{k^+}{k}\,\partial U\,, \\ \widehat{G}^a &= G^a + 2\,\frac{k^+}{k}\,\partial Q^a\,. \end{aligned} \tag{4.37}$$

On the other hand, one can also form a small $\mathcal{N}=4$ algebra $\check{\mathcal{A}} = (\check{L}, \check{G}^a, A^-)$ with $c = 6k^-$ and

$$\begin{aligned} \check{T} &= T - \frac{k^-}{k}\,\partial U\,, \\ \check{G}^a &= G^a - 2\frac{k^-}{k}\partial Q^a. \end{aligned} \tag{4.38}$$

We find that $\widehat{\mathcal{A}} = \widehat{\mathcal{A}}_1 \oplus \widehat{\mathcal{A}}_2$ and $\check{\mathcal{A}} = \check{\mathcal{A}}_1 \oplus \check{\mathcal{A}}_2$ are small $\mathcal{N}=4$ algebras; and now note that p and q in (4.34) have opposite signs.

4. It is useful to state the combination rule in terms of an effective bosonizing field defined by

$$U := \sqrt{\frac{k}{2}}\,\partial\phi\ . \tag{4.39}$$

Then when combining two algebras we have, by (4.34)

$$\phi_{12} := \sqrt{\frac{k_1}{k_1+k_2}}\,\phi_1 + \sqrt{\frac{k_2}{k_1+k_2}}\,\phi_2\ . \tag{4.40}$$

The orthogonal linear combination is a linear-dilaton field,

$$\phi_L := \sqrt{\frac{k_2}{k_1+k_2}}\,\phi_1 - \sqrt{\frac{k_1}{k_1+k_2}}\,\phi_2 \tag{4.41}$$

contributing to the stress tensor as

$$T = -\frac{1}{2}\,(\partial\phi_L)^2 + \frac{Q_{12}}{2}\,\partial^2\phi_L + \cdots\,, \tag{4.42}$$

where

$$Q_{12} = \sqrt{2}\,\frac{k_1^+k_2^- - k_2^+k_1^-}{\sqrt{k_1k_2(k_1+k_2)}}\ . \tag{4.43}$$

Note the interesting fact that if we combine three theories then

$$Q_{(12)3} \neq Q_{1(23)}\ . \tag{4.44}$$

So this operation of combining large $N=4$ theories is *nonassociative*!

5. Symmetric product CFT's with large $\mathcal{N} = 4$

As we have mentioned, a natural candidate for the spacetime CFT dual is the symmetric product of a simple CFT $\mathcal{S}_\kappa$ with large $\mathcal{N} = 4$

$$\mathrm{Sym}^N(\mathcal{S}_\kappa) = (\mathcal{S}_\kappa)^N / S_N \ . \tag{5.1}$$

More precisely, we would like to explore the possibility that this symmetric product CFT is on the same moduli space as the supergravity regime of string theory on $AdS_3 \times \mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^1$; the perturbative CFT regime and the supergravity regime are typically well-separated in the moduli space, as befits a strong-weak coupling duality.

5.1. *General structure of symmetric product orbifolds*

To begin, let us recall some of the features of symmetric product orbifolds that suggest their relation to supergravity. First and foremost is the match between the BPS spectra. We will discuss in detail this matching below, for specific examples related to $AdS_3 \times \mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^1$. First let us discuss the common features of all such orbifolds.

BPS states of an orbifold come from the ground states of twisted sectors. Twisted sectors are in one-to-one correspondence with the conjugacy classes of the orbifold group. In the case of the symmetric product orbifold, the conjugacy classes $[g]$ of $g \in S_N$ can be decomposed into combinations of cyclic permutations, $[g] = \prod (n_i)^{m_i}$, where (n) is a cycle of length n in S_N. This carries the structure of a Fock space of identical particles, in that cycles of the same length are symmetrized over, and represent identical objects. One is thus led to the idea that twist operators for single cycles create one-particle states from the CFT vacuum, and that twist operators containing several cycles correspond to multiparticle states.

Of course, the notion of Fock space only makes sense at weak coupling, i.e. large N. Consider the twist operator for a cycle of length n [a]

$$\mathcal{O}_n = \frac{\lambda_n}{N!} \sum_{h \in S_N} \sigma_{h(1\ldots n)h^{-1}} \ , \tag{5.2}$$

where $\sigma_{(1\ldots n)}$ is the normalized twist operator permuting the first n copies of $\mathcal{S}$,

$$\langle \, \sigma^\dagger_{(1\ldots n)} \, \sigma_{(1\ldots n)} \, \rangle = 1 \ . \tag{5.3}$$

[a] The discussion here parallels [54].

We will abbreviate $\sigma_{(1\ldots n)} \equiv \sigma_n$. If we then demand that $\mathcal{O}_n$ is unit normalized, we find

$$\lambda_n = \left[\frac{n\,(N-n)!}{N!}\right]^{1/2} \sim \sqrt{n\,N^{-n}} \quad , \qquad N \to \infty \tag{5.4}$$

by elementary combinatorics. The operator product of twists obeys the selection rules

$$\sigma_m \sigma_n \sim \sum_p C^p_{mn}[\sigma_p] \quad , \qquad p \in \{\,|m-n|+1\,,\,|m-n|+3\,,\,\ldots,m+n-1\,\}\ , \tag{5.5}$$

and one can readily see that at large N one has

$$\langle\, \mathcal{O}_m\, \mathcal{O}_n\, \mathcal{O}_p\,\rangle \sim \sqrt{\frac{mnp}{N}}\; C_{mnp}\,, \tag{5.6}$$

where the leading behavior of C_{mnp} is N-independent. The C_{mnp} can be calculated [54] using an application of the covering space method of [55]. This scaling is consistent with that of the string coupling in the NS background, $g_B^2 \sim 1/Q_1 \propto 1/c$. Note the similarity to the large N scaling of operator products in the AdS_5/SYM duality [56].

When we apply an operator to one of the states of the symmetric product, say for instance the ground state of a cyclic twist, at large N the result will be predominantly states in twisted sectors with two cycles (we assume that the second operator does not simply annihilate the first one) with coefficient $O(N^0)$; there will also be a small admixture at order $N^{-1/2}$ of twist sectors of single cycles according to the interaction (5.6). At large N, the mixing of various twisted sectors is suppressed by $N^{-1/2}$. Consequently, it is natural to identify the cycles of the symmetric orbifold as the analogue of single trace operators in gauge theory, which realize the single particle excitations of supergravity; we may regard the twist operators for cycles as the creation/annihilation operators for single particles. This structure will be important below in understanding the Hilbert space.

5.2. *Twist operators and moduli of the symmetric product*

As explained in Section 4.4, the moduli are BPS states of the form $(\frac{1}{2},\frac{1}{2})_S$. Any large $\mathcal{N}=4$ theory contains at least one modulus, $U\overline{U}$, which changes the radius of the $U(1)$ in the algebra. In a symmetric product orbifold, this yields two moduli in the untwisted sector: $\sum_i U_i \overline{U}_i$, and $|\sum_i U_i|^2$. The latter 'bi-singleton' perturbation does not correspond to a single particle operator in supergravity.[b] We then wish to identify the former with the combination of the dilaton operator and the $\mathbf{S}^1$ radius deformation which is the modulus $\Im\tau$ of supergravity (see Section 2.5). If a symmetric product is related to the spacetime CFT dual to supergravity, we need to find the second modulus corresponding to $\Re\tau$ in supergravity, equation (2.51). A simple theory, such as any of the $\mathcal{S}_\kappa$ theories (4.7), (4.9), has no additional moduli beyond the universal one; any second modulus must come from twisted sectors of the symmetric product orbifold.

In this subsection, we will construct not only this marginal twist operator, but also all the chiral twist operators of the symmetric product (that is, all the operators with $h = \ell^+ = \ell^-$ and $u = 0$ which are chiral under the $\mathcal{N} = 2$ subalgebra of large $\mathcal{N} = 4$). As discussed above, the single-particle chiral operators are built out of the more basic chiral twist operators for $\mathbb{Z}_n$ cyclic twists.

In order to construct cyclic chiral twist fields in the symmetric product CFT (5.1), it is convenient to recall the properties of a generic symmetric product CFT based on $Sym^N(\mathcal{S}_\kappa)$, where $\mathcal{S}_\kappa$ has central charge c.[c] Given an operator in $\mathcal{S}_\kappa$ with dimension h_0 and R-charge R_0, there is an operator in the $\mathbb{Z}_n$ twisted sector of $\mathcal{S}_\kappa^n/\mathbb{Z}_n$ with dimension and R-charge given by [58]

$$h_n = \frac{h_0}{n} + \frac{c}{24}\,\frac{n^2-1}{n}\ , \qquad R_n = R_0\ . \tag{5.7}$$

For example, if we apply this formula to the ground state $h_0 = R_0 = 0$ of the Neveu-Schwarz sector, we obtain a singlet state in the $\mathbb{Z}_n$ twisted sector with conformal dimension

$$h_{n,\mathrm{gd}} = \frac{c}{24}\left(n - \frac{1}{n}\right)\ . \tag{5.8}$$

This state corresponds to a non-chiral twist operator σ_n which permutes the copies of $\mathcal{S}_\kappa$. The dimension (5.8) of the twist operator σ_n can be understood as a difference between the vacuum energy in a theory based on n separate

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[b] It is the analogue of a double-trace operator in the $\mathcal{N} = 4$ SYM/$AdS_5 \times \mathbf{S}^5$ correspondence [57]. Such perturbations also exist in the small $\mathcal{N} = 4$ theory on $AdS_3 \times \mathbf{S}^3 \times T^4$ [52]; one has the 20 moduli from supergravity deformations of the background, and in addition an $8 \times 8 = 64$ dimensional moduli space from the eight left- and right-moving currents coupling to the charges of wrapped branes on T^4.

[c] The following analysis can be made for any symmetric product CFT. Those based on $\mathcal{S}_\kappa$ are of special interest as effective theories for the GKS long strings discussed in Section 8.

copies of $\mathcal{S}_\kappa$ and a theory on a single copy of $\mathcal{S}_\kappa$, defined on the covering space defined by the map $t \sim z^n$ of the parameter space of the CFT.

In order to build the chiral twist spectrum, we must use nontrivial operators of $\mathcal{S}_\kappa$ carrying the appropriate R-charges. Recall the $\mathcal{S}_\kappa$ theory consists of a bosonic SU(2) WZW model at level $\kappa = k^- - 1$, a free boson, and four free fermions. The scale dimensions of the conformal highest weight states of these respective factors, and their contributions to the various R-charges (the $SU(2)$ spins $\ell^\pm$ and the $U(1)$ charge u), are as follows: The bosonic contributions are

$$\begin{aligned} h_b &= \Big[\frac{j(j+1)}{k^-+1} + jw + \frac{w^2(k^- - 1)}{4}\Big] + u^2\,, \\ \ell_b^- &= j + \tfrac{1}{2}w(k^- - 1)\ , \quad j = 0, \tfrac{1}{2}, ..., \tfrac{1}{2}(k^- - 1)\ , \quad w = 0, 1, 2, ...\,, \\ \ell_b^+ &= 0 \end{aligned} \tag{5.9}$$

(here j is the spin of an $SU(2)$ level $k^- - 1$ highest weight representation, and w is a spectral flow index), while the fermionic contributions are

$$\begin{aligned} h_f &= (\ell_f^+)^2 + (\ell_f^-)^2\,, \\ \ell_f^\pm &= 0, \tfrac{1}{2}, 1, ...\ . \end{aligned} \tag{5.10}$$

Then using $h_0 = h_b + h_f$ in (5.7) with the choices

$$\ell_b^- = j + \tfrac{1}{2}w(k^- - 1)\ , \quad u = 0\ , \quad \ell_f^- = \tfrac{1}{2}w\ , \quad \ell_f^+ = j + \tfrac{1}{2}wk^- \tag{5.11}$$

leads to a spectrum of chiral operators with

$$\begin{aligned} h_n = \ell^- = \ell^+ &= j + \tfrac{1}{2}wk^-\,, \\ n &= 2j + 1 + w(k^- + 1)\,, \end{aligned} \tag{5.12}$$

where again $w = 0, 1, ...,$ and $j = 0, \frac{1}{2}, ..., \frac{1}{2}(k^- - 1)$.

Some other important properties of the chiral spectrum are that the chiral spectrum for $k^- = 1$ appears only in the sectors with odd twist, since $n = 2w + 1$. On the other hand, for $k^- > 1$ all twist sectors contribute to the chiral spectrum. Note also that there are no gaps in the chiral spectrum. All values of $\ell^\pm$ occur, up to the bound set by the stringy exclusion principle; the maximum twist $n \le N$ implicitly restricts $\ell^\pm \lesssim N/2$ via (5.12).

In addition to the chiral spectrum of these twist ground states, one can construct chiral operators by applying the fermionic operator Q, which can raise both the spin and the dimension by one half. We can now see explicitly what part of the action of Q is 'one-particle', and what part 'two-particle'.

Consider the twist $(n)(1)^{N-n}$. The operator Q decomposes into

$$Q_n^a \equiv \sum_{i=1}^{n} \psi_i^a \tag{5.13}$$

and the remainder, $Q - Q_n$. The two-particle component is $(Q - Q_n)\sigma_n$, while the one-particle component is $Q_n\sigma_n$. Summing over the symmetric group as in (5.2) and normalizing as in (5.3)-(5.4), we see indeed that the single-particle component is suppressed by a factor of $g_s \sim N^{-1/2}$ at large N. Restricting consideration to the single-particle BPS spectrum, we see that the twisted sector of order $n = 2\ell + 1$ ($n = 4\ell + 1$ for $\kappa = 0$) gives rise to a quartet of chiral operators, with

$$(h, \overline{h}) = (\ell^{\pm}, \overline{\ell}^{\pm}) = (\ell, \ell)\ , \quad (\ell + \tfrac{1}{2}, \ell)\ , \quad (\ell + \tfrac{1}{2}, \ell)\ , \quad (\ell + \tfrac{1}{2}, \ell + \tfrac{1}{2})\ . \tag{5.14}$$

Two are fermionic and two are bosonic, and so their contribution to the index (4.32) cancels.

Note that there is always a chiral twist operator with $h = \ell^- = \ell^+ = \frac{1}{2}$, which we identify with the second modulus corresponding to $\Re\tau$ in supergravity. Generically this modulus is in the $\mathbb{Z}_2$ twisted sector, with $j = \frac{1}{2}$ and $w = 0$; however, in the special case $k^- = 1$ we find the modulus in the $\mathbb{Z}_3$ twisted sector, with $j = 0$ and $w = 1$ [8,59].

One can also see that this modulus is a RR operator. RR fields are odd under $(-1)^{F_L}$; this operation maps to parity of the spacetime CFT. The perturbative regime of the spacetime CFT is the weak coupling limit of the theory in the RR duality frame; thus we should identify the Wess-Zumino term of the SU(2) WZW model with the background RR three-form flux through $\mathbf{S}^3_-$; the distinguishing characteristic of this term is its odd parity.[d] Indeed, the (parity-even) radius modulus of $\mathbf{S}^1$ in the spacetime CFT is identified with the (NS sector) dilaton. The twisted sector modulus is

$$\mathcal{T} = G^{\alpha\dot{\alpha}}_{-\frac{1}{2}} \overline{G^{\beta\dot{\beta}}_{-\frac{1}{2}}} \Phi^{tw}_{\alpha\dot{\alpha};\overline{\beta\dot{\beta}}}\ ; \tag{5.15}$$

[d] Similarly, in the D1-D5 system on e.g. T^4, the (parity-even) metric moduli of the T^4 in the spacetime CFT map to NS moduli – the shape moduli map onto one another, and the T^4 volume of the spacetime CFT maps to the six dimensional string coupling $g_6^2 = g_s^2/V_{T^4}$ in supergravity. On the other hand, the parity-odd moduli (the antisymmetric tensor $B^{(\mathrm{cft})}_{ij}$) maps to the RR deformation $C^{(\mathrm{sugra})}_{ij}$. Furthermore, the twisted sector moduli $\mathcal{T}^{a\overline{b}} = G^{a\alpha}_{-\frac{1}{2}} \overline{G^{b\beta}_{-\frac{1}{2}}} \Phi^{tw}_{\alpha\overline{\beta}}$ of $Sym^N(T^4)$ (here $\Phi^{tw}_{\alpha\overline{\beta}}$ is the $h = \ell = \frac{1}{2}$ highest weight twist field) can be seen to decompose into a parity-odd singlet, which is the RR axion; and a parity-even triplet, which comprises the self-dual NS B-field moduli of T^4.

this operator is parity-odd, since the lowest component $\Phi^{tw}_{\alpha\dot\alpha;\overline{\beta\dot\beta}}$ of the twist field multiplet is parity-even, and parity interchanges the two supercharges G, $\overline{G}$, thus introducing a fermion minus sign. Hence it is natural to identify the twist modulus with the RR axion.

For $k^- > 1$ we also get a geometrical picture of the twist modulus. The bosonic target space $U(2)$ is four dimensional, so as usual the $\mathbb{Z}_2$ twist blows up the diagonal in $U(2)\times U(2)$, which locally looks like $\mathbb{R}^4/\mathbb{Z}_2$. It would appear that the modulus is a B-flux through the $\mathbf{P}^1$ of the resolution, as is familiar from other contexts [60], and turning off this B-flux results in a singular CFT. [b]

Thus the twisted sector modulus acts as a kind of B-flux turned on by a finite amount at the orbifold point, that resolves the geometrical singularities of the orbifold. This B-flux is a periodic modulus; we have argued that it is the RR axion and has period in the given (RR) background is $\Re\tau \sim \Re\tau + d$. Symmetry considerations analogous to those discussed in [61] lead one to suspect that the orbifold locus is the line $\Re\tau = \frac{1}{2}d$. The twist modulus is parity odd, hence at generic points on the moduli space, the spacetime CFT does not respect parity. There are however two points, $\Re\tau = 0$ and $\Re\tau = d/2$ (i.e. the half-period points) at which parity is conserved. The line $\Re\tau = 0$ is the singular locus, thus the (non-singular) symmetric orbifold CFT could lie on the line $\Re\tau = \frac{1}{2}d$.

Are there other BPS multiplets in the cyclic twist spectrum? Apart from the $N^{\rm th}$ twisted sector, the answer is no. Potential BPS multiplets in the symmetric product with $\ell^+ \neq \ell^-$ will not have a contribution $\frac{(\ell^+-\ell^-)^2}{N(k^++k^-)}$ to their energy unless we are in the $N^{\rm th}$ twisted sector, where these states come from applying κ/N-moded fermion oscillators to the chiral twist ground state.[c] This means that as one perturbs across the moduli space from the supergravity regime to the symmetric orbifold regime, states with $\ell^+ \neq \ell^-$ are not protected and move off the BPS bound. This was observed for $k^- = 1$ in [8].

As an aside, the spectrum (5.7) makes it clear why the orbifold locus is outside the geometrical regime of the spacetime CFT. Suppose we are taking the symmetric orbifold of some theory W of central charge c_w. Consider the theory at some fixed energy E. The way to partition this energy that

[b] The triplet of geometrical blowups of $\mathbb{R}^4/\mathbb{Z}_2$ are not moduli in the present context, since $U(2)$ is not hyperkahler.

[c] Below, we will exhibit these BPS states with $\ell^+ \neq \ell^-$ in the $N^{\rm th}$ twisted sector, in the special case $k^- = 1$.

maximizes the entropy is to take half of it to make a long string, that is to go to the twist sector of a single cyle of order E/c_w; then populate that long string with oscillators using the remaining energy (the oscillator gap will be c_w/E and so the string will be thermalized if c_w is not too big). The entropy is thus of order

$$S \sim \sqrt{c_w(E/c_w)E} \sim E \tag{5.16}$$

(here $c_w(E/c_w)$ is the effective central charge of the long string), i.e. the symmetric product has a Hagedorn spectrum as soon as the long strings can be thermalized [56]. For this we need the temperature to be larger than the gap. The temperature is determined by e.g. $S = c_w LT^2$ where L is the length of the long string, which is E/c_w. Since $S \sim E$ we have $T \sim 1$. So as soon as $E > c_w$ we are in the Hagedorn regime – there is no gap parametrically large in the order of the symmetric product between the AdS scale (order one in our conventions) and the string scale.

5.3. *Explicit construction*

One can give an explicit construction of the cyclic twist operators in symmetric product orbifolds, in the case where the component theory is $\mathcal{S} = \mathcal{S}_0$. This theory consists of one free boson ϕ_r and four free fermions ψ_r^a, and so the cyclic twist operators can be built out of standard orbifold twist operators, see for example [55]. Here $r = 1, ..., N$ labels the copies of $\mathcal{S}_0$ of the symmetric product.

The cyclic twist of order n permutes n copies of $\mathcal{S}_0$ labelled by $r = 1, ..., n$ via $r \to r + 1$ with $r + n \equiv r$. A discrete Fourier transform diagonalizes the twist

$$\phi_\nu = \frac{1}{n} \sum_{r=1}^{n} \exp[2\pi i r\nu/n]\, \phi_r \tag{5.17}$$

and similarly for the fermions. The action of the twist on ϕ_ν is then rotation by ω^ν, where $\omega = \exp[2\pi i/n]$; similarly for the fermions. To keep explicit the $SU(2) \times SU(2)$ content, it is convenient to bosonize the fermions. Define bosons H_ν, H'_ν with corresponding exponentials representing the fermions $\exp[\pm i H_\nu]$ and $\exp[\pm i H'_\nu]$. Note that H, H' are *not* the bosons corresponding to the Cartan subalgebra of $SU(2) \times SU(2)$; the latter are $\frac{1}{2}(H \pm H')$.

The standard $\mathbb{Z}_n$ twist operator σ_ν for ϕ_ν has dimension

$$h_\nu = \frac{1}{4}\,\frac{\nu}{n}\left(1 - \frac{\nu}{n}\right) , \tag{5.18}$$

and the full bosonic twist operator is the product of the twist operators for each ϕ_ν, $\nu = 1, ..., n-1$,

$$\sigma_n^{\rm bos} = \prod_{\nu=1}^{n-1} \sigma_\nu \ , \qquad h_n^{\rm bos} = \sum_{\nu=1}^{n-1} h_\nu = \frac{n^2-1}{24n} \ . \tag{5.19}$$

A fermionic twist operator with the appropriate monodromy is

$$\sigma_n^{\rm ferm} = \prod_{\nu=1}^{n-1} \exp[i\tfrac{\nu}{n}(H_\nu + H'_\nu)] \ , \qquad h_n^{\rm ferm} = \frac{(n-1)(2n-1)}{6n} \ . \tag{5.20}$$

The full twist operator is then $\sigma_n = \sigma_n^{\rm bos}\sigma_n^{\rm ferm}$, whose quantum numbers are

$$\begin{aligned} h &= (n-1)(3n-1)/8n \,, \\ \ell^+ &= (n-1)/2 \,, \\ \ell^- &= 0 \ . \end{aligned} \tag{5.21}$$

Note that this operator is on the unitarity bound

$$h = \frac{k^-\ell^+ + k^+\ell^- + (\ell^+ - \ell^-)^2}{k^+ + k^-} \tag{5.22}$$

if we take $k^+ = k^- = n$ for the n copies being wound together; however, this lies above the unitarity bound for $k^+ = k^- = N$ of the full symmetric product. Successive operator products with the antifermions $\exp[-iH'_\nu]$, $\nu = n-1, n-2, \ldots$, lowers ℓ^+ by 1/2 and raises ℓ^- by 1/2 for each applied antifermion, while staying on the bound (5.22) for $k^+ = k^- = n$. When n is odd, applying the $\frac{1}{2}(n-1)$ antifermions for $\nu = \frac{1}{2}(n+1), \ldots, n-1$ yields a BPS twist operator with quantum numbers

$$h = \ell^+ = \ell^- = \tfrac{1}{4}(n-1) \ . \tag{5.23}$$

This is the operator whose existence was inferred from spectral flow arguments in the previous subsection.

To summarize, for the symmetric product $Sym^N(\mathcal{S}_\kappa)$ there are 'single-particle' BPS states in twisted sectors for each $\ell^+ = \ell^- = 0, \frac{1}{2}, ..., \frac{1}{2}[\frac{1}{2}(N-1)]$; in addition, for $\mathcal{S}_\kappa$ at level $\kappa = 0$, we have exhibited BPS states with $\ell^+ + \ell^- = \frac{1}{2}(N-1)$ for $\ell^- = 0, 1, ..., \frac{1}{2}(N-1)$.

5.4. *A conjectural geometrical interpretation of the chiral spectrum*

It is very important to understand what part of the spectrum of the theory is invariant under perturbations by the modulus (5.15). In the next section we will examine the large $\mathcal{N}=4$ index for these theories. While this detects some invariant states, it turns out not to detect all the important ones. In this section we argue for the existence of some protected states, which turn out not to be detected by the index.

The spectrum of chiral operators we have found above bears some similarity to that of the small $\mathcal{N}=4$ symmetric product $Sym^N(T^4)$, which points to a possible geometrical interpretation. So let us recall the single-particle chiral twist spectrum of $Sym^N(T^4)$ [62,36,3]. There is again a chiral twist field $\Phi^{\alpha_1\cdots\alpha_n}_{\bar\alpha_1\cdots\bar\alpha_n}$ for every $(n+1)$-cycle in the symmetric group, $n=1,...,N-1$, with quantum numbers $(h,\overline{h})=(\ell,\overline{\ell})=(\frac{n}{2},\frac{n}{2})$.

The $\mathbb{Z}_{n+1}$ cyclic twist highest weight states of the symmetric product can be given a cohomological interpretation in terms of the hyperkahler resolution of the singularities along the $n+1$-fold diagonal of $Sym^N(T^4)$. This resolution blows up the diagonal, such that the (orbifold) cohomology of the symmetric product has a representative in dimension $2n$.

Furthermore, the isometries of T^4 lead to four $U(1)$ currents $J^{\dot a a}$, and their superpartners $\psi^{\dot a\alpha}$ (and similarly for right-movers). Here α is an $SU(2)$ doublet index under the small $\mathcal{N}=4$ algebra, a is a doublet index for the custodial $SU(2)$, and $\dot a=1,2$. In the symmetric product, the diagonal U(1) fermion field acts much as in (5.13) to generate a collection of single-particle operators built on Φ; starting with the highest weight state, we can act with $\psi^{\dot a+}$ to make two additional states with $\ell=\frac{1}{2}(n+1)$, and act again to make one more state with $\ell=\frac{1}{2}(n+2)$. Combined with the action of the right-moving $\overline{\psi}^{\dot{\bar a}\bar\alpha}$, there are all told $16=8_B+8_F$ states built on Φ, with a spectrum analogous to (5.14).

From the geometrical viewpoint, the chiral operators in $Sym^N(T^4)$ can be interpreted in terms of the cohomology of the (hyperkahler) target. The twist highest weight ground states are identified with the cohomology of the resolution of diagonals in the symmetric product, and the action of the fermions can be identified with the product in cohomology with the eight even and eight odd cohomology classes of T^4 [62,36,63].

Similarly, we would like to identify the chiral twist fields Φ of the large $\mathcal{N}=4$ theory $Sym^N(\mathcal{S}_\kappa)$ with even cohomology elements of some resolution of diagonals of the complex orbifold $Sym^N(\mathbf{S}^3\times\mathbf{S}^1)$.[d] We saw an example of

[d] In this context, note that $\mathbf{S}^3\times\mathbf{S}^1$ is a rather special target. It is the unique WZW model whose left and right complex structures commute [64]; in fact it has a quaternionic structure, with two commuting triplets of complex structures [17,18,64]. Its Dolbeault cohomology is $H^{p,q}=\mathbb{C}$ for $(p,q)=(0,0),(0,1),(2,1)$, and $(2,2)$, and trivial otherwise. Additional interesting facts about $\mathbf{S}^3\times\mathbf{S}^1$ may be found in [16,65].

this above, when we argued that the fixed locus of the $\mathbb{Z}_2$ twist is resolved by a B-flux through a string-sized $\mathbb{P}^1$. We also wish to identify the action of the fermion (5.13) that makes the two bosonic and two fermionic states (5.14), with the action of tensoring with the two even and two odd cohomology classes of $\mathbf{S}^3 \times \mathbf{S}^1$. This would account for all the chiral cohomology states exhibited above.

We expect that just as there is a smooth metric on a small $\mathcal{N} = 4$ resolution of $\mathrm{Sym}^N(T^4)$ and $\mathrm{Sym}^N(K3)$, there is also a smooth metric on a large $\mathcal{N} = 4$ resolution $\widetilde{X} \to \mathrm{Sym}^N(\mathbf{S}^3 \times \mathbf{S}^1)$ which can be used to define an $\mathcal{N} = 2$ sigma model. The chiral primaries of this model will be given by the cohomology of $\widetilde{X}$, and will be invariant under smooth deformations of $\widetilde{X}$, which we suppose to include the perturbations inherited from (5.15). If this interpretation is correct, it would go a long way to explaining why (as we will see below in Section 7) the chiral twist spectrum is seen both in the symmetric product and in the supergravity limit, whereas the BPS states with $\ell^+ \neq \ell^-$ are seen in supergravity but not in the symmetric product. The latter states would not be associated to any particular cohomology of the target, and being paired up into long representations, nothing prevents them from being lifted as we move around the moduli space. On the other hand, the chiral states are, according to the above proposal, associated to cohomology; even though they are invisible to the index, nonetheless they are not lifted as we cross the moduli space unless we move to a singular point where the cohomology disappears (such as the singular locus at $C_0 = 0$).

If the chiral ring is preserved across moduli space, then we can rule out iterated symmetric products such as $Sym^{Q_1}[Sym^{Q_5}(\mathcal{S})]$, as candidate duals.[e] The chiral ring in this situation differs in the states we would call multiparticle BPS states; for example, the two-particle states correspond to words in the symmetric group that are products of two cycles. In the iterated symmetric product, one has a choice of whether these two cycles come from the same $Sym^{Q_5}(\mathcal{S})$ component or different ones. In the single symmetric product, there is only one state of this type. One readily sees that the growth of states is much faster than that of the Fock space of BPS supergravity states. Note that, apart from this problem, the iterated symmetric product appears to pass the other tests of a duality; the central charge is correct, there is a long string sector with gap of order $\frac{1}{Q_1 Q_5}$, and one can show that the index is the same as that of the single symmetric product of order $Q_1 Q_5$ whenever Q_1 and Q_5 are relatively prime [24].

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[e] Or $Sym^{Q_5}[Sym^{Q_1}(\mathcal{S})]$; note that this is distinct from $Sym^{Q_1}[Sym^{Q_5}(\mathcal{S})]$.

6. The index for the theory $\mathrm{Sym}^N(\mathcal{S})$

In this section we summarize briefly the result of some computations of the index defined in Section 4.6 above for the theory $\mathrm{Sym}^N(\mathcal{S})$. Details of the computations can be found in our companion paper [24].

We consider the theory $\mathcal{C} = \mathrm{Sym}^N(\mathcal{S})$ with $\mathcal{A}_\gamma$ symmetry. The formula for $I_2(\mathcal{C})$ is

$$\begin{aligned} I_2(\mathcal{C}) = \sum_{ad=N} \sum_{n_0,m_0=0}^{d-1} & a\Theta^-_{a(4n_0+1),k}(\omega,\tau)\overline{\Theta^-_{a(4m_0+1),k}(\widetilde{\omega},\widetilde{\tau})} \\ & \times \frac{1}{d}\sum_{b=0}^{d-1} e^{2\pi i \frac{b}{d}(n_0-m_0)(2n_0+2m_0+1)} Z_\Gamma\Big(\frac{a\tau+b}{d}\Big), \end{aligned} \tag{6.1}$$

where

$$Z_\Gamma = \sum_{\Gamma^{1,1}} q^{\frac{1}{2}p_L^2}\, \overline{q}^{\frac{1}{2}p_R^2} \tag{6.2}$$

is the standard Siegel–Narain theta function for the compact scalar of radius R in the theory $\mathcal{S}$. In (6.1) we sum over factorizations $N = ad$. As discussed at length in [24] the $(a = N, d = 1)$ term should be identified as a "short string" and the $(a = 1, d = N)$ term as a "long string" contribution.

To simplify matters we assume that N is prime and we restrict attention to the charge zero sector. The result is

$$I_2^0(\mathcal{C}) = (N+1)|\Theta^-_{N,k}|^2 + \sum_{\mu>0,\mathrm{odd}}^{N-2} \Big|\Theta^-_{\mu,k} + \Theta^-_{2N-\mu,k}\Big|^2 . \tag{6.3}$$

Here and below the conjugation operation implied in $|\Theta|^2$ takes $\omega_\pm \to \widetilde{\omega}_\pm$ and acts as complex conjugation.

Turning to I_1, we find the simplest RR spectrum consistent with this index is

$$\begin{aligned} &\oplus_{\ell^-=1/2}^{N/2} \Big|\Big(\frac{N+1}{2}-\ell^-,\ell^-\Big)\Big|^2 \\ &\oplus_{\ell=1/2}^{(N-1)/4} \Big|(\ell,\ell) + \Big(\frac{N+1}{2}-\ell,\frac{N+1}{2}-\ell\Big)\Big|^2 \oplus \Big|\Big(\frac{N+1}{4},\frac{N+1}{4}\Big)\Big|^2, \end{aligned} \tag{6.4}$$

where the first line comes from the short string and the second from the long string contribution. The short string states have $h = \frac{N}{4} + \frac{u^2}{2N}$ for all states, and the gap to the next excited state is order 1. The long string states have

$$h = \frac{N}{8} + \frac{(4\ell-1)^2}{8N} \tag{6.5}$$

and have small gaps $\sim 1/N$ to the first excited state.

Applying spectral flow to the representation (6.4) gives

$$\oplus_{\ell=0}^{(N-1)/2} |(\ell,\ell)_{NS}|^2$$
$$\oplus_{\ell=0}^{(N-3)/4} \left|\Big(\frac{N-1}{2}-\ell,\ell\Big)_{NS}+\Big(\ell,\frac{N-1}{2}-\ell)_{NS}\right|^2 \oplus \left|\Big(\frac{N-1}{4},\frac{N-1}{4}\Big)_{NS}\right|^2 \tag{6.6}$$

where the first line is from the short string contribution $a = N, d = 1$ and the second from the long string contribution $a = 1, d = N$. The short string states have

$$h = \ell \tag{6.7}$$

while the long string states have

$$h = \frac{N-1}{4} + \frac{(N-1-4\ell)^2}{8N} \ . \tag{6.8}$$

In the case of the general $Sym^N(\mathcal{S}_\kappa)$ theory, where the $U(2)$ is at level κ, we have not managed to evaluate the index completely. However, the short-string contribution is amenable to analysis and the simplest (BPS,BPS) spectrum consistent with the index is

$$\oplus_{j=0}^{\kappa/2} \oplus_{a=0}^{N-1} |(Nj+(a+1)/2,(N-a)/2)|^2 \ . \tag{6.9}$$

Upon spectral flow to the NS sector this is

$$\oplus_{j_1=0}^{\kappa/2} \oplus_{j_2=0}^{\frac{1}{2}(N-1)} |\big(Nj_1+j_2,j_2\big)_{NS}|^2 \ . \tag{6.10}$$

The true BPS spectrum of the $\mathrm{Sym}^N(\mathcal{S})$ theory, which can in principle be directly examined on the orbifold line, differs from that above by short representations with cancelling indices. A detailed examination of the states shows that it is most natural to account for the above spectrum in terms of multiparticle states of the singleton Q and the modulus operator for $\Im\tau$, the size of the $\mathbf{S}^1$; these are all operators in the untwisted sector of the symmetric product. The twisted sector BPS states constructed in the previous section always come with partner representations with cancelling index, as argued around equation (5.14).

7. Comparison of BPS spectra

7.1. *The supergravity spectrum*

Let us ask how the spectrum of supergravity single-particle states fits into the representations described in Section 4.

Reference [9] derives the particle content of the KK reduction of the 10-dimensional type II supergravity multiplet decomposed in terms of representations of the $D(2,1|\alpha) \times D(2,1|\alpha)$ super-isometries of spacetime. The KK spectrum is perhaps most clearly written as

$$\bigoplus_{\ell^+,\ell^- \geq 0\,;\, u} \rho(\ell^+,\ell^-,u) \otimes \overline{\rho(\ell^+,\ell^-,u)}\,, \tag{7.1}$$

where the highest weight state in $(0,0;0,0)$ corresponds to the vacuum, and not to a single-particle state.

This result can be understood intuitively as follows. We should be looking for two things: (a) 256 polarization states, and (b) all the on-shell Fourier modes on $AdS_3 \times \mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^1$. Vertex operators will be products of left-moving and right-moving states. Under the $SO(4)_L = SU(2)_{+,L} \times SU(2)_{-,L}$ isometry of $\mathbf{S}^3_+ \times \mathbf{S}^3_-$, a scalar operator in 10-dimensions gives rise to a tower of vertex operators

$$\Phi^{\alpha_1 \cdots \alpha_{n^+}}_{\dot\alpha_1 \cdots \dot\alpha_{n^-}}\,, \tag{7.2}$$

where $n^\pm$ are integers and the tensor is totally symmetric. Since the decomposition of normalizable functions on $\mathbf{S}^3_+ \times \mathbf{S}^3_-$ under $SO(4)_L \times SO(4)_R$ is

$$L^2(\mathbf{S}^3_+ \times \mathbf{S}^3_-) = \bigoplus_{\ell^\pm \geq 0} (\ell^+,\ell^-;\ell^+,\ell^-)\,, \tag{7.3}$$

all tensors in (7.2) occur with degeneracy 1, where $n^\pm = 2\ell^\pm$. Now, for $u \neq 0$ or $\ell^+ \neq \ell^-$, there are no vanishings in the application of $G_{-1/2}$ (the only candidate, equation (4.10), simply relates $G_{-1/2}$ to $Q_{-1/2}$). Raising with the four G's fills out a 16 component base of the representation, leading to

$$16(2\ell^+ + 1)(2\ell^- + 1) \tag{7.4}$$

states that are not descendants under L_{-1}. Combining left and right quantum numbers, we identify this $\mathcal{A}_\gamma$ multiplet with the supergravity multiplet of states carrying angular momentum $(\ell^+,\ell^-;\ell^+,\ell^-)$ and momentum u on $\mathbf{S}^1$.

For $\ell^+ = \ell^-$ and $u = 0$, the action of $G^{+\dot+}_{-1/2}$ vanishes according to (4.10), and we must be more careful. Previously, (4.10) related $G^{+\dot+}_{-1/2}$ to $Q_{-1/2}$; in the present case $G^{+\dot+}_{-1/2}$ vanishes, however we can still act with $Q_{-1/2}$. Thus we can also make the vertex operator

$$\Phi^{(\alpha_1 \cdots \alpha_n}_{(\dot\alpha_1 \cdots \dot\alpha_n} Q^{\alpha)}_{\dot\alpha)} \tag{7.5}$$

(related to the state (4.16)), leading to a $D(2,1|\alpha)$ short multiplet $(\ell+\frac{1}{2}, \ell+\frac{1}{2})_{\mathbf{s}}$.[a] Thus we find the $D(2,1|\alpha)$ representation content

$$(\ell,\ell)_{\mathbf{s}} \oplus (\ell+\tfrac{1}{2}, \ell+\tfrac{1}{2})_{\mathbf{s}} \tag{7.6}$$

obtained by combining (7.2) and (7.5). The number of states in the $D(2,1|\alpha)$ representation (7.6) which are not descendents of L_{-1} is

$$16(2\ell+1)(2\ell+1) \tag{7.7}$$

(this remains true for the special short representations having $\ell^{\pm} = \frac{1}{2}$). Combining left and right quantum numbers, we identify the multiplet (7.6) formed by (7.2) and (7.5) with the supergravity multiplet of states carrying angular momentum $\ell^+ = \ell^-$ and $u = 0$.

We now come to the question of whether the BPS condition (4.10) is sufficient to protect the conformal dimensions of such states as we move along the moduli space. Here we encounter the distinction between $\mathcal{A}_\gamma$ and the super-isometry algebra $D(2,1|\alpha)$; as mentioned above, their BPS conditions are different unless $\ell^+ = \ell^-$ and $u = 0$. This is a situation not encountered in other contexts, such as AdS_3 backgrounds with $\mathcal{N} = 2,3$ or small $\mathcal{N} = 4$. However, this distinction disappears in the classical limit $k^+ + k^- \to \infty$, where the $\mathcal{A}_\gamma$ unitarity bound (4.11) degenerates to the $D(2,1|\alpha)$ bound $h = \frac{k^+\ell^- + k^-\ell^+}{k^+ + k^-}$. Nevertheless, since this classical dimension *violates* the $\mathcal{A}_\gamma$ BPS bound, and since $\mathcal{A}_\gamma$ is the true symmetry of the theory, we know that supergravity states with $\ell^+ \neq \ell^-$ or $u \neq 0$ *must* get a quantum correction to their mass. Moreover, there must be a corrections to the $D(2,1|\alpha)$ BPS condition $G^{++\dot{+}}_{-1/2}|\ell^+, \ell^-\rangle = 0$ for such states. This is a novel situation in which *states which appear to be BPS in the classical approximation, in fact can receive quantum corrections.* This is perhaps an important cautionary tale.

Since we have not computed the corrections $\sim \frac{(\ell^+ - \ell^-)^2}{k^+ + k^-}$ to the masses in string theory on $AdS_3 \times \mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^1$, we do not know if particle states with $\ell^+ \neq \ell^-$ or $u \neq 0$ arrange themselves into long or short representations of $\mathcal{A}_\gamma$. There is no reason that prevents the various single-particle and multiparticle states in the supergravity Fock space from combining to form massive representations that leave the bound (4.11) in this case. The same mechanism that arranges the BPS states with $\ell^+ = \ell^-$ into long multiplets (namely, acting with Q^a) applies also to the states with $\ell^+ \neq \ell^-$. Indeed,

[a] More precisely, in the regime of supergravity weak coupling the product (7.5) decomposes into an operator creating a two-particle state and (with a coefficient g_s) an operator creating a one-particle state. We focus on the single-particle operator component.

in the symmetric product, we saw that typical single-particle states do not contribute to the index, and indeed we also saw that there were no BPS states with $\ell^+ \neq \ell^-$ small.

If such states were to remain BPS then the string theory corrections would have to be *exactly*

$$h = \frac{k^+\ell^- + k^-\ell^+ + (\ell^+ - \ell^-)^2 + u^2}{k^+ + k^-} . \tag{7.8}$$

If true it would be very striking and would suggest some kind of integrability.

Are there additional BPS objects we can consider? The topological classification of D-brane sources is given by the (twisted) K-theory of spatial infinity, modulo classes which extend to the interior [66,67]. For both IIA and IIB theories this group is $\mathbb{Z}_{Q_5^+} \otimes \mathbb{Z}_{Q_5^-}$, where the two torsion factors come from the twisted K-theory of $\mathbf{S}^3$ and we are working in the NS flux picture. These are classes representing D1-branes wrapping the $\mathbf{S}^1$. However, in the familiar way (reviewed, for example, in [68,69]) the D-objects blow up into $\mathbf{S}^2$ spheres in each of the $\mathbf{S}^3$ factors, so the strings blow up into 5-branes of topology $\mathbf{S}^2 \times \mathbf{S}^2 \times \mathbf{S}^1$. One novelty of the present context is that the mathematical identity $\mathbb{Z}_{Q_5^+} \otimes \mathbb{Z}_{Q_5^-} \cong \mathbb{Z}_{gcd(Q_5^+, Q_5^-)}$ implies interesting instabilities of the Chan-Paton degrees of freedom.[b]

7.2. *Comparison of (BPS,BPS) states with the symmetric product*

One of the key tests of any proposal for a duality is the matching of the BPS spectrum. Here we focus on the left and right BPS states and make several loosely connected remarks concerning the comparison between the supergravity background and the proposed dual $\mathrm{Sym}^N(\mathcal{S})$.

First, comparison with the simplest spectrum suggested by the index (6.6) strongly suggests that the spectrum of short representations of $\mathcal{A}_\gamma$ associated with supergravity particles is in fact precisely

$$\oplus_{\ell \geq 0}^{(N-1)/2} \rho(\ell, \ell, 0) \otimes \rho(\ell, \ell, 0) , \tag{7.9}$$

where the upper bound is imposed by hand, in supergravity, as part of the "stringy exclusion principle" [62].

Now, we have actually argued for *two* towers of BPS states in the representations $|(\ell, \ell)|^2$ in the symmetric product CFT. On the one hand there are the multiparticle states made of untwisted sector states, which contribute

[b] Note that if either fivebrane charge is equal to one, the K-theory is trivial.

to the index. On the other hand, there are the twisted sector states constructed in Section 5. The latter states are more naturally identified with the supergravity one-particle states carrying momentum on $\mathbf{S}^3_\pm$. The companion representations which cancel in the index may be understood in terms of boundstates with singletons (c.f. the discussion surrounding equation (5.14)). We also gave a conjectural cohomological interpretation to these twisted sector states which suggests that, even though they cancel in the index, they might nevertheless be preserved along the moduli space.

As we have stressed, there are generically *no* BPS states with $\ell^+ \neq \ell^-$ with small $\ell^\pm$ in the symmetric product.For instance, in the $Sym^N(\mathcal{S}_0)$ theory, the one-particle supergravity states with these quantum numbers get corrections to their mass of order $\delta h \sim \frac{(\ell^+ - \ell^-)^2}{\ell^+ + \ell^-}$ at large N, under the assumption that the states in (5.22) should be identified with the supergravity one-particle states with the corresponding quantum numbers. On the one hand, one might take this result as a cautionary tale regarding the extent to which the BPS property as seen in the supergravity approximation actually extends to a property of the full theory; on the other hand, if the symmetric product orbifolds only describe situations where one of $Q_5^\pm = 1$, supergravity calculations are suspect and there might not be any contradiction.

It should also be noted that the spectrum (7.1) does not depend on whether $N = Q_1 Q_5$ is prime, nor on whether Q_5 is equal to Q_5'. On the other hand, we show in [24] that the BPS spectrum of $\mathrm{Sym}^N(\mathcal{S}_\kappa)$ depends on the prime factorization of N. Moreover, the conjectural holographic dual for $Q_5/Q_5' = \kappa + 1$ has a BPS spectrum which depends on κ. For the case $\kappa > 0$ we found some BPS states in (6.9). The states with $j_1 > 0$ are "new" in comparison to the spectrum at $\kappa = 0$. Note that the conformal weight of these states is very simple,

$$h = j_2 + N(j_1(j_1+1))/(\kappa+2) \ . \tag{7.10}$$

It follows that particles with $j_1 > 0$ are heavy - parametrically of order N. These states should probably not be identified with supergravity particles. It is possible they can be identified with "conical defect geometries" or smooth versions thereof. Thus, the light spectrum remains (j_2, j_2) for all the $U(2)_\kappa$ theories and is insensitive to κ, This is at least consistent with the idea that $\mathrm{Sym}^N(w(\kappa+1,1))$ is the holographic dual for $Q_1 Q_5 = N(\kappa+1)$, $Q_1 Q_5' = N$.

On top of the above considerations we are left with the "long string BPS states" contributing to line 2 in equations (6.4)(6.6). These are unaccounted for on the sugra side. It is possible that these states correspond to conical defect geometries smoothed out into supertubes, along the lines described in

[27]. The solutions of [27] are specific to the T^4 case, but perhaps could be generalized to $\mathbf{S}^3 \times \mathbf{S}^1$. Moreover, when N is not prime, there will be many further BPS states [24]. They will be heavy, parametrically having mass of order N, but still, they must have supergravity duals, since they are BPS. It should be very interesting to see this structure arising in the supergravity side. For N nonprime the construction of [70] might provide some duals to the new BPS states which are associated with nontrivial divisors of N.

To summarize, even in the most promising case where $Q_5^+ = 1$ or $Q_5^- = 1$ there are discrepancies in the BPS spectrum between supergravity and the CFT dual. However, for reasons discussed above one cannot rule out the $\mathrm{Sym}^N(\mathcal{S})$ theory as a CFT dual solely on this basis.

8. The near-BPS spectrum of $AdS_3 \times \mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^1$

The perturbative "long string" spectrum for $AdS_3 \times \mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^1$ provides a point of comparison between the boundary conformal field theory and supergravity that goes beyond the BPS spectrum. States of high spin on the $\mathbf{S}^3$'s, specifically with $j' = j'' \gg 1$ and $h \sim j'$, are near-BPS states, the so-called BMN states. Their dimensions are expected to be slowly varying functions along the moduli space. Thus, we might expect that this portion of the spectrum to remain intact as we move from the orbifold locus to the singular locus in moduli space, where the perturbative GKS description of [39] can be applied.

8.1. *The spectrum of GKS long strings*

In the worldsheet formalism of [39], AdS_3 is described by an $SL(2,R)$ WZW algebra of level k, two $SU(2)$ current WZW models of levels $k' = Q_5^+$ and $k'' = Q_5^-$ (and a free field theory on $\mathbf{S}^1$); recall that the levels are related by $1/k = 1/k' + 1/k''$. Long strings pulled out of the background ensemble, that wind some number w of times around the angular direction of AdS_3, are obtained by w units of spectral flow from primary states in the $SL(2,R)$ WZW model [71]. The standard worldsheet formalism requires the absence of RR backgrounds, and so describes the NS background duality frame with all RR potentials vanishing. This is the singular locus of the spacetime CFT, where the Coulomb branch of separated onebranes and fivebranes meets the Higgs branch of onebrane/fivebrane bound states described by the $AdS_3 \times \mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^1$ background.

The worldsheet formalism of [39] is a perturbative approximation to the structure of the exact spacetime CFT. So for example one builds a Fock space of strings, ignoring back reaction, stringy exclusion, black holes, etc. Backreaction is a higher loop (and/or nonperturbative/collective) effect. To the extent that one can ignore these latter effects, the long string states describe a subspace of the Hilbert space. However, we are on the singular locus of the spacetime CFT and one could wonder whether such a subspace or Hilbert space is even well-defined. We believe that the answer is yes when sugra is weakly coupled, but then the spacetime CFT is strongly coupled and hard to analyze. A rough physical picture is that the long strings are in a corner of the configuration space of the symmetric product sigma model (where a Coulomb branch meets a Higgs branch); if this is a sufficiently deep pocket in the sigma model target space, a state can get trapped there for a long time and we can usefully think of it as a separate entity (like a resonance). In 1+1 dimensions, the sigma model fields cannot have expectation values due to infrared fluctutations; instead they have wavefunctions. We continue to employ the usual terminology 'Higgs' and 'Coulomb' applied to moduli spaces of scalar vevs in higher dimensions; but these are now regions of configuration space of the theory where the wavefunctions may have support. The support of the wavefunctions of long string states is predominantly on the Coulomb branch. In that region of configuration space, it is energetically cheaper for excitations to be carried by the long string – e.g. a U(1) quantum by itself costs energy going like n/R, while on the long string of winding w it costs $(1/w)(n/R)^2$ which is smaller for small enough n and large enough w.

Let the $SL(2) \times SU(2) \times SU(2)$ spins of the worldsheet primaries be denoted j, j', j'', and let the spectral flow winding be w, w', w'' (as usual, $j' < k'/2$, the total $SU(2)$ spin of a state is $\ell' = \frac{1}{2}k'w' + j'$, etc). Then the formula for the spacetime energy and spin of a long string is (c.f. [71], eq. 75, for the bosonic string, and [70], eq. 97, for the superstring)[a]

$$\begin{aligned} h &= \frac{kw}{4} + \frac{1}{w}\Big[-\frac{j(j-1)}{k} + \Big(\frac{j'(j'+1)}{k'} + j'w' + \frac{k'w'^2}{4}\Big) \\ &\qquad\qquad + \Big(\frac{j''(j''+1)}{k''} + j''w'' + \frac{k''w''^2}{4}\Big) + \Delta_{\rm int} - \tfrac{1}{2}\Big]\,, \\ \ell' &= \frac{1}{2}\,k'w' + j'\,, \\ \ell'' &= \frac{1}{2}\,k''w'' + j''\ . \end{aligned} \tag{8.1}$$

[a] To derive this expression, one solves the worldsheet Virasoro condition $L_0 - 1 = 0$ for the spacetime energy $h = m + \frac{1}{2}kw$ in the $SL(2)$ sector of spectral flow winding w, with m the unflowed J_3 of $SL(2)$.

It is implicit in these formulae that $w \geq 1$; the winding number zero sector is the supergravity spectrum (which is given by a different expression). Adding $U(1)$ charge u simply puts $\Delta_{\rm int} = u^2$ inside the square bracket (we use h to denote spacetime energy, Δ to denote worldsheet conformal dimension). The normalization of this term is set by the winding number zero sector, which is the supergravity spectrum. States that satisfy the GSO projection will need at least one fermion excitation, which we choose orthogonal to $\mathbf{S}^3 \times \mathbf{S}^3$ in order not to deal with multiple cases according to the addition of angular momenta (since the fermions along $\mathbf{S}^3 \times \mathbf{S}^3$ are vectors of $SU(2) \times SU(2)$). Henceforth we will add such an orthogonal fermion excitation, and drop the $-1/2$ in the square brackets of (8.1).

The near-BPS states have large w and sufficiently small $\Delta_{\rm int}$ that the fractional excess of energy above the BPS bound is tiny. For simplicity, let us restrict to states with $\ell' = \ell''$, for which the BPS bound is particularly simple:

$$h \geq \frac{k''\ell' + k'\ell''}{k' + k''} + \frac{(\ell' - \ell'')^2 + u^2}{Q_1(k' + k'')} = \ell' + \frac{u^2}{Q_1(k' + k'')} \ . \tag{8.2}$$

All the states in (8.1) satisfy this bound.

8.2. *Comparison with the symmetric product orbifold $Sym^N(\mathcal{S})$*

The symmetric product orbifold $Sym^N(\mathcal{S})$ is a candidate for the boundary CFT dual to the above supergravity, for $k'|k''$. The orbifold is a non-singular CFT, and so if it is at all related to supergravity on $AdS_3 \times \mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^1$, it is at a different point in moduli space. Our working hypothesis is that the deformation of $\Re\tau$ from the orbifold line to the singular locus does not drastically change the near-BPS spectrum, so that a comparison is possible between the computation of the previous subsection and the near-BPS spectrum of the symmetric product.

For simplicity, consider first the special case $k' = k''$ (i.e. $Q_5^+ = Q_5^-$), for which a candidate dual is $Sym^{Q_1Q_5}(\mathcal{S}_0)$ with $\mathcal{S}_0$ the U(2) WZW model at level $\kappa = 0$. In the $n^{\rm th}$ twisted sector of the symmetric product (i.e. the twisted sector for a single cycle of length n in the symmetric group), the spectrum is

$$\begin{aligned} h &= \frac{n-1}{4} + \frac{h_{\rm int}}{n} \ , \\ \ell' = \ell'' &= \frac{n-1}{4} \ . \end{aligned} \tag{8.3}$$

Here n is necessarily odd, $n = 2r + 1$.

For supergravity long strings (8.1) with $k' = k''$, in order to have $\ell' = \ell''$ we must set $w' = w''$ and $j' = j''$. The simplest way to satisfy the BPS bound is to set $w = w' + w'' = 2w'$, and put $j - 1 = j' = j''$ to cancel the $SU(2)$ and $SL(2)$ Casimir terms in (8.1). Then one finds a spectrum of BPS states with $\ell' = \ell''$, one for each value of the spin. The states in the zero-winding sector in $SL(2)$ are BPS supergravity states, whose spins are bounded by $k'/2$; once we add the long string sectors, we can get arbitrary spin. The states are grouped according to the $SL(2)$ winding w in blocks of size $k'/2$. Back-reaction is supposed to lead to the upper cutoff (due to stringy exclusion) of spin less than of order kQ_1, but in the GKS formalism this restriction is nonperturbative and therefore invisible. If we now add U(1) charge, we get a spectrum of BMN type states with

$$h = \frac{1}{2}\, k'w' + j' + \frac{{u'}^2}{2w'} \tag{8.4}$$

with $j' = 0, 1/2, ..., k'/2$ and $w' = 1, 2, 3, ...$. These states are BPS if $u' = 0$, and near-BPS in the BMN sense if ℓ' is large and u' is small.

Let us compare (8.4) to the spectrum (8.3) of the symmetric product of the $\mathcal{S}_0$ theory. The latter has BMN type states with

$$h = \frac{n-1}{4} + \frac{u^2}{n}\ . \tag{8.5}$$

Each increment of $SU(2)$ spin is accompanied by an increment in the winding sector. The order of the winding is deduced from the (assumed large) first term on the RHS: $n = 2(k'w' + 2j' + 2)$. Identifying $u' = u$, we see that the second terms differ by a factor k'. If we think in terms of the "invariant mass" of the state, which is highly boosted along the $\mathbf{S}^3$'s, we have

$$m^2_{\rm inv} \sim (h - \ell)\ell \sim \tfrac{1}{4} u^2 \tag{8.6}$$

for the symmetric product, and

$$m^2_{\rm inv} \sim (h - \ell)\ell \sim \tfrac{1}{4} k' {u'}^2 \tag{8.7}$$

for super(string)gravity.

It is tempting to identify $k' = Q_5^{\pm} = 1$ from this result; however, the spectra are being compared across a distance in moduli space proportional to $gcd(Q_5^+, Q_5^-)$, *cf.* Section 2. If $Q_5^+ = Q_5^-$ then this distance is order 1 for values of $Q_5^{\pm}$ for which supergravity is valid. In this case the deviation from the BPS bound might well vary significantly. If $Q_5^+ = Q_5^- = 1$ the distance is order g_B, and the deviation from the BPS bound should be controllable.

However, in this case the supergravity approximation is not valid. In spite of all these cautionary remarks, we cannot help noting that the best match is for $Q_5^\pm = 1$, reminiscent of the fact [3] that in the D1-D5 system on T^4, the symmetric orbifold was determined to lie in the cusp of the moduli space related to $Q_5 = 1$.

8.3. *Spectra for $k' \neq k''$*

An analysis of the more general backgrounds with $k' \neq k''$ indicates again a discrepancy in the BMN spectrum between supergravity and the symmetric product of $U(2)$.

Consider the special case $j' = j'' = 0$. Then the special BPS states $\ell' = \ell''$ will have $2\ell' = k'w' = k''w''$. Consider the further specialization $w' = pk''$, $w'' = pk'$. The BPS condition is satisfied for $j = 0$, $w = p(k' + k'')$; then the energy of long BMN-type strings with these particular quantum numbers is

$$h = \frac{kw}{2} + \frac{u^2}{w} = \frac{pk'k''}{2} + \frac{u^2}{p(k'+k'')} \ . \tag{8.8}$$

Let us compare this answer to the symmetric product of $U(2)$. The BMN spectrum is easily determined by the analysis of Section 5,

$$h = j + \tfrac{1}{2}\widehat{w}k^- + \frac{u^2}{2j+1+\widehat{w}(k^-+1)} \ . \tag{8.9}$$

We again fix the order of the twisted sector by comparing the large first terms. For j small, we determine

$$m_{\rm inv}^2 \sim \frac{k_-}{2(k_-+1)}\, u^2 \tag{8.10}$$

for the symmetric product, and

$$m_{\rm inv}^2 \sim \frac{k'k''}{2(k'+k'')}\, u^2 \tag{8.11}$$

for super(string)gravity (recall $k' = Q_5^+$, $k'' = Q_5^-$). Again the best match is for one of $Q_5^\pm$ equal to one, but we cannot exclude other possibilities given the considerations mentioned at the end of the previous subsection.

8.4. *Comparison with the PP-wave Limit of $AdS_3 \times \mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^1$*

As a check on the near-BPS spectrum derived using the GKS formalism above, we reproduce that spectrum by taking the Penrose limit of $AdS_3 \times$

$\mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^1$ and analyzing the spectrum along the lines of [25]. For this purpose, it is convenient to write the space-time metric (2.8) as

$$\begin{aligned} ds^2 = &\ell^2 \left(-\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\phi^2\right) \\ &+ \sum_{i=\pm} R_i^2 \left(d\theta_i^2 + \cos^2\theta_i d\psi_i^2 + \sin^2\theta_i d\varphi_i^2\right) + L^2 d\theta^2 \,. \end{aligned} \tag{8.12}$$

The plane wave limit of the geometry (8.12) is obtained by boosting along a null geodesic in $AdS_3 \times \mathbf{S}^3_+ \times \mathbf{S}^3_- \times \mathbf{S}^1$ [25]. Specifically, we consider a limit where some of the radii in (8.12) are taken to infinity, with α' and g_B kept fixed. In the boundary theory, this corresponds to focusing on the sector of the theory spanned by operators with large values of spin. There are many choices of boost; one can associate these choices with a choice of direction inside $SU(2)_+ \times SU(2)_- \times U(1)$. One particular choice of the Penrose limit was considered in [72], but the PP-wave limit considered there does not describe states which are near BPS. Here, we shall consider another limit, given by the rescaling:

$$\begin{aligned} t &= \mu_0 x^+ \,, \\ \psi_\pm &= \mu_\pm x^+ - \frac{x^-}{2\mu_\pm R_\pm^2} \pm \left(\frac{\mu_- R_-}{\mu_+ R_+}\right)^{\pm 1/2} \frac{y_1}{R_\pm} \,, \\ \theta &= \frac{y_2}{L} \,, \\ \rho &= \frac{r}{\ell} \,, \\ \theta_\pm &= \frac{y_\pm}{R_\pm} \,, \end{aligned} \tag{8.13}$$

where μ_0 and $\mu_\pm$ are some parameters. In CFT, this limit corresponds to $\ell^\pm \to \infty$.

Substituting (8.13) into (8.12), and taking the limit $R \to \infty$, we obtain

$$ds^2 = -2dx^+dx^- - \frac{1}{2}(\mu_0^2 r^2 + \mu_+^2 y_+^2 + \mu_-^2 y_-^2) dx^+ dx^+ + d\vec{r}^2 + d\vec{y}_+^2 + d\vec{y}_-^2 + d\vec{y}^2 \,, \tag{8.14}$$

where, in order to cancel the terms of order $R_\pm^2$, we need to take

$$\mu_0^2 \ell^2 = R_+^2 \mu_+^2 + R_-^2 \mu_-^2 \tag{8.15}$$

and ℓ denotes the radius of AdS$_3$ (not to be confused with $SU(2)$ spin). Notice, that the terms $dx^+ dy_1$ cancel automatically due to a particular choice of the coefficients in (8.13). The last four terms in (8.14) describe the usual flat metric on $\mathbb{R}^8$ written in polar coordinates,

$$ds_6^2 = \left(dr^2 + r^2 d\phi^2\right) + \left(dy_+^2 + y_+^2 d\varphi_+^2\right) + \left(dy_-^2 + y_-^2 d\varphi_-^2\right) + dy_1^2 + dy_2^2$$
$$= d\vec{r}^2 + d\vec{y}_+^2 + d\vec{y}_-^2 + d\vec{y}^2 \ .$$

Similarly, the following components of the 3-form flux (2.6) remain non-zero in the pp-wave limit (8.13),

$$H_{+12} = 2\mu_0 \ , \qquad H_{+34} = 2\mu_+ \ , \qquad H_{+56} = 2\mu_- \ . \tag{8.16}$$

Using (8.13), we find the relation between charges in the pp-wave geometry and the charges in the dual CFT,

$$\begin{aligned} p^- &= i\partial_{x^+} = i\mu_0\partial_t + i\mu_+\partial_{\psi_+} = \mu_0 h - \mu_+\ell^+ - \mu_-\ell^- \ , \\ p^+ &= i\partial_{x^-} = -\frac{i}{\mu_+ R_+^2}\,\partial_{\psi_+} - \frac{i}{\mu_- R_-^2}\,\partial_{\psi_-} = \frac{\ell^+}{\mu_+ R_+^2} + \frac{\ell^-}{\mu_- R_-^2} \ . \end{aligned} \tag{8.17}$$

In the light-cone gauge, $x^+ = p^+\tau$, the string world-sheet theory is Gaussian (hence, solvable). The bosonic excitations are described by the Hamiltonian

$$2p^- = -p_+ = H_{\rm l.c.} = \sum_{n=-\infty}^{\infty}\sum_{I=1}^{8} (a_n^I)^\dagger a_n^I \sqrt{\mu_I^2 + \left(\frac{4\pi^2 n}{p^+}\right)^2} \ , \tag{8.18}$$

where, for different values of the space-time index I, we have

$$\mu_I = \begin{cases} \mu_0, \\ \mu_\pm, \\ 0. \end{cases}$$

Substituting (8.17) into (8.18), we find that the string spectrum in the plane wave background (8.14) looks like

$$h - \frac{\mu_+}{\mu_0}\ell^+ - \frac{\mu_-}{\mu_0}\ell^- = \sum_n N_n \sqrt{\left(\frac{\mu_I}{\mu_0}\right)^2 + \left(\frac{4\pi^2 n}{\mu_0 p^+}\right)^2} + \frac{4\pi^2 h_{\rm int}}{\mu_0 p^+} \tag{8.19}$$

in the RR case. Similarly, in the NS frame, we obtain, c.f. [25]:

$$h - \frac{\mu_+}{\mu_0}\ell^+ - \frac{\mu_-}{\mu_0}\ell^- = \sum_n N_n \left(\frac{\mu_I}{\mu_0} + \frac{4\pi^2 n}{\mu_0 p^+}\right) + \frac{4\pi^2 h_{\rm int}}{\mu_0 p^+} \ . \tag{8.20}$$

Let us now consider in more detail the symmetric case, where $R_+ = R_- = \sqrt{2}\ell$. In this case, the constraint (8.15) implies $\mu_+ = \mu_- = \mu_0/2$. The momenta (8.17) take a simple form

$$p^- = \mu_0 \left(h - \frac{1}{2}(\ell^+ + \ell^-) \right) , \qquad p^+ = \frac{2(\ell^+ + \ell^-)}{\mu_0 R_+^2} . \tag{8.21}$$

Correspondingly, the string spectrum (8.19) becomes

$$h - \frac{1}{2}(\ell^+ + \ell^-) = \sum_n N_n \sqrt{1 + \left(\frac{2\pi^2 R_+^2 n}{\ell^+ + \ell^-} \right)^2} + \frac{2\pi^2 R_+^2 h_{\rm int}}{\ell^+ + \ell^-} . \tag{8.22}$$

On the other hand, in the NS-NS case the spectrum (8.20) takes the form

$$h - \frac{1}{2}(\ell^+ + \ell^-) = \sum_n N_n \left(1 + \frac{2\pi^2 R_+^2 n}{\ell^+ + \ell^-} \right) + \frac{2\pi^2 R_+^2 h_{\rm int}}{\ell^+ + \ell^-} . \tag{8.23}$$

In order to compare this with the spectrum of the GKS long strings, we need to write (8.23) in terms of $k' = Q_5^+ = 4\pi^2 R_+^2$. For $\ell^+ = \ell^- = \ell'$ and $N_n = 0$, we obtain

$$h = \ell' + \frac{k' h_{\rm int}}{4\ell'} . \tag{8.24}$$

This agrees with the GKS long string spectrum (8.4) in the limit of large w', and also agrees with the spectrum (8.5) of the symmetric product orbifold provided that $k' = 1$.

Finally, let us briefly describe interactions in the pp-wave geometry (8.14) when $Q_5^+ = Q_5^- = Q_5$. Since the transverse string fluctuations are confined in this geometry, the strings are effectively two-dimensional. Six transverse directions in (8.14) are massive with a characteristic scale $(\mu_I p^+)^{-1/2}$, whereas the other two transverse directions have sizes R_+ and L, respectively. Therefore, the effective two-dimensional string coupling constant is given by

$$g_2^2 = \frac{g_B^2 (\mu p^+)^3}{R_+ L} \sim \frac{(\ell')^3}{N} , \tag{8.25}$$

where $N = Q_1 Q_5$, and in the last equality we expressed p^+, R_+, and L in terms of the background charges. The result (8.25) has to be compared with the genus-counting parameter in the pp-wave limit of $\mathrm{AdS}_3 \times \mathbf{S}^3 \times K3$. Since the latter geometry has only four massive transverse directions, the effective two-dimensional string coupling in this case scales with the $SU(2)$ spin ℓ' as [73]

$$g_2^2 = \frac{g_B^2 (\mu p^+)^2}{\mathrm{Vol}(K3)} \sim \frac{(\ell')^2}{N} . \tag{8.26}$$

It is tempting to speculate that the cubic power in (8.25) is related to the four-string interaction in $AdS_3 \times \mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^1$ suggested by the structure of the twisted sectors in the dual symmetric product CFT (see Section 5.2).

9. The $U(1) \times U(1)$ gauge theory

The low energy supergravity contains a $U(1) \times U(1)$ gauge theory with Chern–Simons term. The study of the associated topological field theory provides further information on the holographic dual of the theory. Indeed, it leads to our strongest argument that $\mathrm{Sym}^N(\mathcal{S})$ can only be the holographic dual for $Q_5 = 1$.

9.1. *Actions*

In the NS flux picture with IIA on $AdS_3 \times \mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^1$ we find a $U(1) \times U(1)$ massive gauge theory for two U(1) gauge fields in AdS_3 by dimensional reduction of the metric and NS B-field on the $\mathbf{S}^1$. The relevant ansatz is

$$\begin{aligned} ds^2 &= \frac{\ell^2}{{x_2}^2}\Big(-dt^2+(dx^1)^2+(dx^2)^2\Big)+\frac{Q_5^+}{4\pi^2}\,ds^2(\mathbf{S}^3_+)+\frac{Q_5^-}{4\pi^2}\,ds^2(\mathbf{S}^3_-)+L^2(d\theta+a)^2\,, \\ H &= \lambda_0\omega_0+\lambda_+\omega_+ +\lambda_-\omega_- + db\wedge d\theta\,, \end{aligned} \tag{9.1}$$

where a and b are gauge fields on AdS_3 and da and db have integral periods. The relevant part of the action for the H-flux is proportional to

$$H\wedge *H = \frac{1}{L} db\wedge(*_{\mathrm{AdS_3}} db)\wedge\omega_+\wedge\omega_-\wedge d\theta - 2L\lambda_0\, db\wedge a\wedge\omega_+\wedge\omega_-\wedge d\theta\,. \tag{9.2}$$

The second term in (9.2) gives a Chern–Simons term in AdS_3. From (2.5), (9.2) we get

$$\frac{16\pi^5}{g_A^2}\frac{LR_+^3R_-^3}{\ell}\int db\wedge a\,. \tag{9.3}$$

Substituting (2.15) gives

$$2\pi Q_1 \int db\wedge a\ . \tag{9.4}$$

On topologically nontrivial three-manifolds we would define this term by $2\pi Q_1 \int_{M_4} f_b \wedge f_a$. Since our field strengths have integer periods, having integral Q_1 is precisely the right topological quantization.

Including the kinetic terms we have an action of the form

$$S_a = \int \frac{-1}{2e_A^2}\,da * da + \frac{-1}{2e_B^2}\,db * db + 2\pi Q_1 a db\ . \tag{9.5}$$

It is very useful to introduce $\mu := |e_B/e_A|$ and the linear combinations

$$\begin{aligned} A^{(+)} &:= \frac{1}{\sqrt{2}}\Big(\mu^{-1/2}b + \mu^{1/2}a\Big)\,, \\ A^{(-)} &:= \frac{1}{\sqrt{2}}\Big(\mu^{-1/2}b - \mu^{1/2}a\Big)\,. \end{aligned} \tag{9.6}$$

In terms of these fields we may write

$$\begin{aligned} S_s = &\int\left[\frac{-1}{2|e_Ae_B|}\,dA^{(+)} * dA^{(+)} + \pi Q_1 A^{(+)}dA^{(+)}\right] \\ &+ \int\left[\frac{-1}{2|e_Ae_B|}\,dA^{(-)} * dA^{(-)} - \pi Q_1 A^{(-)}dA^{(-)}\right]. \end{aligned} \tag{9.7}$$

The equation of motion is:

$$\begin{aligned} d * dA^{(+)} &= 2\pi Q_1 |e_Ae_B| dA^{(+)}\,, \\ d * dA^{(-)} &= -2\pi Q_1 |e_Ae_B| dA^{(+)} \end{aligned} \tag{9.8}$$

and therefore there are two propagating vector fields of $m^2 = (2\pi Q_1 e_A e_B)^2$.

From straightforward Kaluza-Klein reduction we find

$$e_B^2 = \frac{g_A^2 L}{8\pi^5 R_+^3 R_-^3}\,. \tag{9.9}$$

(Note that it is important to work at $C_0 = 0$ here. Otherwise $R_3 = -C_0 H$ and the term in the action $\sim \int R_3 * R_3$ lead to a correction $\sim C_0^2$ to $1/e_B^2$.) For the Kaluza-Klein gauge field we obtain

$$e_A^2 = \frac{g_A^2}{8\pi^5 R_+^3 R_-^3 L^3}\,. \tag{9.10}$$

Note that this means that

$$\mu^2 := \frac{e_B^2}{e_A^2} = L^4\,. \tag{9.11}$$

The gauge group must be $U(1)\times U(1)$ and not $\mathbb{R}\times\mathbb{R}$ because we know there are KK monopoles and H-monopoles.

The gauge fields (9.6) are

$$A^{\pm} = \pm\sqrt{\frac{\mu}{2}}\left(a \pm \frac{1}{L^2}\,b\right) \tag{9.12}$$

and these are indeed the combinations which appear in the covariant derivatives for left- and right-moving supersymmetry transformations. Moreover, equation (9.8) above shows that $A^{\pm}$ have mass-squared in AdS units

$$m^2\ell^2 = (2\pi Q_1 e_A e_B)^2 \ell^2 = 4\,, \tag{9.13}$$

where we used the fixed-values for the radii. There is a nice check on our formulae. Equation 41 of [74] says that in the AdS/CFT correspondence a vector field satisfying

$$\ell * dA = \mp(h + \overline{h} - 1)A \tag{9.14}$$

corresponds to a primary of dimension $(h, \overline{h})$, where $h - \overline{h} = \pm 1$. This is to be compared with equation (9.8). Using the above values for e_A, e_B, that equation reads

$$\ell * dA^{\pm} = \pm 2A^{\pm}\ . \tag{9.15}$$

So, the massive scalar mode of A^+ is dual to a primary field of dimension $(1, 2)$ and A^- is dual to a primary field of dimension $(2, 1)$. Meanwhile $(1, 0)$ and $(0, 1)$ primaries, i.e., the currents, correspond to $dA = 0$, i.e. the flat fields.

Remark: Very similar considerations apply for $AdS_3 \times \mathbf{S}^3 \times T^4$. If we choose the background

$$ds^2 = \ell^2 ds^2_{AdS} + R^2 ds^2(\mathbf{S}^3) + \sum_{i=1}^{4} L_i^2 (d\theta_i + a_i)^2\,, \tag{9.16}$$

$$H = \lambda_0\omega_0 + \lambda_1\omega_1 + \sum_{i=1}^{4} db_i d\theta_i \tag{9.17}$$

then the Einstein equations give $\lambda_0^2 = \lambda_1^2$, and charge quantization gives $2\pi^2\lambda_1 R^3 = Q_5$ The Chern–Simons interaction is

$$2\pi Q_1 \int \sum db_i \wedge a_i \tag{9.18}$$

and $(e_{a_i}/e_{b_i})^2 = L_i^4$. Meanwhile, the RR fields give another set of $4+4$ $U(1)$ gauge fields β_i, α_i with Chern–Simons term

$$2\pi Q_5 \int \sum d\beta_i \wedge \alpha_i\ . \tag{9.19}$$

The formulae for the charges change when we turn on the background but the Chern–Simons terms are quantized. For the general U-duality invariant formula, valid for all backgrounds, see Sec. 7 of [67].

9.2. *Path integral on the torus*

Imagine doing the path integral on the solid torus for the theory (9.5) with hyperbolic metric. In the topology $D \times \mathbf{S}^1$ let ρ be the radial coordinate on the disk. Consider the path integral where we just integrate over fields for $\rho \leq \rho_1$. The path integral over a, b defines some state $\Psi(a, b; \rho_1)$ in the Hilbert space of the massive Chern–Simons theory, as a function of the gauge fields a, b on the boundary torus at $\rho = \rho_1$. Now consider the path integral at $\rho_2 > \rho_1$. How is the new state $\Psi(a, b; \rho_2)$ related to the old state? We view evolution in ρ as a Euclidean time evolution. Since the hyperbolic metric is of the form

$$ds^2 \sim d\rho^2 + \frac{e^{2\rho}}{4}|d\phi + \tau dt|^2 \tag{9.20}$$

for large ρ, and since the Hamiltonian is conformally invariant (for the flat gauge fields) we find that

$$\Psi(a, b; \rho_2) = e^{-(\rho_2 - \rho_1)H}\Psi(a, b; \rho_1) \ .$$

Thus, if we let $\rho_2 \to \infty$ the wavefunction $\Psi(a, b; \rho_2)$ is projected onto the lowest energy level of the Hamiltonian. It is therefore a linear combination of the gauge invariant wavefunctions for quantization on the torus.

In the companion paper [26] we work through the exercise of implementing the above procedure in detail for the theory (9.5). The result is that the gauge invariant wavefunctions may be understood in terms of two Gaussian models with radius

$$R_A^2 = \frac{1}{4\pi^2}Q_1\mu = \frac{1}{4\pi^2}Q_1 L^2 \tag{9.21}$$

and

$$R_B^2 = \frac{Q_1}{4\pi^2\mu} = \frac{Q_1}{4\pi^2 L^2} \ . \tag{9.22}$$

The partition function can be written in terms of "higher level Siegel–Narain theta functions." It takes the form

$$\sum_{\beta \in \Lambda^*/\Lambda} \Psi_\beta(A)\Psi_{\overline{\beta}}(\lambda)\,, \tag{9.23}$$

where

$$\begin{aligned}\Psi_\beta(A) = \sqrt{\frac{1}{Q_1}}\,\frac{1}{|\eta(\tau)|^2}\, & e^{-2\pi Q_1 \mathrm{Im}\tau\left[A_z^+ A_{\overline{z}}^+ + A_z^- A_{\overline{z}}^- + A_z^{(+)} A_{\overline{z}}^{(-)} - A_z^{(-)} A_{\overline{z}}^{(+)}\right]} \\ & \times \Theta_\Lambda(\tau, 0, \beta; P; \xi(A))\,,\end{aligned} \tag{9.24}$$

$$\Psi_{\overline{\beta}}(\lambda) = \sqrt{\frac{2\tau_2}{Q_1}}\Theta_\Lambda(\tau, 0, \overline{\beta}; P; \xi(\lambda)) \ . \tag{9.25}$$

Here

$$\Lambda = e_1\mathbb{Z} + f_1\mathbb{Z} \cong \sqrt{Q_1}II^{11} \tag{9.26}$$

is a lattice with hyperbolic metric: $e_1^2 = f_1^2 = 0$, $e_1 \cdot f_1 = Q_1$. Θ_Λ is a Siegel–Narain theta function for the embedding P of $\Lambda \otimes \mathbb{R}$ into $\mathbb{R}^{1,1}$ defined as usual by left- and right-moving momenta $(p_L; p_R)$, with metric $p_L^2 - p_R^2$. Also, $\beta = \rho/Q_1 e_1 - \widetilde{\rho}/Q_1 f_1$ and $\overline{\beta} = \rho/Q_1 e_1 + \widetilde{\rho}/Q_1 f_1$ are representatives of the dual quotient group $\Lambda^*/\Lambda \cong (\mathbb{Z}/Q_1\mathbb{Z})^2$ while

$$\begin{aligned} \xi(A) &= \left(\sqrt{Q_1}2i\tau_2 A_{\overline{z}}^{(-)}; -\sqrt{Q_1}2i\tau_2 A_z^{(+)}\right), \\ \xi(\lambda) &= \left(-\frac{\overline{\lambda}}{2\pi i\sqrt{Q_1}}; \frac{\lambda}{2\pi i\sqrt{Q_1}}\right). \end{aligned} \tag{9.27}$$

Here $\lambda, \overline{\lambda}$ are arbitrary constants that depend, for example, on what kind of operators have been inserted in the solid torus. See [26] for further details.

The "conformal blocks" $\Psi_\beta(A)$ in (9.25) predict conformal weights that give an explicit realization to the "level Q_1 $U(1)$ current algebra" in the sense of [75,3].

9.3. *Comparison to* $I_2(\mathrm{Sym}^N(\mathcal{S}))$

The path integral for the gauge fields A, B that we have discussed is only part of the bulk superstring path integral dual to - say - the index I_2 of the boundary CFT. First, the gauge fields couple to the charged supergravity modes, and hence perturbative string interactions should be taken into account. One might naively think that since there are no couplings between the $SU(2)^4$ gauge fields and the $U(1) \times U(1)$ then (9.23) would have to be an overall factor in the partition function. This is not true when one takes into account instanton effects such as NS5-brane instantons and KK monopole instantons. These effects can lead to a correlation between the $U(1)$ and $SU(2)$ quantum numbers of the spectrum computed from the supergravity viewpoint.

Nevertheless, since the topological theory is expected to dominate at long distance it is very natural to conjecture that the $A^{(+)}, A^{(-)}$-dependent wavefunctions are valid in the full AdS/CFT duality of string theory. That is, if Z^{ab} is the partition function on the solid torus and is written as in

(9.25) as a linear combination of some finite dimensional space of "conformal blocks",

$$Z^{ab} = \sum_\beta \zeta^\beta \Psi_\beta(A) \qquad (9.28)$$

where ζ^β are constants (the λ-dependent terms in (9.25)) then the full string theory partition function is of the form

$$Z^{string} = \sum_\beta Z^\beta(\Phi_\infty)\Psi_\beta(A)\,, \qquad (9.29)$$

where Φ_∞ are the boundary values of the other fields in the supergravity theory. That is, the exact $A^{(+)}, A^{(-)}$ dependence is given by a linear combination of the same "conformal blocks" as in the massive gauge theory.

If we accept the above conjecture, then we can compare to the proposed holographic dual $Z(\mathrm{Sym}^N(\mathcal{S}))$. Let us consider the index I_2, for simplicity. Then from (6.1) we can deduce that the dependence on $A^{(+)}, A^{(-)}$ - defined to be the coordinates $(\chi_L;\chi_R)$ dual to the charges $u, \widetilde{u}$ - is given by higher level Siegel–Narain theta functions for $\Lambda_{cft} = \sqrt{N} II^{1,1}$. On the other hand, the wavefunctions appearing in (9.29) are Siegel–Narain theta functions for $\Lambda_{sg} = \sqrt{Q_1} II^{1,1}$. Comparing with the "conformal blocks" (9.25) of the theory (9.5) suggests that we must identify $N = Q_1$. Since $N = Q_1 Q_5$, this conjecture supports the idea that the orbifold theory $\mathrm{Sym}^N(\mathcal{S})$ is only on the moduli space of the supergravity theory for $Q_5 = 1$.

Let us comment on possible subtleties that could invalidate the conclusion that the partition functions can only match for $Q_5 = 1$. First, it is possible that one loop determinants associated with charged fermions on AdS_3 induce a renormalization of the Chern–Simons (9.4). We think this unlikely, but it bears further thought. Second, it is possible to change the level of a theta function by summing over certain vectors $\beta \in \Lambda^*/\Lambda$. In this way, one can express a theta function of level k in terms of a theta function of level $k\Delta^2$, where Δ is an integer. In our present example we would require

$$\sqrt{Q_1} II^{1,1} \subset \frac{1}{\sqrt{Q_1 Q_5}} II^{1,1}\,. \qquad (9.30)$$

This, in turn, is true iff Q_5 is a perfect square. Thus, when Q_5 is not a perfect square, we cannot evade the conclusion. It is conceivable that some unknown physical mechanism changes the basic periodicity of the large gauge transformations of the a, b fields to be multiplied by Q_5. In this case, the supergravity partition function would be expressed in terms of level $Q_1 Q_5$ theta functions. However, we cannot see any justification for this. Thus, we conclude that $Q_5 = 1$ is necessary to match to the simplest proposal for the holographic dual $\mathrm{Sym}^N(\mathcal{S})$.

It is worth stressing that the above argument does *not* apply to the case of $AdS_3 \times \mathbf{S}^3 \times T^4$. Here the enlarged U-duality group allows one to redefine a basis of gauge fields so that (9.18) and (9.19) are rearranged into level 1 and level $Q_1 Q_5$ Chern–Simons theories. This redefinition, of course, depends on the cusp in moduli space.

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10. Appendix A: Proof of marginality

As mentioned in Section 4.4, although the candidate modulus operator preserves large $\mathcal{N} = 4$ supersymmetry, it cannot be written as a superspace integral, and so the proof of [23] that this operator is truly a modulus must be reconsidered. Since we will be using the methodology of [23] in an essential way, and this article may not be readily available to the reader, let us reproduce (more or less verbatim) the proof there of marginality for $\mathcal{N} = 2$ massless perturbations. We will then adapt this proof to our modified circumstances, and show that the candidate modulus $\mathcal{T}$ preserves conformal invariance to all orders in conformal perturbation theory.

10.1. *Dixon's proof for* $\mathcal{N} = 2$

The reasoning of [23] runs as follows. Consider an $\mathcal{N} = 2$ theory with an $h = \ell = \frac{1}{2}$ chiral primary field with lower component Φ_0^+ and upper component $\Phi_1^+ = G^-_{-\frac{1}{2}} \Phi_0^+$, and and antichiral field Φ_0^- with upper component$\Phi_1^- = G^+_{-\frac{1}{2}} \Phi_0^-$. The k^{th} term in conformal perturbation theory involves a correlation function

$$\Big\langle \prod_{i=1}^{k} \mathcal{T}(z_i) \Big\rangle \tag{10.1}$$

of the modulus deformation $\mathcal{T} = \Phi_1^+ + \Phi_1^-$, integrated over $k-3$ arguments.

Consider the term with m operators Φ_1^+ and n operators Φ_1^-, $m+n=k$. We suppress the right-moving structure except as needed. Embed this CFT correlator in a string scattering amplitude [a]

$$\begin{aligned} \int \prod_{i,j} d^2 z_i d^2 w_j \Big(\prod_i \big(\Phi_1^+(z_i) + ik_i \cdot \psi_i \Phi_0^+(z_i)\big) e^{ik_i \cdot X(z_i)} \\ \prod_j \big(\Phi_1^-(w_j) + ik_j \cdot \psi_j \Phi_0^-(w_j)\big) e^{ik_j \cdot X(w_j)} \Big) \ . \end{aligned} \tag{10.2}$$

The leading term as $k_i \to 0$ comes from taking $\Phi_1^\pm$ in each factor, leading to the correlation function

$$F(z_i, w_j) = \Big\langle \prod_{i=1}^{m} \Phi_1^+(z_i) \prod_{j=1}^{n} \Phi_1^-(w_j) \Big\rangle, \tag{10.3}$$

where, despite the slightly confusing notation, $\Phi_1^\pm$ carry zero R-charge so the correlators are non-vanishing even when $m \neq n$. It will turn out that F is a total derivative; integration by parts brings down factors of $k_i \cdot k_j$ from the correlator of the exponentials, and one can choose the kinematics such that the surface terms vanish in the integration by parts [76]. Replacing pairs of $\Phi_1^\pm$ by pairs of $ik_i \cdot \psi \Phi_0^\pm$ also leads to terms with at least two powers of momenta.

The scattering amplitude can develop poles $\frac{1}{k_i \cdot k_j}$ from on-shell intermediate states. This would lead to contact terms at zero momentum and a non-vanishing effective potential for the candidate modulus, as the pole cancels the quadratic vanishing of the numerator. But fortunately F also picks up a total derivative in $(\overline{z}_i, \overline{w}_j)$ from its right-moving superstructure. The amplitude behaves as $k^4/k^2 \to 0$ as the momenta are uniformly scaled to zero, and no effective potential is generated for $\mathcal{N} = (2,2)$ supersymmetry.

To complete the proof, one must show that F is indeed a total derivative with three of the coordinates (z_i, w_j) held fixed. Consider the expression for

[a] One could worry that this restricts the CFT to have $\widehat{c} = \frac{2}{3}c \leq 9$, but at least in tree level string amplitudes one can admit larger $\widehat{c}$ together with a compensating timelike linear dilaton. The tree level scattering amplitudes are unlikely to exhibit any pathology.

the upper component

$$\Phi_1^{\pm}(z_1) = \oint_{z_1} dz\; 2G^{\mp}(z)\Phi_0^{\pm}(z_1) = \frac{1}{z_1}\oint_{z_1} dz\; z\; 2G^{\mp}(z)\Phi_0^{\pm}(z_1) \tag{10.4}$$

and deform the integration contour so that it surrounds the other vertices in the correlator (these pick out the only two modes of $G(z)$ that are regular at $z = 0, \infty$). The relevant operator products are

$$\begin{aligned} &G^+(z)\Phi_0^+(w) \sim 0\,, \\ &G^+(z)\Phi_1^+(w) \sim 2\,\frac{\partial}{\partial w}\left(\frac{1}{z-w}\Phi_0^+(w)\right), \\ &G^-(z)\Phi_0^+(w) \sim \frac{1}{z-w}\,\Phi_1^+(w)\,, \\ &G^-(z)\Phi_1^+(w) \sim 0\;, \end{aligned} \tag{10.5}$$

and similarly for $\Phi_{0,1}^-$. One finds

$$\begin{aligned} F(z_i, w_j) &= -\sum_{r=1}^{n} \partial_{w_r} F_r(z_i, w_j)\,, \\ z_1 F(z_i, w_j) &= -\sum_{r=1}^{n} \partial_{w_r}[w_r F_r(z_i, w_j)]\;, \end{aligned} \tag{10.6}$$

where

$$F_r \equiv \langle \Phi_0^+(z_1)\Phi_1^+(z_2)\cdots\Phi_1^+(z_m)\Phi_1^-(w_1)\cdots\Phi_0^-(w_r)\cdots\Phi_1^-(w_n)\rangle\;. \tag{10.7}$$

If $m \geq 3$, we can fix three of the z_i and then F is a total derivative with respect to the w_j, which are integrated. Similarly for $n \geq 3$. Thus one need only examine the cases $(m, n) = (2, 1)$, $(1, 2)$, and $(2, 2)$. Without loss of generality we can assume $m = 2$ (otherwise just interchange the roles of chiral and antichiral in the following). Use $SL(2, C)$ invariance to fix z_1, z_2, and w_1. Multiply the first of (10.6) by w_1 and subtract from the second to obtain

$$F(z_i, w_j) = \frac{1}{w_1 - z_1}F_1(z_i, w_j) + \frac{1}{w_1 - z_1}\partial_{w_2}[(w_2 - w_1)F_1] \tag{10.8}$$

(we have assumed the most complicated case $n = 2$; if $n = 1$, replace $\Phi_{0,1}^-(w_2)$ by the identity operator). The problem boils down to showing that F_1 is a total derivative. Apply the expressions (10.4) to $\Phi_1^+(z_2)$ and deform contours to obtain

$$\begin{aligned} F_1(z_i,w_j) &= -H - \partial_{w_2} H_2\,, \\ z_2 F_1(z_i,w_j) &= -z_1 H - \partial_{w_2}[w_2 H_2]\ , \end{aligned} \tag{10.9}$$

where

$$\begin{aligned} H(z_i,w_j) &= \langle \Phi_1^+(z_1)\Phi_0^+(z_2)\Phi_0^-(w_1)\Phi_1^-(w_2)\rangle\,, \\ H_2(z_i,w_j) &= \langle \Phi_0^+(z_1)\Phi_0^+(z_2)\Phi_0^-(w_1)\Phi_0^-(w_2)\rangle\ . \end{aligned} \tag{10.10}$$

Now eliminate H from (10.9) to obtain

$$F_1(z_i,w_j) = \frac{1}{z_1 - z_2}\,\partial_{w_2}\big[(w_2 - z_1)H_2\big]\ , \tag{10.11}$$

a total derivative with respect to the integrated variable w_2 if $n = 2$, or vanishing if $n = 1$. Thus the effective potential for $\mathcal{N} = 2$ massless chiral fields vanishes, and they are moduli.

Note that the key here is the last OPE in (10.5), which says that the contour deformation of G^- does not act on any Φ_1^+. Then in (10.7), there are no derivatives with respect to any of the z_i, in particular the unintegrated ones; if there were, the expression could not be reduced further and we could not show that the correlator is a total derivative with respect to integrated variables. This is for instance why the argument does not apply to $\mathcal{N} = 1$ supersymmetry, where all the operator insertions are on the same footing.

10.2. *Application to large* $\mathcal{N} = 4$

The above proof relied essentially on the properties (10.5) of chiral superderivatives. The modulus appeared in the combination $\mathcal{T} = g_+\Phi_1^+ + g_-\Phi_1^-$ (where $g_- = g_+^*$), and the terms with m chiral operators $\Phi_1^+ = G^-_{-\frac{1}{2}}\Phi_0^+$ and n antichiral operators $\Phi_1^- = G^+_{-\frac{1}{2}}\Phi_0^-$ were analyzed separately.

For large $\mathcal{N} = 4$, the modulus deformation has the form

$$\mathcal{T} = G^{\beta\dot\beta}_{-\frac{1}{2}}\overline{G}^{\overline{\alpha\dot\alpha}}_{-\frac{1}{2}}\Phi_{\beta\dot\beta;\overline{\alpha\dot\alpha}}\ ; \tag{10.12}$$

expanding in components, one has two canonical $\mathcal{N} = 2$ substructures

$$\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2\ , \tag{10.13}$$

where (again suppressing the anti-holomorphic structure)

$$\begin{aligned} \mathcal{T}_1 &= \Big(G^{+\dot+}\Phi^{-\dot-} + G^{-\dot-}\Phi^{+\dot+}\Big)\,, \\ \mathcal{T}_2 &= -\Big(G^{+\dot-}\Phi^{-\dot+} + G^{-\dot+}\Phi^{+\dot-}\Big)\ ; \end{aligned} \tag{10.14}$$

the problem is that $\Phi^{\pm\dot{\mp}}$ are neither chiral nor anti-chiral under the $\mathcal{N}=2$ generated by $G^{\pm\dot{\pm}}$, and similarly $\Phi^{\pm\dot{\mp}}$ are nonchiral under the $\mathcal{N}=2$ generated by $G^{\mp\dot{\pm}}$; so we cannot directly apply Dixon's argument. Fortunately, we will be able to find an appropriate modification.

The correlation function (10.1) of conformal perturbation theory can be broken apart into contributions

$$\left\langle \prod_{i=1}^{m} \mathcal{T}_1(z_i) \prod_{j=1}^{n} \mathcal{T}_2(w_j) \right\rangle . \tag{10.15}$$

Embedding the problem again in string theory via an expression of the sort in (10.2), the problem again boils down to showing that all of these contributions are total derivatives with respect to the integrated variables.

We now claim that there is a rearrangement lemma, namely

$$\left\langle \prod_{j=1}^{n} \mathcal{T}_2(w_j) \prod_{i=1}^{m} \mathcal{T}_1(z_i) \right\rangle = \left\langle \mathcal{T}_1(w_1) \prod_{j=2}^{n} \mathcal{T}_2(w_j) \prod_{i=1}^{m} \mathcal{T}_1(z_i) \right\rangle . \tag{10.16}$$

This Ward identity allows us to reduce the large $\mathcal{N}=4$ moduli problem to the $\mathcal{N}=2$ subalgebra, and then we can invoke Dixon's theorem.

In order to prove (10.16) we will need the following OPE's:

$$\begin{aligned}
G^{+\dot{+}}(z)\mathcal{T}_1(w) &= G^{+\dot{+}}(z)\Big(G^{-\dot{-}}\Phi^{+\dot{+}}\Big) = -\frac{\partial}{\partial w}\left(\frac{1}{z-w}\Phi^{+\dot{+}}(w)\right), \\
G^{+\dot{+}}(z)\mathcal{T}_2(w) &= -\frac{\partial}{\partial w}\left(\frac{1}{z-w}\Phi^{+\dot{+}}(w)\right), \\
G^{-\dot{-}}(z)\mathcal{T}_1(w) &= G^{+\dot{+}}(z)\Big(G^{+\dot{+}}\Phi^{-\dot{-}}\Big) = -\frac{\partial}{\partial w}\left(\frac{1}{z-w}\Phi^{-\dot{-}}(w)\right), \\
G^{-\dot{-}}(z)\mathcal{T}_2(w) &= -\frac{\partial}{\partial w}\left(\frac{1}{z-w}\Phi^{-\dot{-}}(w)\right)
\end{aligned} \tag{10.17}$$

as well as

$$\begin{aligned}
A^{+,\dot{-}\dot{-}}(z)\mathcal{T}_1(w) &\sim 0\,, \\
A^{+,\dot{-}\dot{-}}(z)\mathcal{T}_2(w) &\sim 0\,, \\
A^{-,--}(z)\mathcal{T}_1(w) &\sim 0\,, \\
A^{-,--}(z)\mathcal{T}_2(w) &\sim 0\,.
\end{aligned} \tag{10.18}$$

The equations (10.18) are proved using the identities (4.19).

Now we write

$$
\begin{aligned}
&\Big\langle \prod_{j=1}^{n} \mathcal{T}_2(w_j) \prod_{i=1}^{m} \mathcal{T}_1(z_i) \Big\rangle \\
&= -\Big\langle \Big(\oint_{w_1} G^{+\dot{-}} \Phi^{-\dot{+}} + \oint_{w_1} G^{-\dot{+}} \Phi^{+\dot{-}} \Big) \prod_{j=2}^{n} \mathcal{T}_2(w_j) \prod_{i=1}^{m} \mathcal{T}_1(z_i) \Big\rangle \\
&= -\sum_{r=2}^{n} \frac{\partial}{\partial w_r} \Big\{ \Big\langle \Phi^{-\dot{+}}(w_1) \Phi^{+\dot{-}}(w_r) \prod{}'' \mathcal{T}_2 \prod \mathcal{T}_1 \Big\rangle \\
&\qquad + \Big\langle \Phi^{+\dot{-}}(w_1) \Phi^{-\dot{+}}(w_r) \prod{}'' \mathcal{T}_2 \prod \mathcal{T}_1 \Big\rangle \Big\} \\
&\qquad - \sum_{i=1}^{m} \frac{\partial}{\partial z_i} \Big\{ \Big\langle \Phi^{-\dot{+}}(w_1) \Phi^{+\dot{-}}(z_i) \prod{}' \mathcal{T}_2 \prod{}' \mathcal{T}_1 \Big\rangle \\
&\qquad + \Big\langle \Phi^{+\dot{-}}(w_1) \Phi^{-\dot{+}}(z_i) \prod{}' \mathcal{T}_2 \prod{}' \mathcal{T}_1 \Big\rangle \Big\} \qquad (10.19) \\
&= +\sum_{r=2}^{n} \frac{\partial}{\partial w_r} \Big\{ \Big\langle \Phi^{-\dot{-}}(w_1) \Phi^{+\dot{+}}(w_r) \prod{}'' \mathcal{T}_2 \prod \mathcal{T}_1 \Big\rangle \\
&\qquad + \Big\langle \Phi^{+\dot{+}}(w_1) \Phi^{-\dot{-}}(w_r) \prod{}'' \mathcal{T}_2 \prod \mathcal{T}_1 \Big\rangle \Big\} \\
&\qquad + \sum_{i=1}^{m} \frac{\partial}{\partial z_i} \Big\{ \Big\langle \Phi^{-\dot{-}}(w_1) \Phi^{+\dot{+}}(z_i) \prod{}' \mathcal{T}_2 \prod{}' \mathcal{T}_1 \Big\rangle \\
&\qquad + \Big\langle \Phi^{+\dot{+}}(w_1) \Phi^{-\dot{-}}(z_i) \prod{}' \mathcal{T}_2 \prod{}' \mathcal{T}_1 \Big\rangle \Big\} \\
&= \Big\langle \mathcal{T}_1(w_1) \prod_{j=2}^{n} \mathcal{T}_2(w_j) \prod_{i=1}^{m} \mathcal{T}_1(z_i) \Big\rangle \; .
\end{aligned}
$$

The primes on the products indicate that the appropriate factor is deleted from the product. In the first equality, we have written the definition of $\mathcal{T}_1, \mathcal{T}_2$. In the second we have deformed contour integrals of G and used (10.17). In the third we have used the Ward identity following from contour deformation of integrals of $A^{\pm,--}$ and made use of (10.18). Finally, in last equality, we have used again a Ward identity following from deformation of contour integrals of G. Thus we can systematically reduce the correlators (10.15) to correlators of only $\mathcal{T}_1$.

Now let us separate the product over $\mathcal{T}_1(z_i)$ in (10.15) (with $n = 0$ now) into its separate contributions from m_+ operators $G^{--\dot{-}}\Phi^{++\dot{+}}(z_i)$, and m_- operators $G^{++\dot{+}}\Phi^{--\dot{-}}(z_i')$. Since $G^{--\dot{-}}$ in the OPE (10.17) treats $G^{++\dot{+}}\Phi^{--\dot{-}}(z_i')$ in the same way that the antichiral operators $\Phi_1^-(w)$ behaved in Dixon's analysis, we are done; we can simply apply the same analysis with $G^{--\dot{-}}\Phi^{++\dot{+}}$ playing the role of Φ_1^+ and $G^{++\dot{+}}\Phi^{--\dot{-}}$ playing the role of Φ_1^-.

11. Appendix B: $\mathcal{N}=4$ algebra in bispinor notation

Spinor conventions: $\epsilon^{+-}=\epsilon_{-+}=1$ raises/lowers spinor indices from the left. The adjoint of $su(2)$ is a bispinor according to

$$\begin{aligned} x^{A\dot{B}} &= x^j\sigma_j^{A\dot{B}}\,,\\ x^{++} &= x^1+ix^2\,,\\ x^{+-} &= -x^3\,,\\ x^{--} &= -(x^1-ix^2)\,. \end{aligned} \tag{11.1}$$

In four dimensions,

$$\begin{aligned} \sigma^\mu_{A\dot{B}} &\equiv (i,\vec{\sigma})\,,\\ \overline{\sigma}^{\mu\dot{A}B} &\equiv (i,-\vec{\sigma})\,,\\ v^{\dot{B}A}=v^{A\dot{B}} &\equiv -\frac{1}{2}\,(\overline{\sigma}^\mu)^{\dot{B}A}v_\mu\,,\\ v_\mu &= (\sigma_\mu)_{A\dot{B}}v^{A\dot{B}}\,. \end{aligned} \tag{11.2}$$

In terms of $\gamma=k^-/k$, $1-\gamma=k^+/k$ we have

$$\begin{aligned} \left\{G_m^{A\dot{B}},G_n^{C\dot{D}}\right\} =& -\frac{1}{2}\,\epsilon^{\dot{B}\dot{D}}\epsilon^{AC}\left[2L_{n+m}+\frac{c}{3}\delta_{n+m,0}(m^2-\frac{1}{4})\right]\\ &+i(n-m)\left[-\gamma A_{n+m}^{+,\dot{B}\dot{D}}\epsilon^{AC}+(1-\gamma)A_{n+m}^{-,AC}\epsilon^{\dot{B}\dot{D}}\right],\\ \left[A_m^{+,j},G_n^{\dot{B}A}\right] =& \frac{i}{2}\,(\sigma^j)^{\dot{B}}{}_{\dot{C}}\left(G_{n+m}^{\dot{C}A}-2(1-\gamma)mQ_{n+m}^{\dot{C}A}\right),\\ \left[A_m^{-,j},G_n^{\dot{B}A}\right] =& -\frac{i}{2}\left(G_{n+m}^{\dot{B}C}+2\gamma mQ_{n+m}^{\dot{B}C}\right)(\sigma^j)_C{}^A \end{aligned} \tag{11.3}$$

where

$$G^{A\dot{B}}=G^{\dot{B}A}=\frac{1}{2}\begin{pmatrix} G^3-iG^4 & G^1-iG^2\\ G^1+iG^2 & -G^3-iG^4\end{pmatrix}\ . \tag{11.4}$$

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