A NEW HAT FOR THE $c = 1$ MATRIX MODEL

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We consider two-dimensional supergravity coupled to $\hat{c} = 1$ matter. This system can also be interpreted as noncritical type 0 string theory in a two-dimensional target space. After reviewing and extending the traditional descriptions of this class of theories, we provide a matrix model description. The 0B theory is similar to the realization of two-dimensional bosonic string theory via matrix quantum mechanics in an inverted harmonic oscillator potential; the difference is that we expand around a non-perturbatively stable vacuum, where the matrix eigenvalues are equally distributed on both sides of the potential. The 0A theory is described by a quiver matrix model.

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1. Introduction

Matrix quantum mechanics of an $N \times N$ Hermitian matrix $M(x)$ describes two-dimensional bosonic string theory (for reviews see [1–3]). Briefly, the large N planar graph expansion of the Euclidean matrix path integral

$$\mathcal{Z} = \int D^{N^2} M(x) \exp\left[-\beta \int_{-\infty}^{\infty} dx \, \mathrm{Tr}\left(\frac{1}{2} \left(D_x M \right)^2 + V\left(M\right) \right) \right] \tag{1.1}$$

discretizes a closed string worldsheet embedded in the target space (Euclidean) time direction parametrized by x [4]; a spatial direction grows out of the matrix eigenvalue coordinate [5]. The coupling β is the inverse Planck constant in the matrix quantum mechanics. The continuum ("double scaling") limit [6] isolates a quadratic maximum of the potential

$$V(M) = -\frac{1}{2\alpha'} M^2. \tag{1.2}$$

This formulation of 2d string theory has long been suspected of being an example of exact open/closed string duality. Recent work [7–11] has established this idea concretely through computations of D-brane dynamics in the two-dimensional bosonic string. This progress builds on studies of holography in linear dilaton backgrounds (see, e.g. Ref. [12]), and on construction of D-branes in such backgrounds. Of particular importance for formulating a precise open/closed string duality, as emphasized in Ref. [13], is the discovery of boundary states in Liouville theory localized at large ϕ [14].

Let us summarize the basic facts about two-dimensional string theory. The closed string background for two-dimensional strings consists of a free scalar field x, with conformal invariance maintained by coupling to a Liouville field theory [15], often thought of as worldsheet gravity. The closed string "tachyon" is the principal effective field in spacetime; the tachyon mass is in fact lifted to zero by a linear dilaton background, so that the theory is perturbatively stable. There is in addition a set of discrete physical degrees of freedom which appear at special momenta.

The bosonic string has unstable D-branes. The open string tachyon field on N such D0-branes is a Hermitian matrix $M(x)$. The curvature of the potential at the quadratic maximum is given by the open string tachyon mass-squared $m_T^2 = -1/\alpha'$, in agreement with the matrix model value (1.2) which was originally established by comparison with closed string amplitudes [1]. This serves as a consistency check on the identification of $M(x)$ with the tachyon field localized on the D0-branes. The open string spectrum also includes a non-dynamical gauge field A, corresponding to the vertex operator $\dot{x}$. It enters the covariant derivative in (1.1)

$$D_x M = \partial_x M - [A, M] \ . \tag{1.3}$$

A acts as a lagrange multiplier that projects onto $SU(N)$ singlet wave functions which depend on the N matrix eigenvalues only. After a Vandermonde Jacobian is taken into account, the eigenvalues act as free fermions [16]. In the bosonic string theory the Fermi sea is asymmetric; it fills only one side of the upside down harmonic oscillator potential to Fermi level $-\mu$ (as measured from the top of the potential).

The proposal is then that the matrix model description simply *is* the effective dynamics of the bosonic string D0-branes, and that the closed string state that they 'decay' into is the background described by $c = 1$ matter coupled to Liouville gravity. This proposal has been subjected to a number of nontrivial checks, including the computation of closed string radiation from a decaying D0-brane [7–9], as well as the leading instanton corrections to the partition function [8, 10, 11].

This model is unstable nonperturbatively. Various stable nonperturbative definitions of the theory were studied in Refs. [17–19] and were critically discussed in Ref. [3]. One of the attractive ways to stabilize the model is to fill the two sides of the potential to the same Fermi level. The authors of Ref. [19] argued that the problems raised in Ref. [3] could be avoided in this case.

In this paper we argue that this two-sided matrix model in fact describes two-dimensional NSR strings. Let us describe the general structure of this theory. The closed string background is again described on the worldsheet by Liouville theory coupled to a free scalar field, now with $N = 1$ worldsheet supersymmetry. Here it is natural to carry out a non-chiral GSO projection, so that we have type 0 strings. The closed string "tachyon" is again lifted to zero mass by the linear dilaton of Liouville theory, so the theory is perturbatively stable.

The R-R spectrum depends on the GSO projection. In the type 0B theory the R-R spectrum consists of a massless scalar field. There are also discrete physical states appearing at special momenta. Like the bosonic string, the 0B theory also has unstable D0-branes whose worldline dynamics is described again by Hermitian matrix quantum mechanics about an unstable quadratic maximum. It is natural to conjecture that the same sort of matrix path integral (1.1) also describes the 2d type 0B theory. The difference is that the open string potential is a stable double-well, with the ground state filling up both wells with equal numbers of eigenvalues.[a,b] It is natural to suspect that this theory is non-perturbatively well-defined. The eigenvalue density perturbations that are even and odd under the Z_2 reflection symmetry of the potential correspond to the NS-NS and R-R scalars of type 0B theory, respectively.

In contrast, the type 0A theory has no propagating modes in the R-R sector, but has two vector fields (the two electric fields are examples of discrete states). To obtain type 0A theory from the type 0B matrix model, we need to quotient by a Z_2 acting in the manner familiar from quiver gauge theories [20]. This leads in general to a matrix quantum mechanics of a complex $M \times N$ matrix, with $U(M) \times U(N)$ (gauge) symmetry.

This relation between the matrix model and open strings on unstable D0-branes of type 0 theory,[c] provides a clear motivation for this duality and allows for additional precise checks.

The plan of the paper is as follows. Section 2 introduces two-dimensional type 0 string theory, and its worldsheet description as $\hat{c} = 1$ superconformal field theory of a free superfield X, whose lowest component is x, coupled to super-Liouville theory. We discuss the possible GSO projections at finite and infinite radius of x, and their spectra.

Sections 3 and 4 explore some perturbative aspects of the theory. Section 3 investigates the ground ring structure [26–28] and tree level S-matrix of the 2d bosonic and type 0B strings. The ground ring is a collection of BRST invariant operators of the worldsheet theory satisfying a closed operator algebra, which is related to the phase space of the eigenvalues in the matrix model description. Recent progress in (super-)Liouville theory enables one to explore this relation in detail. Section 4 calculates the torus partition function, which includes an important contribution from the odd spin structure.

Sections 5 , 6 , 7 and 8 discuss the D-branes of the theory from various points of view. Section 5 contains a semiclassical description of boundary conditions in super-Liouville theory. In Section 6 we study the minisuperspace approximation. Section 7 discusses the boundary states of the full

[a] A conjecture relating double-well $c = 1$ matrix model to ten-dimensional superstrings was made in Ref. [7].

[b] Equivalently, this model can be described as quantum mechanics of a unitary matrix U with potential $\mathrm{Tr}(U + U^\dagger)$. After the double scaling limit is taken, both models are described by fermions in an upside down harmonic oscillator potential filling both sides of the potential to Fermi level $-\mu$.

[c] The spectrum of D-branes in 10-d type 0 theories was studied in Refs. [21–25].

quantum theory, and their relation to the minisuperspace wavefunctions. The annulus partition function is also considered. Section 8 discusses some aspects of the worldvolume dynamics of D-branes and their interactions with closed string fields.[d]

The precise nature of our proposal for a type 0B matrix model is presented in Section 9, where we check the torus worldsheet calculation against a computation of the finite temperature matrix model free energy. Section 10 introduces the type 0A matrix model and performs the analogous check. In Section 11 we study the relation of tachyon condensation and rolling matrix eigenvalues [7–9] through a calculation of the disk expectation value of the ground ring generators. We also explore the radiation of closed strings produced by the decay. Three Appendices contain useful technical results.

It is also possible to show that $\hat{c} < 1$ superconformal minimal models coupled to 2d supergravity are dual to unitary matrix models,[e] as one might expect on the basis of gravitational RG flow [30–32]. These matrix models were solved in the planar limit in Ref. [33], and in the double scaling limit in Refs. [34–36]. These models will be the subject of a separate publication.

Note Added: While completing this manuscript, we learned of work by T. Takayanagi and N. Toumbas [37] where the matrix model for 2d type 0 strings is also proposed.

2. The 2d Fermionic String

Fermionic strings are described by $N = 1$ supersymmetric world sheet field theories coupled to worldsheet supergravity. The construction of type II string theories requires the existence of a non-anomalous chiral $(-1)^{F_L}$ symmetry of the worldsheet theory. The generic background may only admit a nonchiral $(-1)^F$; the use of this class of (GSO) projection gives type 0 string theory (see Ref. [38] for a review).

There are in fact two distinct choices, depending on how the $(-1)^F$ symmetry is realized in the closed string Ramond-Ramond (R-R) sector; the closed string spectrum admits the sectors

$$\begin{aligned} &\text{type 0A}: \ (NS-, NS-) \oplus (NS+, NS+) \oplus (R+, R-) \oplus (R-, R+)\,, \\ &\text{type 0B}: \ (NS-, NS-) \oplus (NS+, NS+) \oplus (R+, R+) \oplus (R-, R-)\,, \end{aligned} \tag{2.1}$$

[d] Sections 2-8 review known material but also describe many new results. The reader who is impatient to get to the matrix model can move directly to Section 9.

[e] A suggestion for string duals of the unitary matrix models, which involves open and closed strings, was made in Ref. [29].

where $\pm$ refers to worldsheet fermion parity. As is familiar from type II, the two theories are interchanged under orbifolding by $(-1)^{\mathbf{F}_L}$ where $\mathbf{F}_L$ is left-moving NSR parity.[a] Because there are no (NS, R) or (R, NS) sectors, there are no closed strings that are spacetime fermions. In addition to the usual NS sector fields appearing in the $(NS+, NS+)$ sector, one has a closed string tachyon T appearing in the $(NS-, NS-)$ sector. The R-R spectrum is doubled; in particular, at the massless level there are two R-R gauge fields C_{p+1}, $\tilde{C}_{p+1}$ of each allowed rank $p+1$ (p is even for type 0A, odd for type 0B).[b] The spacetime effective action for these fields is [22, 38, 39]

$$S = \frac{1}{2\kappa^2}\int d^d x\sqrt{-G}\left[e^{-2\Phi}\left(\frac{10-d}{\alpha'} + R + 4(\nabla\Phi)^2 - \frac{1}{2}|H_3|^2 - \frac{1}{2}(\nabla T)^2 + \frac{1}{\alpha'}T^2\right)\right.$$
$$\left. - \frac{1}{2}\left(\sum_p |F_{p+2}|^2 + |\tilde{F}_{p+2}|^2 + T\,|F_{p+2}\tilde{F}_{p+2}|\right) + \ldots\right], \quad (2.2)$$

where $|F_n\tilde{F}_n| = \frac{1}{n!}\,F^{m_1\ldots m_n}\tilde{F}_{m_1\ldots m_n}$ (there is an additional factor of $\frac{1}{2}$ for $n = d/2$).

The specialization of this theory to $d = 2$ is particularly simple. There is no NS B field, and the dilaton gravity sector of Φ and G has no field theoretic degrees of freedom. A closed string background which solves the equations of motion of (2.2) is (we set $\alpha' = 2$ unless indicated otherwise)

$$G_{\mu\nu} = \eta_{\mu\nu}\,, \quad \Phi = \phi\,, \quad T = \mu e^{\phi}, \quad (2.3)$$

where ϕ is the spatial coordinate. The (possibly Euclidean) time coordinate will be denoted by x.

In two dimensions, there are no transverse string oscillations, and longitudinal oscillations are unphysical at generic momenta. Thus the tachyon is the only physical NS sector field.[c] In the R-R sector, the spectrum consists of two vector fields $C_1, \tilde{C}_1$ for type 0A, while the type 0B theory has a scalar C_0 (the self-dual part of C_0 comes from the $(R+, R+)$ sector and

[a] Note that $(-1)^{\mathbf{F}_L}$ reverses the sign of the left moving spin field. This should not be confused with the worldsheet Z_2 operation $(-1)^{F_L}$ giving a minus sign to the leftmoving worldsheet fermion $\psi_L \to -\psi_L$.

[b] For the middle rank $p + 2 = d/2$, there is *one* gauge field, but it has both self-dual and anti-self-dual components.

[c] When we consider operators in the spacetime theory, i.e. non-normalizable deformations, then there are additional operators (usually called discrete "states") at special values of the momenta. For instance, when x is compactified, the radius deformation $V_G = \partial x\bar{\partial}x$ corresponds to a non-normalizable physical vertex operator on the worldsheet.

the anti-self-dual part comes from the $(R-,R-)$ sector) and a pair of two-forms $C_2\,,\tilde{C}_2\,.$ Only the scalar C_0 gives a field-theoretic degree of freedom. The rest of the fields give rise to discrete states.

Let us discuss the action of the 0B theory in more detail. Equation (2.2) becomes

$$S=\int d\phi\, dt\,\sqrt{-G}\,\Bigg[\frac{e^{-2\Phi}}{2\kappa^2}\Big(\frac{8}{\alpha'}+R+4\big(\nabla\Phi\big)^2-f_1\big(T\big)\big(\nabla T\big)^2+f_2\big(T\big)+\dots\Big) \\ -\frac{1}{8\pi}\,f_3\big(T\big)\big(\nabla C_0\big)^2+\dots\Bigg], \tag{2.4}$$

with $f_i(T)$ functions of the tachyon field T. To all orders in perturbation theory the action is invariant under shifts of the R-R scalar $C_0\,.$ In the matrix model this shift symmetry corresponds to multiplying the wave functions of all the fermions on one side of the potential by a common phase while not doing that to the fermions on the other side of the potential. Non-perturbatively, the fermions on the two sides affect each other and this shift symmetry is violated by instantons. Correspondingly, the theory is invariant only under a discrete shift of $C_0\,.$ This discrete shift symmetry is a gauge symmetry, i.e. C_0 is compact. We normalize $f_3(T=0)=1\,.$ We will later argue that in this normalization the field C_0 is a two-dimensional field at the self-dual radius. Also, we will give evidence that $f_3(T)=e^{-2T}$ (see also Ref. 40). With this form of f_3 we derive a few consequences:

1. The effective radius of C_0 goes to zero as $T\to\infty$ – the circle parametrized by C_0 is pinched to a point. Therefore, the theory based on the action (2.4) violates the winding symmetry in C_0 at tree level. D-instantons which are not included in (2.4) violate also the momentum symmetry. This can be seen by computing a disk amplitude with one insertion of $C_0\,.$
2. The theory has a peculiar "S-like" duality under which $T\to -T$ and the compact boson C_0 is dualized. In terms of the dual field $\tilde{C}_0$ the classical theory has no momentum symmetry because the field $\tilde{C}_0$ is pinned at infinity where the coefficient of its kinetic term is infinite. In the matrix model this S-duality corresponds to changing the sign of μ; i.e. to lifting the Fermi level above the maximum of the potential. In the worldsheet description of the theory this operation is $(-1)^{F_L}$ where F_L is the worldsheet fermion number.
3. The field $\chi=e^{-T}C_0$ has a canonical kinetic term

$$-\frac{1}{8\pi}\left(\nabla\chi+\chi\nabla T\right)^2 \tag{2.5}$$

but the shift symmetry is more complicated $\chi \to \chi + e^{-T}\alpha$ for constant α. We can think of $F = d\chi + \chi dT$ as a one form field strength of χ. Then, in the background (2.3) the equation of motion and the Bianchi identity are

$$\begin{aligned} \Big(-\frac{\partial}{\partial\phi} + \mu e^{\phi}\Big) F_\phi &= -\frac{\partial}{\partial t} F_t \,, \\ \Big(\frac{\partial}{\partial\phi} + \mu e^{\phi}\Big) F_t &= \frac{\partial}{\partial t} F_\phi \,. \end{aligned} \tag{2.6}$$

4. Because of the coupling to T for each sign of μ there is only one solution of the equations of motion with time independent F

$$C_0 = \begin{cases} \int^\phi d\phi' e^{2T} = \int^\phi d\phi' \exp\big(2\mu e^{\phi'}\big) & \mu < 0 \\ t & \mu > 0 \end{cases} \tag{2.7}$$

or equivalently
$$F = \begin{cases} \exp\big(\mu e^{\phi}\big)\, d\phi & \mu < 0 \\ \exp\big(-\mu e^{\phi}\big)\, dt & \mu > 0\,. \end{cases} \tag{2.8}$$

The other solution is unacceptable because it diverges at $\phi \to +\infty$. The solution for $\mu > 0$ and the solution for $\mu < 0$ are exchanged by the S-duality we mentioned above. These solutions represent one possible non-normalizable deformation of the theory, which is different depending on the sign of μ.

5. In the matrix model with positive μ this deformation corresponds to changing the two Fermi levels on the two sides of the potential.[d] Nonperturbatively, eigenvalues can tunnel from one side of the potential to the other and the only stable situation is when the Fermi levels are the same. This means that nonperturbatively this zero mode of C_0 is fixed (or more precisely, it is quantized according to the periodicity of C_0) and cannot be changed.

The situation in the 0A theory is analogous. In fact, by compactifying the Euclidean time direction on a circle of radius R, T-duality relates these two theories. The target space action is

$$\begin{aligned} S = \int d\phi\, dt\, \sqrt{-G}\, \Bigg[&\frac{e^{-2\Phi}}{2\kappa^2} \Big(\frac{8}{\alpha'} + R + 4\big(\nabla\Phi\big)^2 - f_1(T)\big(\nabla T\big)^2 + f_2(T) + \ldots \Big) \\ &- \frac{(2\pi)\alpha'}{4} f_3(T)\big(F^{(+)}\big)^2 - \frac{(2\pi)\alpha'}{4} f_3(-T)\big(F^{(-)}\big)^2 + \ldots \Bigg], \end{aligned} \tag{2.9}$$

[d] For negative μ these are the two distinct Fermi levels of the left movers and the right movers.

where we see two different one form gauge fields and their field strengths $F^{(\pm)}$. Our S-duality transformation $T\to -T$ exchanges them. We have normalized the gauge fields so that the coupling to a unit charge is $e^{i\int A}$. The normalization of the kinetic terms for the gauge fields in (2.9) follows from T-duality and the normalization of the C field, which is argued to be at the self-dual radius in Appendix C. Ordinarily the only degree of freedom in a two-dimensional gauge theory is the zero mode, and one might expect the possibility of turning on time independent $F^{(\pm)}$. However, in this system, because of the coupling $f_3(\pm T)$, time independent field strength can be turned on only for that gauge field whose coefficient $f_3(\pm T)$ is nonzero at infinity. This means that we can only turn on $F^{(-)}=\exp(-2\mu e^{\phi})$ for μ positive and $F^{(+)}=\exp(2\mu e^{\phi})$ for μ negative. This statement is T-dual to the analogous fact of having only one zero energy solution in the 0B theory. Nonperturbatively, this zero mode is quantized, as in the 0B theory. It corresponds to the background field of D0-branes whose charge is quantized.

2.1. *Worldsheet Description*

The worldsheet description of the background (2.3) involves two scalar superfields

$$\begin{aligned} \Phi &= \phi + i\theta\psi + i\bar\theta\bar\psi + i\theta\bar\theta F\,, \\ X &= x + i\theta\chi + i\bar\theta\bar\chi + i\theta\bar\theta G\,. \end{aligned} \tag{2.10}$$

Our 2d supersymmetry conventions are as follows. The covariant derivatives, supercharges and algebra are

$$\begin{aligned} &D=\frac{\partial}{\partial\theta}+\theta\partial\,, \quad \bar D=\frac{\partial}{\partial\bar\theta}+\bar\theta\bar\partial\,, \quad \{D,D\}=2\partial\,, \quad \{\bar D,\bar D\}=2\bar\partial\,, \\ &Q=\frac{\partial}{\partial\theta}-\theta\partial\,, \quad \bar Q=\frac{\partial}{\partial\bar\theta}-\bar\theta\bar\partial\,, \quad \{Q,Q\}=-2\partial\,, \quad \{\bar Q,\bar Q\}=-2\bar\partial\,, \end{aligned} \tag{2.11}$$

and all other (anti)commutators vanish. We define $z=x+iy$, $\bar z=x-iy$ and therefore $\partial=(\partial_x-i\partial_y)/2$, $\bar\partial=(\partial_x+i\partial_y)/2$. Finally, the integration measure is $\int d^2z\,d^2\theta=2\int dx\,dy\,d\bar\theta\,d\theta$.

The action for X is the usual free field action

$$S_X=\frac{1}{4\pi}\int d^2z\,d^2\theta\,DX\bar DX\,, \tag{2.12}$$

while the dynamics of Φ is governed by the super-Liouville action [41–45]

$$S=\frac{1}{4\pi}\int d^2z\,d^2\theta\left[D\Phi\,\bar D\Phi+2i\mu_0 e^{b\Phi}\right]. \tag{2.13}$$

There is implicitly a dilaton linear in Φ, of slope

$$Q = b + \frac{1}{b}\ , \tag{2.14}$$

which makes its usual appearance as an improvement term in the superstress tensor. The action (2.13) gives rise to an $N=1$ superconformal field theory with central charge

$$\hat{c}_L = 1 + 2Q^2. \tag{2.15}$$

The case of interest here will be $b=1$, $\hat{c}_L=9$. Exponential operators have dimension

$$\Delta\big(e^{\alpha\Phi+ikX}\big) = \frac{1}{2}\,\alpha\big(Q-\alpha\big) + \frac{1}{2}\,k^2\,; \tag{2.16}$$

in particular, the Liouville interaction is scale invariant.

2.2. *Compactification*

In the Euclidean theory we may replace the non-compact free scalar superfield theory of X by any other $\hat{c}=1$ superconformal field theory. The set of $\hat{c}=1$ superconformal field theories was classified in Refs. [46, 47]. Apart from orbifolds and isolated theories which will not concern us here, there are two lines of theories parametrized by the radius of compactification R of the lowest component of the superfield X. The first line of models is called the 'circle' theory; in it, the sum over the spin structures of $\chi\,,\bar{\chi}$ is independent of the (left and right) momentum $k\,,\,\bar{k}$ of x, giving rise to a tensor product of a compact boson and an Ising model. The momentum takes the usual form $\big(k,\bar{k}\big) = \big(\frac{p}{R} + \frac{wR}{2}\,,\,\frac{p}{R} - \frac{wR}{2}\big)$; hence the lattice of momenta is $\Lambda = \{(p,w)|\,p\,,w\in Z\}$. The circle model has the usual $R\to 2/R$ T-duality symmetry. At the self-dual point ($R=\sqrt{2}$) an affine $SU(2)\times SU(2)$ symmetry appears for the boson x. This symmetry does not commute with worldsheet supersymmetry and therefore is not a symmetry of the spacetime theory.

The second line, the 'super affine' theory, is obtained from the circle by modding out by a Z_2 symmetry $(-)^{\mathbf{F}_L}e^{i\pi p}$, where $(-)^{\mathbf{F}_L}=1$ (-1) on NS-NS (R-R) states. This correlates the sum over spin structures with the momentum and winding of x. To describe the resulting spectrum it is convenient to define the following sublattices:

$$\begin{aligned}
\Lambda^+ &= \Big\{(p,w)|\,p\in 2Z\,,w\in Z\Big\}\,,\\
\Lambda^- &= \Big\{(p,w)|\,p\in 2Z+1\,,w\in Z\Big\}\,,\\
\Lambda^+_\delta &= \Big\{(p,w)|\,p\in 2Z\,,w\in Z+\tfrac{1}{2}\Big\}\,,\\
\Lambda^-_\delta &= \Big\{(p,w)|\,p\in 2Z+1\,,w\in Z+\tfrac{1}{2}\Big\}\,.
\end{aligned} \tag{2.17}$$

The NS states in the super affine theory have momenta in $\Lambda^+ \cup \Lambda_\delta^-$, whereas Ramond states inhabit $\Lambda^- \cup \Lambda_\delta^+$ [47]. The super affine model is self-dual under $R \to \frac{4}{R}$. At the self-dual point, $R=2$, an $SU(2) \times SU(2)$ super affine algebra appears, making this the most symmetric point in the moduli space of $\hat{c}=1$ theories. These symmetries commute with worldsheet supersymmetry so they translate into symmetries of the spacetime theory. Despite the fact that the super affine line of theories is conveniently described by twisting the circle theory, it is not in any sense "less fundamental". The super affine theory has the following interesting property. Recall that degenerate superconformal representations at $\hat{c}=1$ occur at $k=(r-s)/2$ with $(r-s) \in 2Z\,(2Z+1)$ corresponding to NS (R) degenerate representations; we see that the self-dual super affine model at $R=2$ has the property that all the representations that occur in it are degenerate, as is the case for the self-dual $(R=\sqrt{2})$ bosonic circle model; there is no point on the superconformal circle line with these properties.

2.3. *Spectrum and Vertex Operators*

Corresponding to the two lines of SCFT's described above are four different two-dimensional string theories obtained by coupling X (2.12) to the super-Liouville field (2.13). Consider first the circle line. Physical states corresponding to momentum modes of the NS "tachyon" field have the form (in the $(-1\,,-1)$ picture)

$$T_k^{(\pm)} = c\bar{c}\ \exp\left[-(\varphi+\bar{\varphi}) + ik(x_L + x_R) + (1 \mp k)\phi\right], \tag{2.18}$$

where as explained above, $k=\bar{k}=p/R$, $p \in Z$.[e] Note that, due to the linear dilaton in (2.3), the dispersion relation of the "tachyon" T is in fact massless. The Ramond vertex has the form (in the $(-1/2\,,-1/2)$ picture)

$$V_k^{(\pm)} = c\bar{c}\ \exp\left[-\frac{\varphi+\bar{\varphi}}{2} \mp \frac{i}{2}\big(H+\bar{H}\big) + ik(x_L + x_R) + \big(1 \mp k\big)\phi\right] \tag{2.19}$$

with $k=p/R$ as in (2.18) ; we have bosonized the fermions in (2.10) as usual via $\psi + i\chi = \sqrt{2}\, e^{iH}$. Physical winding modes $\hat{T}_k\,, \hat{V}_k$ have the same form as (2.18) , (2.19) with $x_R\,,\ \bar{H} \to -x_R\,, -\bar{H}$, and $k=wR/2\,,\ w \in Z$.

All the NS states above have even fermion number and are physical. In the Ramond sector there are two allowed fermion number projections which lead to different spectra. Projecting to states invariant under $H \to H + \pi$, $\bar{H} \to \bar{H} - \pi$ keeps the momentum modes V_k of (2.19) while projecting out

[e] The fields $\varphi\,,\ \bar{\varphi}$ bosonize the spinor ghosts of the fermionic string.

the winding modes $\hat{V}_k$. The opposite projection $H(\bar{H}) \to H(\bar{H}) + \pi$ has the opposite effect. These are the type 0B and type 0A circle theories, respectively. As is familiar from higher dimensions, they are dual to each other under $R \to 2/R$. At infinite R the physical spectrum of the type 0B theory includes two (massless) field-theoretic degrees of freedom, the tachyon T_k and the R-R scalar V_k; the type 0A theory has only the NS scalar field T_k.

Repeating the analysis above for the case of the super affine theory, we are led to the following spectrum: in the NS sector we find *even* momentum modes T_k of (2.18) with $k = 2n/R$ and *integer* winding modes $\hat{T}_k$ with $k = mR/2$. In the Ramond sector we find *odd* momentum modes V_k of (2.19) with $k = (2n+1)/R$ and *half integer* windings $\hat{V}_k$ with $k = (m + 1/2)R/2$. The GSO projection in this theory is momentum sensitive. One gets two super affine theories, depending on the sign of the projection: the type 0B theory allows all the Ramond states listed above (momentum and winding) and the type 0A theory projects them all out (the NS states are again physical). Each of the two theories is separately self-dual under $R \to 4/R$. At infinite radius they give rise to the same two theories as the infinite radius circle ones.

3. The Ground Ring and Tree Level Scattering

The BRST cohomology of string theory in two dimensions includes a set of operators of dimension zero and ghost number zero, known as the ground ring. E. Witten [26] proposed that the ground ring provides insight into the relation between the continuum formulation and the matrix model. In this section we discuss this idea, first in the bosonic string, and then for the NSR case. Progress in Liouville field theory since [26] makes it possible to say more about the properties of the ring and its relation to the matrix model (a subject to which we will return in Section 11).

3.1. *The Ground Ring of the 2d Bosonic String and the Matrix Model*

We start by listing a few results on Liouville field theory which will be useful later (see e.g. Refs. [14, 48, 49]). In this subsection we set $\alpha' = 1$. The Liouville central charge is

$$c_L = 1 + 6Q^2 = 13 + 6b^2 + \frac{6}{b^2} \,, \tag{3.1}$$

where we used the parametrization

$$Q = b + \frac{1}{b}\,. \tag{3.2}$$

The cosmological term in the Liouville Lagrangian is

$$\delta\mathcal{L} = \mu_0 e^{2b\phi}\,. \tag{3.3}$$

A set of natural observables in the theory is

$$V_\alpha(\phi) = e^{2\alpha\phi}\,, \tag{3.4}$$

whose scaling dimension is $\Delta_\alpha = \bar{\Delta}_\alpha = \alpha(Q-\alpha)$. Degenerate representations of the Virasoro algebra occur for $\alpha = \alpha_{m,n}$, such that

$$\Delta^{(L)}_{m,n} = \Delta(\alpha_{m,n}) = \frac{1}{4}\,Q^2 - \frac{1}{4}\Big(\frac{m}{b} + nb\Big)^2. \tag{3.5}$$

The corresponding $\alpha_{m,n}$ are

$$\alpha_{m,n} = \frac{1}{2b}(1-m) + \frac{b}{2}(1-n)\,; \qquad m\,,\, n = 1\,,\,2\,,\,\ldots\,. \tag{3.6}$$

The null state in the representation (3.5) occurs at level mn. The first few cases are:

$$\begin{aligned}
\alpha_{1,1} &= 0\,, & \partial V^{(L)}_{1,1} &= 0\,;\\
\alpha_{1,2} &= -\frac{b}{2}\,, & \big(\partial^2 + b^2\,T^{(L)}\big)V_{-\frac{b}{2}} &= 0\,;\\
\alpha_{2,1} &= -\frac{1}{2b}\,, & \big(\partial^2 + \frac{1}{b^2}\,T^{(L)}\big)V_{-\frac{1}{2b}} &= 0\,.
\end{aligned} \tag{3.7}$$

Here $T^{(L)}$ is the Liouville stress tensor.

The matter theory in $c \le 1$ string theory can be described by taking $b \to ib$ in the above formulae. Thus, one has (see (3.1))

$$c_M = 13 - 6b^2 - \frac{6}{b^2}\,, \qquad c_L + c_M = 26\,, \tag{3.8}$$

and the dimensions of degenerate operators $V^{(M)}_{m,n}$ are (see (3.5))

$$\Delta^{(M)}_{m,n} = -\,\frac{1}{4}\Big(b - \frac{1}{b}\Big)^2 + \frac{1}{4}\Big(\frac{m}{b} - nb\Big)^2. \tag{3.9}$$

In particular, one has

$$\Delta^{(M)}_{m,n} + \Delta^{(L)}_{m,n} = 1 - mn\,. \tag{3.10}$$

The ground ring operators are obtained by applying raising operators of level $(mn-1)$ to $V^{(L)}_{m,n}V^{(M)}_{m,n}$. The first few examples (corresponding to (3.7)) are the following [26, 50]:

$$
\begin{aligned}
\mathcal{O}_{1,1} &= V_{1,1}^{(L)} V_{1,1}^{(M)} = 1\,, \\
\mathcal{O}_{1,2} &= \left|\, cb - \frac{1}{b^2}\big(L_{-1}^{(L)} - L_{-1}^{(M)}\big)\right|^2 V_{1,2}^{(L)} V_{1,2}^{(M)}\,, \\
\mathcal{O}_{2,1} &= \left|\, cb - b^2\big(L_{-1}^{(L)} - L_{-1}^{(M)}\big)\right|^2 V_{2,1}^{(L)} V_{2,1}^{(M)}\,.
\end{aligned}
\tag{3.11}
$$

Note that here b stands for two unrelated quantities: the reparametrization ghost and the parameter introduced in (3.2). Hopefully, it is clear from context which is which.

There are also current operators obtained by tensoring standard holomorphic vertex operators with dimension $(1,0)$ with these antiholomorphic fields of dimension $(0,0)$ to form left moving currents of dimension $(1,0)$ (and similarly right moving currents of dimension $(0,1)$). These satisfy a W_∞ algebra [51, 52]. These are spacetime symmetries which reflect the fact that the dual theory can be written in terms of free fermions.

The operators (3.11) have $\Delta = \bar{\Delta} = 0$ and are in the BRST cohomology of the model. One can show that $\partial_z \mathcal{O}_{m,n}$ and $\partial_{\bar{z}} \mathcal{O}_{m,n}$ are BRST exact. Therefore, any amplitude that involves $\mathcal{O}_{m,n}$ and other BRST invariant operators does not depend on the position of $\mathcal{O}_{m,n}$. Below we will use the freedom to move these operators around when calculating amplitudes.

We will be mostly interested in the case $c_M = 1$ corresponding to $b = 1$. In that case one has

$$
\begin{aligned}
V_{1,2}^{(L)} V_{1,2}^{(M)} &= e^{ix-\phi}\,, \\
V_{2,1}^{(L)} V_{2,1}^{(M)} &= e^{-ix-\phi}\,,
\end{aligned}
\tag{3.12}
$$

and the ground ring generators (3.11) are

$$
\begin{aligned}
\mathcal{O}_{1,2} &= \big(cb + \partial\phi + i\partial x\big)\big(\bar{c}\,\bar{b} + \bar{\partial}\phi + i\bar{\partial}x\big)e^{ix-\phi}\,, \\
\mathcal{O}_{2,1} &= \big(cb + \partial\phi - i\partial x\big)\big(\bar{c}\,\bar{b} + \bar{\partial}\phi - i\bar{\partial}x\big)e^{-ix-\phi}\,.
\end{aligned}
\tag{3.13}
$$

It is argued [26] that for the case of non-compact x, which can be continued to Minkowski time by replacing $x \to it$, the operators $\mathcal{O}_{1,2}$, $\mathcal{O}_{2,1}$ are nothing but the phase space variables of the inverted harmonic oscillator Hamiltonian $H = p^2 - q^2$,

$$
\begin{aligned}
\mathcal{O}_{1,2} &= (q+p)e^{-t}, \\
\mathcal{O}_{2,1} &= (q-p)e^{t}.
\end{aligned}
\tag{3.14}
$$

The quantities on the r.h.s. are constants of motion in the inverted harmonic oscillator potential. Their product is the matrix model Hamiltonian,

$$\mathcal{O}_{1,2}\,\mathcal{O}_{2,1} = q^2 - p^2 = -H\,. \tag{3.15}$$

One expects [26] that in the perturbative string sector of the theory

$$\mathcal{O}_{1,2}\,\mathcal{O}_{2,1} \simeq \mu\,. \tag{3.16}$$

Comparing to (3.15) we see that $-\mu$ is the level of the Fermi sea of the matrix model (measured from the top of the potential), on which the perturbative string excitations live.

Equation (3.16) can be verified directly by using the fact that the Liouville operators that enter $\mathcal{O}_{1,2}$ and $\mathcal{O}_{2,1}$ are degenerate (see (3.7)). For example, the OPE of $V^{(L)}_{1,2} = V_{-\frac{b}{2}}$ with any other V_α has the form

$$V_{-\frac{b}{2}} \cdot V_\alpha = V_{\alpha-\frac{b}{2}} + C_{-}(\alpha) V_{\alpha+\frac{b}{2}} + \dots\ , \tag{3.17}$$

where the "…" stands for Virasoro descendants of the operators on the r.h.s. of (3.17), and [53, 54]

$$C_{-}(\alpha) = -\mu_0\, \frac{\pi\gamma(2b\alpha - 1 - b^2)}{\gamma(-b^2)\,\gamma(2b\alpha)}\ , \tag{3.18}$$

where $\gamma(x){=}\Gamma(x)/\Gamma(1-x)$. Repeated application of $V_{-\frac{b}{2}}$ can take a vertex operator that satisfies the bound $\alpha{<}\frac{Q}{2}$ to one that violates it [55]. One can still use (3.17) in this case, with the understanding that Liouville operators satisfy the reflection property (see *e.g.* Ref. [49])

$$\begin{aligned} &V_\alpha = S(\alpha) V_{Q-\alpha}\ , \\ &S(\alpha) = \big(\pi\mu_0\,\gamma(b^2)\big)^{-(Q-2\alpha)/b}\, \frac{\Gamma\big(\frac{1}{b}(Q-2\alpha)\big)\,\Gamma\big(b(Q-2\alpha)\big)}{\Gamma\big(-\frac{1}{b}(Q-2\alpha)\big)\,\Gamma\big(-b(Q-2\alpha)\big)}\ . \end{aligned} \tag{3.19}$$

The limit $b{\to}1$ (or $c_M{\to}1$) is singular. This can be dealt with as in Ref. [8]. Define

$$\mu = \pi\mu_0\gamma(b^2) \tag{3.20}$$

and hold μ fixed as $b{\to}1$ (and $\mu_0{\to}\infty$). This gives

$$C_{-}(\alpha) = \frac{\mu}{(2\alpha-2)^2(2\alpha-1)^2}\ . \tag{3.21}$$

Using (3.17), one can calculate the OPE $\mathcal{O}_{1,2}\,\mathcal{O}_{2,1}$ (3.13). The first term on the r.h.s. of (3.17) does not contribute. The second term gives

$$\mathcal{O}_{1,2}(z)\mathcal{O}_{2,1}(w) = \Big(1 - \frac{28}{4}\Big)^2 \frac{\mu}{9{\cdot}4} = \mu\,. \tag{3.22}$$

We see that the ground ring generators (3.13) satisfy the relation (3.16), in agreement with their identification with the matrix model objects (3.14).

The tachyon vertex operators form a module under the action of the ground ring. The vertex operator of a tachyon of momentum k is

$$T_k^{(\pm)} = c\bar{c}e^{ikx+(2\mp k)\phi}, \tag{3.23}$$

where the superscript $(\pm)$ refers to the spacetime chirality. For positive (negative) k, only the $-(+)$ signs define good spacetime operators (see the discussion following (3.18)). By using (3.17) with $\alpha = 1 \mp k/2$ one can check that the action of the ground ring on the tachyon modules is

$$\begin{aligned} \mathcal{O}_{1,2}T_k^{(+)} &= k^2\, T_{k+1}^{(+)} , \\ \mathcal{O}_{1,2}T_k^{(-)} &= \frac{\mu}{(k+1)^2}\, T_{k+1}^{(-)} , \\ \mathcal{O}_{2,1}T_k^{(-)} &= k^2\, T_{k-1}^{(-)} , \\ \mathcal{O}_{2,1}T_k^{(+)} &= \frac{\mu}{(k-1)^2}\, T_{k-1}^{(+)} . \end{aligned} \tag{3.24}$$

These equations can be simplified by redefining T_k as

$$\tilde{T}_k^{(\pm)} = \frac{\Gamma(\pm k)}{\Gamma(1\mp k)}\, T_k^{(\pm)} . \tag{3.25}$$

In terms of $\tilde{T}_k$, one finds

$$\begin{aligned} \mathcal{O}_{1,2}\tilde{T}_k^{(+)} &= -\tilde{T}_{k+1}^{(+)} , \\ \mathcal{O}_{1,2}\tilde{T}_k^{(-)} &= -\mu\tilde{T}_{k+1}^{(-)} , \\ \mathcal{O}_{2,1}\tilde{T}_k^{(-)} &= -\tilde{T}_{k-1}^{(-)} , \\ \mathcal{O}_{2,1}\tilde{T}_k^{(+)} &= -\mu\tilde{T}_{k-1}^{(+)} . \end{aligned} \tag{3.26}$$

The relations (3.25), (3.26) were previously derived for 'bulk' correlation functions [27, 28]. We now see that they are exact properties of the full CFT. Note that (3.26) implies (3.22); this is consistent with the fact that the tachyons $T_k^{(\pm)}$ correspond to infinitesimal deformations of the Fermi surface.

These relations lead to certain periodicity properties of the three point functions of tachyons (3.23) in two-dimensional string theory. In four and higher point functions, (3.24) receive corrections due to the presence of integrated vertex operators, or from the spacetime point of view, due to the deformation of the Fermi surface (3.22) caused by the propagation of tachyons on it.

3.2. *The Ground Ring of the 2d Fermionic String*

In the fermionic case it is convenient to return to the conventions $\alpha' = 2$. In this case there are two classes of observables, corresponding to Neveu-Schwarz and Ramond vertex operators

$$\begin{aligned} N_\alpha &= e^{\alpha\phi} , \\ R_\alpha &= \sigma e^{\alpha\phi} , \end{aligned} \tag{3.27}$$

where σ is a spin field. Correspondingly, there are two kinds of degenerate operators: the NS sector operators

$$N_{m,n} = e^{\alpha_{m,n}\phi} , \tag{3.28}$$

with $\alpha_{m,n}$ given again by (3.6), $n, m = 1, 2, 3, \ldots$, and $(m-n) \in 2Z$ (note that $n=m=1$ is the identity operator). We also have Ramond sector operators

$$R_{m,n} = \sigma e^{\alpha_{m,n}\phi} , \tag{3.29}$$

with $(m-n) \in 2Z+1$. The natural analog of the operators $V_{-\frac{b}{2}}$, $V_{-\frac{1}{2b}}$ (see (3.7)) in this case are the Ramond operators

$$\begin{aligned} R_{1,2} &= \sigma e^{-\frac{b}{2}\phi} , \\ R_{2,1} &= \sigma e^{-\frac{1}{2b}\phi} , \end{aligned} \tag{3.30}$$

which satisfy the (level one) null state conditions

$$\begin{aligned} \big(L_{-1} + b^2 G_{-1} G_0\big) R_{1,2} &= 0 , \\ \big(L_{-1} + \frac{1}{b^2} G_{-1} G_0\big) R_{1,2} &= 0 . \end{aligned} \tag{3.31}$$

There are similar degenerate operators in the matter theory obtained by taking $b \to ib$. As in the bosonic case, these degenerate operators lead to nontrivial BRST cohomology at ghost number zero. For $\hat{c}_M = 1$ (which again corresponds to $b \to 1$) these can be written as [56]

$$\begin{aligned} \mathcal{O}_{1,2} &= \left(e^{-\frac{\varphi}{2} + \frac{i}{2}H} - \frac{c}{\sqrt{2}}\,\partial\xi e^{-\frac{3\varphi}{2} - \frac{i}{2}H} \right) \left(e^{-\frac{\bar\varphi}{2} + \frac{i}{2}\bar H} - \frac{\bar c}{\sqrt{2}}\,\bar\partial\bar\xi e^{-\frac{3\bar\varphi}{2} - \frac{i}{2}\bar H} \right) e^{\frac{i}{2}x - \frac{1}{2}\phi} , \\ \mathcal{O}_{2,1} &= \left(e^{-\frac{\varphi}{2} - \frac{i}{2}H} - \frac{c}{\sqrt{2}}\,\partial\xi e^{-\frac{3\varphi}{2} + \frac{i}{2}H} \right) \left(e^{-\frac{\bar\varphi}{2} - \frac{i}{2}\bar H} - \frac{\bar c}{\sqrt{2}}\,\bar\partial\bar\xi e^{-\frac{3\bar\varphi}{2} + \frac{i}{2}\bar H} \right) e^{-\frac{i}{2}x - \frac{1}{2}\phi} , \end{aligned} \tag{3.32}$$

where φ is the bosonized superconformal ghost and H bosonizes the world-sheet fermions ψ, χ as in (2.19).

A few comments are in order here.

(1) The operators $\mathcal{O}_{1,2}$, $\mathcal{O}_{2,1}$ are present in the type 0B theory, where they generate the ground ring. These operators are projected out in the type 0A theory. There, the ring is generated by the lowest NS sector operators $\mathcal{O}_{1,3}$, $\mathcal{O}_{3,1}$, and $\mathcal{O}_{2,2}$.
(2) As in the bosonic string, one can also form left moving and right moving currents [57–59] which satisfy an interesting current algebra.
(3) Upon compactification, one again finds a structure similar to the bosonic one. As mentioned in Section 2, the most symmetric point in moduli space is the 0B super affine theory at the self-dual radius $R = 2$. In this theory, we can define the (anti-)chiral vertex operators

$$\begin{aligned}
\mathbf{x} &= \left(e^{-\frac{\varphi}{2}+\frac{i}{2}H} - \frac{c}{\sqrt{2}}\,\partial\xi e^{-\frac{3\varphi}{2}-\frac{i}{2}H} \right) e^{\frac{i}{2}x_L-\frac{1}{2}\phi_L} \,, \\
\mathbf{y} &= \left(e^{-\frac{\varphi}{2}-\frac{i}{2}H} - \frac{c}{\sqrt{2}}\,\partial\xi e^{-\frac{3\varphi}{2}+\frac{i}{2}H} \right) e^{-\frac{i}{2}x_L-\frac{1}{2}\phi_L} \,, \\
\bar{\mathbf{x}} &= \left(e^{-\frac{\bar\varphi}{2}+\frac{i}{2}\bar H} - \frac{\bar c}{\sqrt{2}}\,\bar\partial\bar\xi e^{-\frac{3\bar\varphi}{2}-\frac{i}{2}\bar H} \right) e^{\frac{i}{2}x_R-\frac{1}{2}\phi_R} \,, \\
\bar{\mathbf{y}} &= \left(e^{-\frac{\bar\varphi}{2}-\frac{i}{2}\bar H} - \frac{\bar c}{\sqrt{2}}\,\bar\partial\bar\xi e^{-\frac{3\bar\varphi}{2}+\frac{i}{2}\bar H} \right) e^{-\frac{i}{2}x_R-\frac{1}{2}\phi_R} \,,
\end{aligned} \tag{3.33}$$

and the ground ring is generated by the four operators $a_1 = \mathbf{x}\bar{\mathbf{x}}$, $a_2 = \mathbf{y}\bar{\mathbf{y}}$, $a_3 = \mathbf{x}\bar{\mathbf{y}}$, and $a_4 = \bar{\mathbf{x}}\mathbf{y}$, all of which are in the spectrum. These operators are subject to the relation $a_1a_2 - a_3a_4 = const$. The rich mathematical structure associated to this ring was explored in Ref. [26].

In order to repeat the discussion of the bosonic case, we need the analog of (3.17) for this case [60, 61]:

$$\begin{aligned}
R_{1,2}N_\alpha &= R_{\alpha-\frac{b}{2}} + C_-^{(N)}(\alpha)\, R_{\alpha+\frac{b}{2}} \,, \\
R_{1,2}R_\alpha &= N_{\alpha-\frac{b}{2}} + C_-^{(R)}(\alpha)\, N_{\alpha+\frac{b}{2}} \,,
\end{aligned} \tag{3.34}$$

where

$$\begin{aligned}
C_-^{(N)}(\alpha) &= \frac{\mu_0 b^2 \gamma\big(\alpha b - \frac{1}{2}b^2 - \frac{1}{2}\big)}{4\gamma\big(\frac{1-b^2}{2}\big)\,\gamma\big(\alpha b\big)} \,, \\
C_-^{(R)}(\alpha) &= \frac{\mu_0 b^2 \gamma\big(\alpha b - \frac{1}{2}b^2\big)}{4\gamma\big(\frac{1-b^2}{2}\big)\,\gamma\big(\alpha b + \frac{1}{2}\big)} \,.
\end{aligned} \tag{3.35}$$

In the limit $b\to 1$, and keeping

$$\mu = \frac{1}{4}\,\mu_0\,\gamma\Big(\frac{1+b^2}{2}\Big) \tag{3.36}$$

fixed, one has

$$\begin{aligned} C_-^{(N)}(\alpha) &= -\frac{\mu}{(\alpha-1)^2}\,, \\ C_-^{(R)}(\alpha) &= -\mu\Big(\alpha-\frac{1}{2}\Big)^2. \end{aligned} \tag{3.37}$$

We would next like to compute the OPE of $\mathcal{O}_{1,2}$ and $\mathcal{O}_{2,1}$ using the Liouville results (3.34), (3.37). As in the bosonic case, only the term proportional to C_- on the second line of (3.34) contributes. Moreover, it is not difficult to see that in multiplying the two lines of (3.32), only the cross-terms are non-zero

$$\mathcal{O}_{1,2}\,\mathcal{O}_{2,1} = 2\cdot\frac{c\,\bar c}{2}\,\partial\xi\bar\partial\bar\xi\,e^{-\frac{3}{2}(\varphi+\bar\varphi)-\frac{i}{2}(H+\bar H)+\frac{i}{2}x-\frac{\phi}{2}}\cdot e^{-\frac{1}{2}(\varphi+\bar\varphi)-\frac{i}{2}(H+\bar H)-\frac{i}{2}x-\frac{\phi}{2}}\,. \tag{3.38}$$

By using the second line of (3.34) with (3.37), $C_-^{(R)}(-1/2) = -\mu$, one finds

$$\mathcal{O}_{1,2}\,\mathcal{O}_{2,1} = -\mu e^{-2(\varphi+\bar\varphi)}c\,\bar c\,\partial\xi\bar\partial\bar\xi\,. \tag{3.39}$$

The r.h.s. of (3.39) is the identity operator in disguise. More precisely, by applying picture changing, using the term[a]

$$Q_{\rm BRST} = -\oint\frac{dz}{2\pi i}\,\gamma^2 b+\ldots \tag{3.40}$$

in the BRST charge, one finds that (3.39) is equivalent to

$$\mathcal{O}_{1,2}\,\mathcal{O}_{2,1} = \mu\,. \tag{3.41}$$

We next move on to the action of the ground ring generators on the two massless scalar fields of two-dimensional (0B) string theory. Applying the ring generators $\mathcal{O}_{1,2}$ and $\mathcal{O}_{2,1}$ from Eq. (3.32) to the R-R vertex operator $V_k^{(\pm)}$ of

[a] We use the conventions of [38].

Eq. (2.19), one finds (using (2.18), the second line of (3.34), and (3.37))

$$\begin{aligned}
\mathcal{O}_{1,2}\cdot V_k^{(+)} &= T^{(+)}_{k+\frac{1}{2}} \ ,\\
\mathcal{O}_{2,1}\cdot V_k^{(+)} &= -\frac{\mu}{\left(k-\frac{1}{2}\right)^2}\, T^{(+)}_{k-\frac{1}{2}} \ ,\\
\mathcal{O}_{1,2}\cdot V_k^{(-)} &= -\frac{\mu}{\left(k+\frac{1}{2}\right)^2}\, T^{(-)}_{k+\frac{1}{2}} \ ,\\
\mathcal{O}_{2,1}\cdot V_k^{(-)} &= T^{(-)}_{k-\frac{1}{2}} \ .
\end{aligned} \tag{3.42}$$

The action of $\mathcal{O}_{1,2}$ and $\mathcal{O}_{2,1}$ on $T^{(\pm)}$ can be deduced from (3.41) and (3.42)

$$\begin{aligned}
\mathcal{O}_{1,2}\cdot T_k^{(+)} &= -k^2 V^{(+)}_{k+\frac{1}{2}} \ ,\\
\mathcal{O}_{2,1}\cdot T_k^{(+)} &= \mu V^{(+)}_{k-\frac{1}{2}} \ ,\\
\mathcal{O}_{1,2}\cdot T_k^{(-)} &= \mu V^{(-)}_{k+\frac{1}{2}} \ ,\\
\mathcal{O}_{2,1}\cdot T_k^{(-)} &= -k^2 V^{(-)}_{k-\frac{1}{2}} \ .
\end{aligned} \tag{3.43}$$

One can diagonalize the action (3.42), (3.43) by redefining

$$\begin{aligned}
\tilde{T}_k^{(\pm)} &= \frac{\Gamma(\pm k)}{\Gamma(1\mp k)}\, T_k^{(\pm)} \ ,\\
\tilde{V}_k^{(\pm)} &= \frac{\Gamma(\frac{1}{2}\pm k)}{\Gamma(\frac{1}{2}\mp k)}\, V_k^{(\pm)} \ ,
\end{aligned} \tag{3.44}$$

and changing variables to

$$\begin{aligned}
T_R^{(\pm)}(k) &= \frac{1}{2}\left(\tilde{T}_k^{(\pm)}+\tilde{V}_k^{(\pm)}\right) \ ,\\
T_L^{(\pm)}(k) &= \frac{1}{2}\left(\tilde{T}_k^{(\pm)}-\tilde{V}_k^{(\pm)}\right) \ .
\end{aligned} \tag{3.45}$$

In terms of these variables, the action (3.42), (3.43) breaks up into two copies of the bosonic one. That is, one has

$$\begin{aligned}
\mathcal{O}_{1,2}\cdot T_R^{(+)}(k) &= T_R^{(+)}\left(k+\frac{1}{2}\right) ,\\
\mathcal{O}_{1,2}\cdot T_L^{(+)}(k) &= -T_L^{(+)}\left(k+\frac{1}{2}\right) ,\\
\mathcal{O}_{2,1}\cdot T_R^{(-)}(k) &= T_R^{(-)}\left(k-\frac{1}{2}\right) ,\\
\mathcal{O}_{2,1}\cdot T_L^{(-)}(k) &= -T_L^{(-)}\left(k-\frac{1}{2}\right) ;
\end{aligned} \tag{3.46}$$

these relations, together with (3.41), specify the ring action completely. We will see later that T_L and T_R are excitations living on the two sides of the inverted harmonic oscillator potential of the matrix model. Equation (3.46) indicates the side of the potential algebraically by the sign of $\mathcal{O}_{1,2}$, $\mathcal{O}_{2,1}$.

One can further show [62] that the scattering amplitudes of T_L and T_R factorize, at least at tree level. One has

$$\begin{gathered}\Big\langle \prod_{i=1}^{n} T_L(k_i) \prod_{j=1}^{m} T_R(p_j) \Big\rangle = 0 \quad (n\,,\, m \geq 1)\,, \\ \Big\langle \prod_{i=1}^{n} T_L(k_i) \Big\rangle = \Big\langle \prod_{i=1}^{n} T_R(k_i) \Big\rangle = \frac{1}{4} \Big\langle \prod_{i=1}^{n} \tilde{T}^{(B)}(\sqrt{2}k_i) \Big\rangle_B\,,\end{gathered} \tag{3.47}$$

where $\tilde{T}^{(B)}$ is the bosonic string vertex operator (3.25), and the correlator $\langle \cdots \rangle_B$ is computed in bosonic string theory.

The first line of (3.47) follows from the action of the ground ring on the tachyons T_L , T_R . One can show, using similar methods to those in Ref. [28], that bulk amplitudes satisfy the selection rule on the first line of (3.47), and then extend the result to non-bulk amplitudes as in Ref. [62]. The second line of (3.47) is the statement that correlators of excitations living on a given side of the potential are the same as in the bosonic string, up to the usual rescaling of α' by a factor of two. The overall factor of 1/4 can be thought of as due to a rescaling by a factor of two of g_s between the bosonic and fermionic string.

4. The Torus Partition Function

The one-loop string path integral provides a wealth of information about the theory. We will consider, following [63, 64], the compactified theory with X living on a circle of radius R. As discussed in Section 2, the compactification of the scalar superfield $X(z, \bar{z}, \theta, \bar{\theta})$ can be done in two ways. In the circle theory one simply sums over all windings and momenta in each spin structure, while in the superaffine theory [46, 47] one correlates the windings and momenta with the fermion boundary conditions.

We begin however with a review of the bosonic theory.[a]

4.1. *Liouville on the Torus*

In the torus path integral of the 2d bosonic string, the oscillator contributions cancel among Liouville, matter, and ghosts, leaving a zero mode sum and

[a] In this section we work with the Lagrangian $\frac{1}{4\pi\alpha'}(dx)^2 + \frac{1}{8\pi}(d\phi)^2 + \cdots$, i.e. we make the scale of x explicit, while fixing the scale of ϕ. In the matrix model the scale of ϕ is not visible and the curvature of the potential sets the scale of the x-field.

an integral over the torus moduli

$$\frac{\mathcal{Z}_1\big(R/\sqrt{\alpha'}\big)}{V_L} = \frac{R}{\sqrt{\alpha'}}\frac{1}{4\pi\sqrt{2}}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\sum_{n,m}\exp\left(-\frac{\pi R^2|n-m\tau|^2}{\alpha'\tau_2}\right); \tag{4.1}$$

here $\mathcal{F}$ is the fundamental domain for the torus modular parameter, and $V_L = -(\ln\mu)/\sqrt{2}$ is the volume of the zero mode of the Liouville field.

The integral in (4.1) may be evaluated by trading the summation over (n,m) for $(n,m)\neq 0$ for extending the integration region from $\mathcal{F}$ to the entire strip $\frac{1}{2}\leq\tau_1<\frac{1}{2}$ and summing only over $(n,m)=(n,0)$, $n>0$ [65],

$$\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\sum_{n,m}\exp\left(-\frac{\pi R^2|n-m\tau|^2}{\alpha'\tau_2}\right) = \int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2} + 2\sum_{n=1}^{\infty}\frac{d\tau_2}{\tau_2^2}\exp\left(-\frac{\pi R^2 n^2}{\alpha'\tau_2}\right).$$

Evaluating these integrals we get [66]

$$\mathcal{Z}_1\left(\frac{R}{\sqrt{\alpha'}}\right) = -\frac{1}{24}\ln\mu\left(\frac{R}{\sqrt{\alpha'}}+\frac{\sqrt{\alpha'}}{R}\right). \tag{4.2}$$

Another useful, though somewhat formal, way of evaluating this integral is to first perform a Poisson resummation

$$\frac{R}{\sqrt{\alpha'}}\sum_{n,m}\exp\left(-\frac{\pi R^2|n-m\tau|^2}{\alpha'\tau_2}\right) = \sqrt{\tau_2}\sum_{s,\,t\in Z} q^{\left(s\sqrt{\alpha'}/R+tR/\sqrt{\alpha'}\right)^2/4} \times \bar{q}^{\left(s\sqrt{\alpha'}/R-tR/\sqrt{\alpha'}\right)^2/4}.$$

Now let us formally extend the integration region to the entire strip [64]. Then we find

$$\frac{\mathcal{Z}_1\big(R/\sqrt{\alpha'}\big)}{V_L} = \frac{1}{4\pi\sqrt{2}}\sum_s\int_0^\infty\frac{d\tau_2}{\tau_2^{3/2}}\,e^{-\pi\tau_2 s^2\alpha'/R^2} + \left(R\to\frac{\alpha'}{R}\right). \tag{4.3}$$

Since

$$-\frac{1}{4\pi\sqrt{2}}\int_0^\infty\frac{d\tau_2}{\tau_2^{3/2}}\,e^{-2\pi\tau_2\omega^2}$$

is a proper time representation for the quantum mechanical zero-point energy $\omega/2$, we find

$$\frac{\mathcal{Z}_1\big(R/\sqrt{\alpha'}\big)}{V_L} = -\left(\frac{R}{\sqrt{\alpha'}}+\frac{\sqrt{\alpha'}}{R}\right)\frac{1}{\sqrt{2}}\sum_{s=1}^{\infty} s\ .$$

After using the standard zeta-function regularization

$$\sum_{s=1}^{\infty} s = -\frac{1}{12} ,$$

we find that the sum over toroidal surfaces is again (4.2) [1, 66]. Although this approach is more formal, it suggests that the terms proportional to $1/R$ and R arise from the momentum and winding modes, respectively.

The factor $\ln\mu$ in $\mathcal{Z}_1$ should be interpreted as $\ln(\mu/\Lambda)$ with $\Lambda \gg \mu$, a cutoff at large negative ϕ. The effective length of the Liouville direction is $\frac{1}{\sqrt{2}}\ln(\Lambda/\mu)$. Λ is a UV cutoff on the worldsheet and an IR cutoff in spacetime. In the equations below we will suppress an additive (infinite) constant proportional to $\ln\Lambda$.

4.2. *Super-Liouville on the Torus: the Uncorrelated GSO Projection*

First consider the standard type 0 non-chiral projection (the 'circle theory'), with no correlation between the fermionic boundary conditions and the soliton winding numbers. Let (r, s) label the spin structures as $(e^{i\pi r}, e^{i\pi s})$. The three even spin structures have $(r, s) = (0, 1), (1, 0), (1, 1)$. Let $\mathcal{D}_{r,s}$ be the corresponding fermionic determinants divided by the square root of the scalar determinant. Then the partition function in the (r, s) sector is

$$\tilde{Z}^{(S)}_{r,s}\big(R/\sqrt{\alpha'}, \tau, \bar{\tau}\big) = \frac{R}{\sqrt{\alpha'}} \frac{1}{\sqrt{\tau_2}} |\mathcal{D}_{r,s}|^2 \sum_{m,n\in Z} e^{-S_{m,n}} . \tag{4.4}$$

In coupling such a matter system to supergravity we have to multiply Eq. (4.4) by the path integrals over the super-Liouville field and over the superghost field. The contribution of the super-Liouville theory is equal to $\big(V_L/2\pi\sqrt{2\tau_2}\big)|\mathcal{D}_{r,s}|^2$, where $V_L = -\ln|\mu|$; the contribution of the superghost sector equals $(1/2\tau_2)|\mathcal{D}_{r,s}|^{-4}$. One can see that the determinants due to all the excitations again cancel out, and only the zero modes contribute to the full partition function. Coupling this system to supergravity and counting each spin structure with factor $1/2$, we find that the contribution of the even spin structures to the genus one amplitude is

$$\begin{aligned} \frac{\tilde{\mathcal{Z}}_{\text{even}}\left(R/\sqrt{\alpha'}\right)}{V_L} &= \frac{1}{2} \frac{R}{\sqrt{\alpha'}} \frac{1}{4\pi\sqrt{2}} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \left(\sum_{(r,s)} \sum_{m,n\in Z} e^{-S_{m,n}} \right) \\ &= \frac{3}{2} f\big(R/\sqrt{\alpha'}\big) , \end{aligned} \tag{4.5}$$

where

$$f(x) = \frac{1}{12\sqrt{2}}\left(x + \frac{1}{x}\right). \tag{4.6}$$

The peculiar factor of 3/2 is due to the fact that we have included 3 even spin structures. This is inconsistent because in the type 0B string there are twice as many fields as in the bosonic string. The coefficient of $1/R$ is related to the $\sim T^2$ term in the thermal free energy, which can be calculated in 2d free field theory and does not require a stringy regularization. Therefore, in type 0B theory this coefficient should be double that of the bosonic string. Indeed, the odd spin structure contributes in this string theory, since the zero modes of the fermions are cancelled by those of the ghosts. As shown in Appendix A, its contribution is

$$\frac{\tilde{\mathcal{Z}}_{\rm odd}\big(R/\sqrt{\alpha'}\big)}{V_L} = \pm\frac{1}{24\sqrt{2}}\left(\frac{R}{\sqrt{\alpha'}} - \frac{\sqrt{\alpha'}}{R}\right). \tag{4.7}$$

The sign of (4.7) is chosen such that the coefficient of $1/R$ is that of two massless free fields in the type 0B theory, and one such field for type 0A. This leads to the result

$$\begin{aligned} \mathcal{Z}_B &= -\frac{\ln|\mu|}{12\sqrt{2}}\left(\frac{R}{\sqrt{\alpha'}} + 2\,\frac{\sqrt{\alpha'}}{R}\right), \\ \mathcal{Z}_A &= -\frac{\ln|\mu|}{12\sqrt{2}}\left(2\,\frac{R}{\sqrt{\alpha'}} + \frac{\sqrt{\alpha'}}{R}\right). \end{aligned} \tag{4.8}$$

Note that T-duality $R \to \alpha'/R$ properly interchanges the two models.

These results can also be derived via zeta-function regularization from the spectra of 0B and 0A circle theories discussed in Section 2. For example, in the 0B case, we may use this logic to derive

$$\frac{\mathcal{Z}_B\big(R/\sqrt{\alpha'}\big)}{V_L} = -\left(\frac{R}{\sqrt{\alpha'}} + 2\,\frac{\sqrt{\alpha'}}{R}\right)\frac{1}{\sqrt{2}}\sum_{s=1}^{\infty} s\,,$$

which agrees with (4.8). The factor of 2 in the second term is due to the doubling of momentum states.

4.3. *Correlated GSO Projection*

Now consider the super affine theory. Here certain instanton sectors are weighted with the opposite sign relative to others [46]

$$\begin{aligned} Z^{(S)}_{r,s}(\tau,\bar\tau) &= \frac{\mathcal{R}}{\sqrt{\alpha'\tau_2}}\,|\mathcal{D}_{r,s}|^2\left(\sum_{m,n\in Z} e^{-S_{m,n}} - 2\sum_{i,j\in Z} e^{-S_{2i+r,2j+s}}\right), \\ S_{m,n} &= \frac{\pi\mathcal{R}^2}{\alpha'\tau_2}\,|n-m\tau|^2\,. \end{aligned} \tag{4.9}$$

Counting each even spin structure with a factor of $1/2$, we get after coupling to supergravity

$$\begin{aligned} \frac{\mathcal{Z}_{\rm even}}{V_L} &= \frac{1}{2}\,\frac{\mathcal{R}}{\sqrt{\alpha'}}\,\frac{1}{4\pi\sqrt{2}}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\left(\sum_{m,n\in Z} e^{-S_{m,n}} + 2\sum_{i,j\in Z} e^{-S_{2i,2j}}\right) \\ &= \frac{1}{2}\,f\big(\mathcal{R}/\sqrt{\alpha'}\big) + \frac{1}{2}\,f\big(2\mathcal{R}/\sqrt{\alpha'}\big)\,. \end{aligned} \tag{4.10}$$

Therefore, for the super affine theory we find

$$\mathcal{Z}_{\rm even} = -\frac{\ln|\mu|}{8\sqrt{2}}\left(\frac{\mathcal{R}}{\sqrt{\alpha'}} + \frac{\sqrt{\alpha'}}{2\mathcal{R}}\right). \tag{4.11}$$

This theory is a Z_2 orbifold of the usual type 0 theory; and in terms of the covering space radius $R = 2\mathcal{R}$,

$$\mathcal{Z}_{\rm even} = -\frac{\ln|\mu|}{16}\left(\frac{R}{\sqrt{2\alpha'}} + \frac{\sqrt{2\alpha'}}{R}\right). \tag{4.12}$$

The duality $R \to 2\alpha'/R$ is the T-duality of the super affine theory discussed in Section 2.

We also need to include the contribution of the odd spin structure. Consider first the super affine theory before coupling to supergravity. The contributions of the NS-NS and NS-NS $(-1)^{F+\tilde F}$ spin structures to the partition function [46] are[b]

$$\begin{aligned} \frac{1}{2\,|\eta(q)|^2}&\left[\,|\chi_0+\chi_{1/2}|^2\left(\sum_{E,M\ \rm even} + \sum_{E,M\ \rm odd}\right) + |\chi_0-\chi_{1/2}|^2\left(\sum_{E,M\ \rm even} - \sum_{E,M\ \rm odd}\right)\right] \\ &\times q^{\left(E\sqrt{2\alpha'}/R+MR/\sqrt{2\alpha'}\right)^2/8}\,\bar q^{\left(E\sqrt{2\alpha'}/R-MR/\sqrt{2\alpha'}\right)^2/8}\,, \end{aligned} \tag{4.13}$$

[b] Note that the momentum number is E, but the winding number is $M/2$.

while the R-R spin structure gives

$$\frac{1}{|\eta(q)|^2}\,|\chi_{1/16}|^2\Bigg(\sum_{\substack{M\ \text{odd}\\ E\ \text{even}}}+\sum_{\substack{M\ \text{even}\\ E\ \text{odd}}}\Bigg)q^{(E\sqrt{2\alpha'}/R+MR/\sqrt{2\alpha'})^2/8}\,\bar{q}^{\,(E\sqrt{2\alpha'}/R-MR/\sqrt{2\alpha'})^2/8}. \tag{4.14}$$

Note that each of these partition functions is explicitly symmetric under the T-duality $R\to 2\alpha'/R$ which interchanges E and M. Here χ_0 , $\chi_{1/2}$, and $\chi_{1/16}$ are the characters of the $c = 1/2$ Majorana fermion theory. In each spin structure they cancel after we include the Liouville and ghost factors. Thus, the sum over even spin structures gives

$$\begin{aligned}\frac{\mathcal{Z}_{\text{even}}}{V_L} = \frac{1}{4\pi\sqrt{2}}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^{3/2}}\Bigg(\sum_{E,M\ \text{even}}+\frac{1}{2}\sum_{\substack{E\ \text{even}\\ M\ \text{odd}}}+\frac{1}{2}\sum_{\substack{M\ \text{even}\\ E\ \text{odd}}}\Bigg)\\ \times q^{(E\sqrt{2\alpha'}/R+MR/\sqrt{2\alpha'})^2/8}\,\bar{q}^{\,(E\sqrt{2\alpha'}/R-MR/\sqrt{2\alpha'})^2/8}.\end{aligned} \tag{4.15}$$

We see that in the NS-NS sector each state with even E, M enters with correct normalization, while in the R-R sector there are incorrect factors of $1/2$.

Naively, to correct this the contribution of the odd spin structure has to be

$$\begin{aligned}\pm\frac{1}{4\pi\sqrt{2}}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^{3/2}}\Bigg(\frac{1}{2}\sum_{\substack{E\ \text{even}\\ M\ \text{odd}}}-\frac{1}{2}\sum_{\substack{M\ \text{even}\\ E\ \text{odd}}}\Bigg)\\ \times q^{(E\sqrt{2\alpha'}/R+MR/\sqrt{2\alpha'})^2/8}\,\bar{q}^{\,(E\sqrt{2\alpha'}/R-MR/\sqrt{2\alpha'})^2/8}.\end{aligned} \tag{4.16}$$

An interesting property of this expression is that, after Poisson resummation, it is equal to

$$\pm\frac{1}{8\pi\sqrt{2}}\frac{\mathcal{R}}{\sqrt{\alpha'}}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\Bigg(\sum_{m,n\in Z}e^{-S_{m,n}}-2\sum_{i,j\in Z}e^{-S_{2i,2j}}\Bigg), \tag{4.17}$$

which establishes its modular invariance. This is precisely the contribution of the $(r, s) = (0, 0)$ sector in (4.9) after we strip off the fermionic determinant. This expression is simply the sum over solitons with phases, and evaluating the integral in (4.17), we find

$$\pm\frac{1}{2}\Big(f\big(\mathcal{R}/\sqrt{\alpha'}\big)-f\big(2\mathcal{R}/\sqrt{\alpha'}\big)\Big)=\pm\frac{1}{48}\left(-\frac{R}{\sqrt{2\alpha'}}+\frac{\sqrt{2\alpha'}}{R}\right). \tag{4.18}$$

As shown in Appendix A, this result is actually the one-point function of the dilaton operator, which is $R^2\frac{\partial}{\partial R^2}$ acting on the contribution of the odd spin structure. Thus,

$$\frac{\mathcal{Z}_{\rm odd}}{V_L} \sim \pm\frac{1}{48}\left(\frac{R}{\sqrt{2\alpha'}}+\frac{\sqrt{2\alpha'}}{R}\right). \tag{4.19}$$

Using (4.10) and (4.19), we find the torus amplitudes in the two theories:

$$\mathcal{Z}_{\rm A}^{\text{super aff.}} = -\frac{\ln|\mu|}{12}\left(\frac{R}{\sqrt{2\alpha'}}+\frac{\sqrt{2\alpha'}}{R}\right), \tag{4.20}$$

$$\mathcal{Z}_{\rm B}^{\text{super aff.}} = -\frac{\ln|\mu|}{24}\left(\frac{R}{\sqrt{2\alpha'}}+\frac{\sqrt{2\alpha'}}{R}\right). \tag{4.21}$$

These results can again be derived via zeta-function regularization, using the spectra discussed in Section 2 . For example, the complete physical spectrum of the super affine A theory contains integer windings and even momenta. This explains (4.20); in particular, the sum over even integers is $-1/6$, while the sum over all integers is $-1/12$, which explains the relative factor of 2 . Similarly, (4.21) follows from the spectrum of super affine B theory via zeta-function regularization.

Note that the answer in the type B theory is twice smaller than in the type A theory, even though it contains physical R-R modes. The reason is that these modes are odd-integer valued, and in the zeta-function regularization, the sum over odd integers is $1/12$ while the sum over all integers is $-1/12$. Thus, in the model with physical R-R modes the partition function is smaller.

The fact that the type B affine theory is half of the circle answer can also be understood in the following way. As we discussed before, the term proportional to $1/R$ in the torus partition function arises from the thermal free energy of the fields we have in the bulk spacetime. In the affine B theory we have two massless scalar fields, but one of them has odd boundary conditions as we go around the thermal circle. This thermal partition function can be easily computed and indeed it matches with the $1/R$ term in (4.21). The term proportional to R can be obtained by T duality.

5. Semiclassical Picture of D-branes in Super-Liouville

Having discussed the perturbative structure of 2d fermionic string theory, we now turn to the structure of its D-branes. We begin with a general discussion of boundary conditions in Lagrangian field theory with $N=1$ worldsheet supersymmetry.

The Lagrangian density is a superfield $\mathcal{L}$ which has a component expansion as in (2.10). The action is $\int d^2z d^2\theta \mathcal{L}$. The supersymmetry variation of a superfield Φ is given by its commutator with $\zeta Q+\bar{\zeta}\bar{Q}$ (Eq. (2.11)), yielding the transformation laws

$$\begin{aligned}
\delta\phi &= i\zeta\psi + i\bar{\zeta}\bar{\psi}\,,\\
\delta\psi &= -i\zeta\partial\phi + \bar{\zeta}F\,,\\
\delta\bar{\psi} &= -\zeta F - i\bar{\zeta}\bar{\partial}\phi\,,\\
\delta F &= -\zeta\partial\bar{\psi} + \bar{\zeta}\bar{\partial}\psi\,.
\end{aligned} \tag{5.1}$$

The supersymmetry variation of $\int d^2\theta\mathcal{L}$ is then a total derivative

$$\delta\int d^2\theta\mathcal{L} = -\zeta\partial\mathcal{L}\big|_{\bar{\theta}} + \bar{\zeta}\bar{\partial}\mathcal{L}\big|_{\theta}\,. \tag{5.2}$$

If the worldsheet has a boundary at $z=\bar{z}$, the surface term from (5.2) is

$$-i\Big(\zeta\mathcal{L}\big|_{\bar{\theta}} + \bar{\zeta}\mathcal{L}\big|_{\theta}\Big)\Big|_{y=0}\,. \tag{5.3}$$

If we add to the action an integral along the boundary

$$i\eta\oint_{y=0} dx\mathcal{L}(\theta=\bar{\theta}=0)\,, \qquad \eta=\pm 1\,, \tag{5.4}$$

then its variation $i\eta\big(\zeta\mathcal{L}\big|_{\theta} + \bar{\zeta}\mathcal{L}\big|_{\bar{\theta}}\big)$ cancels (5.3) for $\zeta=\eta\bar{\zeta}$; i.e. only $Q+\eta\bar{Q}$ is preserved. It is important that without the boundary interaction (5.4) no supersymmetry is preserved.

In the presence of the boundary, the preserved super-translation operator is

$$D_t = D + \eta\bar{D}d\,, \qquad D_t^2 = \partial_x\,; \tag{5.5}$$

the conjugate coordinate is $\theta_t = \frac{1}{2}(\theta+\eta\bar{\theta})$. Similarly, $D_n = D-\eta\bar{D}$ is the superderivative in the normal direction.

We should make some comments about the parameter η. In the type II theory we gauge the worldsheet Z_2 R-symmetry under which $Q\to -Q$ and $\bar{Q}$ is invariant. This is a symmetry under which the Lagrangian $\mathcal{L}$ and θ are odd, but $\bar{\theta}$ is even and therefore $\int d^2\theta\mathcal{L}$ is invariant. Since this symmetry is gauged, we sum over worldsheets obtained by the action of this group. This sum is included in the sum over spin structures. The parameter η is odd under the symmetry, and therefore, whenever there is a boundary, we must sum over worldsheets with different values of η. In the type 0 theory this symmetry is not gauged, and it might not even be a global symmetry on the worldsheet. Therefore, we do not sum over η and different values of η correspond to distinct D-branes. If it is a worldsheet global symmetry, as in the flat ten-dimensional background, the D-branes with $\eta=\pm 1$ are related by a local Z_2 spacetime symmetry. In other cases, where there is no such symmetry, the two values of η correspond to two different branes. It might happen that one of them is infinitely heavy and then should be thought of as absent.

Now let us turn to the specific case of super-Liouville theory. The Lagrangian density in superspace, $\mathcal{L}$, is given by equation (2.13). The Liouville interaction in (2.13) breaks the Z_2 chiral R-symmetry mentioned above. Therefore, the sign of the real constant μ_0 can be changed by a field redefinition, but the sign of $\eta\mu_0$ cannot be changed. In order not to clutter the equations, we will let μ_0 be positive.

The action is

$$\begin{aligned} S &= \int d^2z\, d^2\theta \mathcal{L} + i\eta \oint_{y=0} dx\, \mathcal{L}(\theta=\bar\theta=0) \\ &= \frac{1}{4\pi}\int d^2z \left(\partial\phi\bar\partial\phi + \psi\bar\partial\psi + \bar\psi\partial\bar\psi - F^2 - 2\mu_0 b e^{b\phi}F + 2i\mu_0 b^2 e^{b\phi}\psi\bar\psi\right) \\ &\qquad - \frac{\eta}{2\pi}\oint_{y=0} dx \left(\frac{i}{2}\,\psi\bar\psi + \mu_0 e^{b\phi}\right). \end{aligned} \tag{5.6}$$

It is interesting that the theory must have a boundary cosmological constant $\rho = -\mu_0\eta/2\pi$ [42, 44]. The variation of the fields leads to the bulk equations of motion

$$\begin{aligned} \partial\bar\partial\phi &= \mu_0^2 b^3 e^{2b\phi} + i\mu_0 b^3 e^{b\phi}\psi\bar\psi\,, \\ \bar\partial\psi &= -i\mu_0 b^2 e^{b\phi}\bar\psi\,, \\ \partial\bar\psi &= i\mu_0 b^2 e^{b\phi}\psi\,, \\ F &= -\mu_0 b\, e^{b\phi}\,, \end{aligned} \tag{5.7}$$

we used the last equation to simplify the first one. The boundary variation is proportional to

$$i(\psi - \eta\bar\psi)(\delta\psi + \eta\delta\bar\psi) + (\partial_y\phi + 2\eta\mu_0 b e^{b\phi})\delta\phi\,. \tag{5.8}$$

There can be two kinds of boundary conditions.

1. Fixed boundary conditions correspond to imposing

$$\phi(y=0) = \phi_0\,, \qquad \psi(y=0) + \eta\bar\psi(y=0) = 0\,. \tag{5.9}$$

These boundary conditions are not conformally invariant except for $\phi_0 = \pm\infty$ since the Liouville field shifts under scale transformations. From a target space point of view this can be understood by remembering that the tension of such a brane is proportional to $1/g_s(\phi_0) = e^{\frac{-Q}{2}\phi_0}$, and therefore the stable finite tension brane is at $\phi_0 \to \infty$. Therefore we limit the discussion to this case. Such a brane which is localized at infinity, is the supersymmetric version of the ZZ brane [14, 67, 68]. Semiclassically, the worldsheet looks asymptotically near $y=0$ as AdS_2

$$e^{2b\phi} = -\frac{1}{\mu_0^2 b^4 (z-\bar{z})^2} \ ; \quad \psi = \bar{\psi} = 0 \ ; \quad F = -\mu_0 b e^{b\phi} = -\frac{1}{2by} \ ; \tag{5.10}$$

(recall that we took μ_0 to be positive), which is a solution of the bulk equations of motion (5.7). The supersymmetry variation (5.1) of (5.10) is

$$\begin{aligned} \delta\phi &= 0 \,, \\ \delta\psi &= \frac{\zeta - \bar{\zeta}}{2by} \,, \\ \delta\bar{\psi} &= \frac{\zeta - \bar{\zeta}}{2by} \,, \\ \delta F &= 0 \,. \end{aligned} \tag{5.11}$$

The variation vanishes for $\zeta = \bar{\zeta}$ and therefore $Q + \bar{Q}$ is unbroken. This corresponds to $\eta = 1$ above. More generally, it is $\mu_0 \eta > 0$. We see that that semiclassical ZZ branes prefer one sign of η.

2. Free boundary conditions lead to the supersymmetric version of the FZZT branes [48, 69]. The boundary equations of motion are

$$\begin{aligned} \psi - \eta\bar{\psi}\big|_{y=0} &= 0 \,, \\ \partial_y \phi + 2\eta\mu_0 b e^{b\phi}\big|_{y=0} &= 0 \,, \end{aligned} \tag{5.12}$$

where we recognize the contribution of the boundary cosmological constant. The classical Euclidean equations of motion have no regular solutions with these boundary conditions.[a]

The two types of boundary condition (5.9) and (5.12) are simply expressed

[a] The solution is essentially (5.10), so both $\partial_y \phi$ and $e^{b\phi}$ diverge at the boundary. In particular, the length of the boundary is infinite. The choice $\eta = 1$ barely misses being a solution; the boundary cosmological constant is at the critical value – if it were infinitesimally larger, regular solutions would exist. Bulk vertex operators modify the classical solution, but it still has infinite boundary length. For $\eta = -1$ there is no classical solution.

in terms of the tangential and normal super-derivatives at the boundary:

$$\begin{aligned} D_t\Phi &= (D+\eta\bar{D})\Phi = 0 \qquad \text{Dirichlet}\,, \\ D_n\Phi &= (D-\eta\bar{D})\Phi = 0 \qquad \text{Neumann}\,. \end{aligned} \tag{5.13}$$

which are equivalent to (5.9) , (5.12) modulo the equations of motion. Here, one is instructed to take the derivatives, and then set $z=\bar{z}$, $\theta=\eta\bar{\theta}$.

An open string "tachyon" interaction can be added to the action of the FZZT branes to make regular the classical solution with Neumann boundary conditions. It is convenient to write it with the aid of a fermionic boundary superfield $\Gamma=\gamma+i\theta_t f$, which realizes a Chan-Paton Hilbert space,

$$\begin{aligned} S_{\rm bry} &= \frac{1}{2\pi}\oint dx d\theta_t \left(\Gamma D_t\Gamma + 2i\mu_B\Gamma e^{\frac{b}{2}\Phi}\right) \\ &= \frac{1}{2\pi}\oint dx \Big[\gamma\partial_x\gamma - f^2 - \mu_B\Big(b\gamma(\psi+\eta\bar{\psi})e^{\frac{b}{2}\phi} + 2fe^{\frac{b}{2}\phi}\Big)\Big] \\ &= \frac{1}{2\pi}\oint dx \Big[\gamma\partial_x\gamma - \mu_B\, b\,\gamma(\psi+\eta\bar{\psi})e^{\frac{b}{2}\phi} + \mu_B^2\, e^{b\phi}\Big]\,, \end{aligned} \tag{5.14}$$

where in the last line we have eliminated the auxiliary field f. Note that μ_B is odd under $(-1)^{\mathbf{F}_L}$. The quantization of the fermion γ realizes it as a Pauli matrix acting on a two-dimensional Hilbert space living on the boundary. Besides introducing this Chan-Paton space, another effect of the boundary interaction is to shift the boundary cosmological constant $-\eta\mu_0$ introduced by the bulk cosmological term by $-\eta\mu_0 \to -\eta\mu_0+\mu_B^2$. The introduction of μ_B changes the boundary conditions in such a way that (when it is sufficiently large) classical solutions exist; they have finite ϕ on the boundary, and hence finite boundary length.

In the matrix model realization of bosonic Liouville theory, the boundary cosmological constant can be identified with the continuum limit of the coupling to the (redundant) operator which shifts the matrix by a constant [70, 71]. The analogous KPZ scaling of μ_B in the fermionic string motivates us to identify it with the analogous redundant coupling in the two-sided matrix model sketched in the introduction.

The open string "tachyon" interaction (5.14) is not present on "stable" Dp-branes (p even in type 0A, odd in type 0B), due to the GSO projection. It is present on brane-antibrane pairs, and on "unstable" branes (p odd in type 0A, even in type 0B).[b]

[b] The reason for the quotes is that the linear dilaton can in some cases stabilize a brane by lifting the open string tachyon to zero mass.

Let us now list the D-branes of the 2d fermionic string, described by Liouville theory coupled to a free scalar superfield, each obeying either Dirichlet or Neumann boundary conditions (5.13). In the type 0B theory, D(-1)-branes and D1-branes are stable. The D(-1)-branes source the R-R scalar C_0. The D1-branes produce tadpoles for the R-R two-forms which cannot be cancelled.[c] Thus we consider in Lorentzian spacetime only D1-$\overline{\mathrm{D1}}$ pairs, or D0-branes. The latter come in two varieties: dynamical branes, which are Neumann in X and Dirichlet in Φ; and spacelike branes, D in X and N in Φ. It was argued [8] that the spacelike D0 in the bosonic string is an observable $W(\mu_B, x) = \mathrm{Tr}\log[M(x) - \mu_B]$ in the matrix path integral of the open string tachyon on dynamical D0's. One expects that the spacelike D0-brane of the type 0 theory corresponds to similar observables in the two-sided matrix model.

For type 0A the situation is as follows. The stable D0-branes source a R-R gauge field. However, in this case the resulting tadpole merely leads to a constant R-R electric field; after cancelling this energy, a sensible theory remains behind. There is a discrete family of theories labelled by the net D0 charge. A spacelike D0-$\overline{\mathrm{D0}}$ pair gives a macroscopic loop observable of the matrix model (the boundary interaction (5.14) is naturally expressed in terms of a complex fermionic superfield Γ). There are also sphaleron-like D(-1)'s in the theory.

The type 0A D1-brane, as well as the type 0B D1-$\overline{\mathrm{D1}}$ pair, does not carry R-R charge, but the open string "tachyons" on them are in fact massless due to the contribution of the linear dilaton. Consideration of this open-closed string theory lies beyond the scope of our discussion here.

6. Minisuperspace Wavefunctions

The minisuperspace approximation truncates the dynamics to the zero modes on the strip or the cylinder. In bosonic Liouville theory, this truncation serves as an important source of intuition about the dynamics of the full theory. We will perform the analogous truncation of the fermionic string.

In the NS sector, there are no fermion zero modes, and the dynamics is much the same as in the bosonic theory. In the R sector, one has supersymmetric Liouville quantum mechanics. More specifically, we have $\mathcal{N}=2$ supersymmetric quantum mechanics with Euclidean time τ (which differs by a factor of 2 from its value above). The supersymmetry operators and

[c] Except perhaps if we include orientifold planes; we leave this interesting possibility for future work.

supercovariant derivatives are

$$\begin{aligned} D &= \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial\tau}\,, \qquad \bar{D} = \frac{\partial}{\partial\bar\theta} + \bar\theta\frac{\partial}{\partial\tau}\,; \\ Q &= \frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial\tau}\,, \qquad \bar{Q} = \frac{\partial}{\partial\bar\theta} - \bar\theta\frac{\partial}{\partial\tau}\,. \end{aligned} \tag{6.1}$$

Taking the superpotential[a] $W(\Phi) = \mu e^{b\Phi}$, the action is

$$\begin{aligned} S &= \int d\tau d^2\theta\left[\frac{1}{2}D\Phi\bar{D}\Phi + iW(\Phi)\right] \qquad (6.2)\\ &= \int d\tau\left[\frac{1}{2}\Big(\frac{d}{d\tau}\,\phi\Big)^2 + \frac{1}{2}\,\psi\,\frac{d}{d\tau}\,\psi + \frac{1}{2}\,\bar\psi\,\frac{d}{d\tau}\,\bar\psi - \frac{1}{2}F^2 - \mu b\,e^{b\phi}F + i\mu b^2 e^{b\phi}\psi\bar\psi\right] \end{aligned}$$

and supersymmetry variation is

$$\begin{aligned} \delta\phi &= i\zeta\psi + i\bar\zeta\bar\psi\,, \\ \delta\psi &= -i\zeta\frac{d}{d\tau}\phi + \bar\zeta F\,, \\ \delta\bar\psi &= -\zeta F - i\bar\zeta\frac{d}{d\tau}\phi\,, \\ \delta F &= -\zeta\frac{d}{d\tau}\bar\psi + \bar\zeta\frac{d}{d\tau}\psi\,. \end{aligned} \tag{6.3}$$

After rotating to Lorentzian time and performing canonical quantization the supercharges are $Q = -p\psi + W'(\phi)\bar\psi$, $\bar{Q} = -p\bar\psi - W'(\phi)\psi$. The two fermions ψ and $\bar\psi$ are represented by two-dimensional matrices, say

$$\psi = \frac{1}{\sqrt{2}}\,\sigma_1\,, \quad \bar\psi = \frac{1}{\sqrt{2}}\,\sigma_2\,, \quad \text{and} \quad \psi\bar\psi = \frac{i}{2}\,\sigma_3 = \frac{i}{2}\,(-1)^F.$$

They can be represented as

$$\begin{aligned} \frac{1}{\sqrt{2}}(Q - i\bar{Q}) &= i\begin{pmatrix} 0 & \frac{\partial}{\partial\phi} + W'(\phi) \\ 0 & 0 \end{pmatrix} = i\begin{pmatrix} 0 & \frac{\partial}{\partial\phi} + b\mu e^{b\phi} \\ 0 & 0 \end{pmatrix}; \\ \frac{1}{\sqrt{2}}(Q + i\bar{Q}) &= -i\begin{pmatrix} 0 & 0 \\ -\frac{\partial}{\partial\phi} + W'(\phi) & 0 \end{pmatrix} = -i\begin{pmatrix} 0 & 0 \\ -\frac{\partial}{\partial\phi} + b\mu e^{b\phi} & 0 \end{pmatrix}. \end{aligned} \tag{6.4}$$

For every positive energy $E = \frac{1}{2}\,b^2p^2 > 0$, there are two states,

$$|p+\rangle = \begin{pmatrix} \Psi_{p+}(\phi) \\ 0 \end{pmatrix} \quad \text{and} \quad |p-\rangle = \begin{pmatrix} 0 \\ \Psi_{p-}(\phi) \end{pmatrix},$$

[a] In this section only, we ignore the distinction between the bare cosmological constant μ_0 and he physical quantity μ from (3.36).

which satisfy

$$\begin{aligned}\left(\frac{\partial}{\partial\phi}+b\mu e^{b\phi}\right)\Psi_{p-}(\phi)&=bp\Psi_{p+}(\phi)\ ,\\ \left(-\frac{\partial}{\partial\phi}+b\mu e^{b\phi}\right)\Psi_{p+}(\phi)&=bp\Psi_{p-}(\phi)\ ;\end{aligned}\tag{6.5}$$

thus

$$\left(-\frac{\partial^2}{\partial\phi^2}\pm b^2\mu e^{b\phi}+b^2\mu^2e^{2b\phi}\right)\Psi_{p\pm}(\phi)=b^2p^2\Psi_{p\pm}(\phi)\,.\tag{6.6}$$

Recall our convention that μ is positive. Then, in terms of $z=\mu e^{b\phi}$ we can write equations (6.5) , (6.6) as

$$\begin{aligned}\left(z\frac{\partial}{\partial z}+z\right)\Psi_{p-}&=p\Psi_{p+}\ ,\\ \left(-z\frac{\partial}{\partial z}+z\right)\Psi_{p+}&=p\Psi_{p-}\,,\\ \left(-\left(z\frac{\partial}{\partial z}\right)^2\pm z+z^2-p^2\right)\Psi_{p\pm}&=0\end{aligned}\tag{6.7}$$

for the R minisuperspace dynamics; for the NS sector we have simply

$$\left(-\left(z\frac{\partial}{\partial z}\right)^2+z^2-p^2\right)\Psi_{p0}=0\ .\tag{6.8}$$

The latter is identical to the corresponding equation in bosonic Liouville theory (the theory with $\psi=\bar\psi=0$) [55,72,73]. It lacks the term proportional to z which arises from the fermions.

We impose that the wave functions $\Psi_{p\pm}$, Ψ_{p0} decay deep under the Liouville potential ($\phi\to+\infty$). These equations with these boundary conditions are solved by Whittaker functions or Bessel functions

$$\begin{aligned}\Psi_{p+}&=\frac{p}{\sqrt{2z}}\,W_{\substack{\lambda=-1/2\\ \nu=ip}}(2z)=-i\sqrt{\frac{z}{2\pi}}\left(K_{ip+\frac12}(z)-K_{ip-\frac12}(z)\right)\\ &=\frac{pe^{-z}}{2z}\left(1-\frac{1+p^2}{2z}+\frac{(p^2{+}1)(p^2{+}4)}{8z^2}-\frac{(p^2{+}1)(p^2{+}4)(p^2{+}9)}{48z^3}+\mathcal{O}\Big(\frac{1}{z^4}\Big)\right),\\ \Psi_{p-}&=\frac{1}{\sqrt{2z}}\,W_{\substack{\lambda=1/2\\ \nu=ip}}(2z)=\sqrt{\frac{z}{2\pi}}\left(K_{ip+\frac12}(z)+K_{ip-\frac12}(z)\right)\\ &=e^{-z}\left(1-\frac{p^2}{2z}+\frac{p^2(p^2{+}1)}{8z^2}-\frac{p^2(p^2{+}1)(p^2{+}4)}{48z^3}+\mathcal{O}\Big(\frac{1}{z^4}\Big)\right),\\ \Psi_{p0}&=\frac{1}{\sqrt{2z}}\,W_{\substack{\lambda=0\\ \nu=ip}}(2z)=\frac{1}{\sqrt{\pi}}\,K_{\nu=ip}(z)\,.\end{aligned}\tag{6.9}$$

We would like to make a few comments about these functions.

1. All the wave functions decay at large $\ell = e^{b\phi}$ as $e^{-z} = e^{-\mu\ell}$. For small $z=\mu\ell$ and generic p all these functions are linear combinations of terms of the form $z^{ip}\big(1+\mathcal{O}(z)\big)$ and $z^{-ip}\big(1+\mathcal{O}(z)\big)$. These represent incoming and outgoing waves in the Liouville coordinate ϕ.
2. For $p=iN$ with integer N the Ramond functions $\Psi_{iN,\pm}(z)$ are elementary functions. This is analogous to the fact that for $p = i\big(N+\frac{1}{2}\big)$ the NS functions $\Psi_{i(N-\frac{1}{2}),0}(z)$ are elementary.
3. For $p = i\big(N+\frac{1}{2}\big)$ the Ramond functions exhibit a resonance phenomenon between the growing solution $z^{|N+\frac{1}{2}|}\big(1+\mathcal{O}(z)\big)$ and the decaying solution $z^{-|N+\frac{1}{2}|}\big(1+\mathcal{O}(z)\big)$. These appear as corrections proportional to $\log z$ to the decaying solution. This is analogous to a similar phenomenon in the NS sector for $p=iN$. These logarithms are related to the "leg poles" (3.44) and to the discrete states of $\hat{c}=1$ [62].
4. In the NS sector, the delta function normalizable spectrum includes all positive energy states but there is no zero energy state. The zero energy eigenfunction

$$\Psi_{p0} = \frac{1}{\sqrt{2z}}\, W_{\substack{\lambda=0\\ \nu=0}}(2z) = \frac{1}{\sqrt{\pi}}\, K_{\nu=0}(z)$$

   satisfies the correct boundary conditions at $\phi \to +\infty$ but it grows linearly in $\phi \sim \log z$ near $\phi \to -\infty$, and therefore it is not normalizable. (This is an example of the resonance phenomenon we mentioned above.) Surprisingly, the situation is different in the Ramond sector, which does not exhibit resonances at integer ip. Restoring the possibility of μ of different signs, we find a single Ramond state for $p=0$

$$|p=0\rangle = \begin{cases} |+\rangle = \begin{pmatrix} \Psi_{0+}(\phi) = e^{W(\phi)} = e^{\mu e^{b\phi}} \\ 0 \end{pmatrix} & \mu < 0\,, \\ |-\rangle = \begin{pmatrix} 0 \\ \Psi_{0-}(\phi) = e^{-W(\phi)} = e^{-\mu e^{b\phi}} \end{pmatrix} & \mu > 0\,. \end{cases} \tag{6.10}$$

   This wave function is (delta function) normalizable and represents a supersymmetric zero energy ground state.

For the application to $\hat{c}=1$ strings, we should append to the minisuperspace quantum mechanics the dynamics of the free coordinate X from (2.10), whose supercharges analogous to (6.4) act in a two-dimensional Hilbert space representation of the fermions χ and $\bar{\chi}$. The full fermion Hilbert space is then the four-dimensional tensor product space, with $Q=Q_\phi \otimes \mathbb{1} + \mathbb{1} \otimes Q_x$, and so on. The choice of fermion number projection $(-1)^F = \sigma_3 \otimes (\pm\sigma_3)$ determines whether we are in type 0B or type 0A.

Equations (6.5) are similar to (2.6) which were derived from the spacetime Lagrangian. This provides a motivation for our assertion in Section 2 that $f_3(T)$ of (2.4) is proportional to e^{-2T}.

Since the wave functions $\Psi_{p\pm}$ satisfy the same equations as the components of the field strength $F = d\chi + \chi dT = e^{-T} d\big(e^T \chi\big)$ (see discussion after (2.5)), they are identified with them rather than with the fundamental fields C_0 or χ. This one-form F is used in vertex operators in the $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ picture, while χ is used in $\left(-\frac{1}{2}, -\frac{3}{2}\right)$ or $\left(-\frac{3}{2}, -\frac{1}{2}\right)$ picture. Therefore, even though the zero energy solution $e^{-|\mu|\ell}$ is a delta function normalizable state in Liouville, the corresponding wave function in string theory, which is computed using the field χ rather than F, is not normalizable. Therefore, it corresponds to a microscopic operator in the terminology of Ref. [55].

More generally, if the theory we consider includes in addition to super-Liouville also another superconformal matter field theory with central charge $\hat{c}$, we can try to form supersymmetric states out of different states in the superconformal field theory. If the state in the conformal field theory is a Ramond primary with dimension $\Delta > \hat{c}/16$, the condition for unbroken supersymmetry states that the total energy of the state vanishes. This means that in the Liouville part of the theory we need a state with negative energy (imaginary p in (6.5), (6.6)) whose wave function is not normalizable [55]. Also, the supercharge $Q + \bar{Q}$ does not annihilate the Liouville part of the wave function nor the matter part of the wave function. Instead it leads to a relation between them. This condition is the spacetime Dirac equation. In our case the differential operator in the Liouville direction in this "Dirac equation" is $\frac{\partial}{\partial\phi} \pm \mu b e^{b\phi}$ of (6.5). Therefore, in terms of $\frac{1}{2}\nu^2 b^2 = \Delta - \hat{c}/16$ the wave function is $\Psi_{i\,\nu\pm}(\phi)$ where the sign is determined according to the fermion number of the matter wave function and whether we study the 0A or the 0B theory. For $\phi \to -\infty$ it behaves like $\Psi_{i\nu\pm} \sim e^{-\nu b\phi}$ with $\nu > 0$. The ϕ dependence of the corresponding vertex operator is determined from the asymptotic behavior at $\phi \to -\infty$ of $V \sim g_s(\phi)\Psi(\phi) \sim e^{(\frac{Q}{2} - \nu b)\phi}$; νb is the momentum in the ϕ direction.

6.1. *The Transform to Free Fields*

One of the intriguing features of bosonic gravity is the observation [71, 73] that the minisuperspace wavefunctions are related to the mode functions of eigenvalue collective field theory via an integral transform

$$
\begin{aligned}
K_{iE}(\mu\ell) &= \int_0^\infty d\tau\,\left(e^{-\mu\ell\,\mathrm{ch}\tau}\right)\,\cos E\tau\,, \\
\frac{\pi\cos E\tau}{E\,\mathrm{sh}\pi E} &= \int_0^\infty \frac{d\ell}{\ell}\,\left(e^{-\mu\ell\,\mathrm{ch}\tau}\right) K_{iE}(\mu\ell)\,.
\end{aligned}
\tag{6.11}
$$

It was noted that this transform bears a striking similarity to the Backlund transform that converts Liouville theory into free field theory. In retrospect this is not so surprising, since the kernel of the full-fledged 1+1 field theory functional integral transform

$$
W[\phi,\psi] = \exp\left[\int d\sigma\left(\phi\psi' - \mu e^{\gamma\varphi/2}\mathrm{sh}\psi\right)\right] \tag{6.12}
$$

has the property that it converts the Liouville Hamiltonian in ϕ into the free field Hamiltonian in ψ, hence the full 2d field theory Wheeler-deWitt equation into a free field one.[b] Thus one may regard (6.11) as the minisuperspace truncation of the Backlund transform.

In the fermionic string the super-Backlund transformation [43] has the same property, that it converts the interacting Liouville super-WdW equation into a free field one. Hence we should search for an integral transform that turns the Ramond wavefunctions into sines and cosines. The answer is

$$
\begin{aligned}
\sqrt{\mu\ell}\big(K_{\frac12+iE}+K_{\frac12-iE}\big)(\mu\ell) &= \int_0^\infty d\tau\left((\mu\ell)^{\frac12}e^{-\mu\ell\mathrm{ch}\tau}\mathrm{ch}\frac{\tau}{2}\right)\cos E\tau\,, \\
\frac{\pi\cos E\tau}{\mathrm{ch}\pi E} &= \int_0^\infty \frac{d\ell}{\ell}\left((\mu\ell)^{\frac12}e^{-\mu\ell\mathrm{ch}\tau}\mathrm{ch}\frac{\tau}{2}\right)\sqrt{\mu\ell}\big(K_{\frac12+iE}+K_{\frac12-iE}\big)(\mu\ell)\,, \\
-\,i\sqrt{\mu\ell}\big(K_{\frac12+iE}-K_{\frac12-iE}\big)(\mu\ell) &= \int_0^\infty d\tau\left((\mu\ell)^{\frac12}e^{-\mu\ell\mathrm{ch}\tau}\mathrm{sh}\frac{\tau}{2}\right)\sin E\tau\,, \\
\frac{\pi\sin E\tau}{\mathrm{ch}\pi E} &= \int_0^\infty \frac{d\ell}{\ell}\left((\mu\ell)^{\frac12}e^{-\mu\ell\mathrm{ch}\tau}\mathrm{sh}\frac{\tau}{2}\right)\big(-i\sqrt{\mu\ell}\big)\big(K_{\frac12+iE}-K_{\frac12-iE}\big)(\mu\ell)\,.
\end{aligned}
\tag{6.13}
$$

Let us now derive this result from the minisuperspace dynamics for super-Liouville.

---

[b] The reason $\mathrm{sh}\,\psi$ appears here and $\mathrm{ch}\,\tau$ appears in (6.11) is due to the relation between Minkowski and Euclidean 2d Liouville theory.

One way to view the transform (6.11) is that in the minisuperspace approximation, the 'boundary state wavefunction' is $\Psi_B = \exp\big[-\rho e^{b\phi}\big]$; it implements the Neumann boundary condition (5.12), since $\dot\phi = \partial_\phi = -b\rho e^{b\phi}$ when acting on the boundary state wavefunction. The 'disk one-point function' is the inner product of this wavefunction with the minisuperspace wavefunction Ψ_{p0} of a 'closed string primary' of the NS sector of the fermionic string, or of the bosonic string. This gives the transform in the second line of (6.11), if we set $\rho = \mu \cosh\tau$.

In the R sector, the boundary state wavefunction includes the boundary (zero mode) part of the bulk Liouville action (5.6), as well as the boundary interaction (5.14),

$$\Psi_B\big(\phi,\psi,\bar\psi\big) = \big(\psi - \eta\bar\psi\big)\exp\Big[\mu_B\, b\,\gamma\big(\psi+\eta\bar\psi\big)e^{\frac{b}{2}\phi} - \big(-\eta\mu + \mu_B^2\big)e^{b\phi}\Big]\,. \quad (6.14)$$

The fermion factor in front of the exponential implements the boundary condition (5.12) on the fermions. We can also organize the Liouville wavefunction (6.9) as $\Psi_p = \Psi_{p,-\eta}(\phi) + (\psi + \eta\bar\psi)\Psi_{p,\eta}(\phi)$ and write the overlap as

$$\langle\Psi_p|\Psi_B\rangle = \int d\phi\, d\psi\, d\bar\psi\, d\gamma\ \Psi_p^\dagger\big(\phi,\psi,\bar\psi\big)\Psi_B\big(\phi,\psi,\bar\psi\big)\,; \quad (6.15)$$

after integrating over Grassmann variables the result is

$$\int\frac{d\ell}{\ell}\, 2\eta\mu_B\sqrt{\ell}\,\exp\big[-\big(-\eta\mu+\mu_B^2\big)\ell\big]\ \Psi^\dagger_{p,-\eta}\big(\phi,\psi,\bar\psi\big)\,. \quad (6.16)$$

Plugging in (6.9), and defining τ via

$$\begin{aligned}\rho &\equiv |\mu|\cosh\tau = -\eta\mu + \mu_B^2\,,\\ \mu_B &= \pm\sqrt{|\mu|\cosh\tau + \mu\eta}\,,\end{aligned} \quad (6.17)$$

we recognize the second and fourth lines of (6.13). Note that for $\eta\mu < 0$, the effective boundary cosmological constant ρ is real and positive all the way down to $\mu_B = 0$, leading to a sensible transform and wavefunctions concentrated at finite semiclassical boundary loop length. In contrast, for $\eta\mu > 0$ one must turn on a finite $\mu_B^2 > 2|\mu|$ for the integral transform to converge.

7. Quantum Super-Liouville Theory

The conformal bootstrap for supersymmetric Liouville theory has been analyzed in Refs. [60,61], and the boundary bootstrap is treated in Refs. [67,68].

The latter works find the exact disk one-point functions and annulus partition functions; these are most conveniently expressed in terms of the boundary state wavefunctions introduced in Refs. [14, 48, 69] in the context of bosonic Liouville theory.

7.1. *Neumann Boundaries*

The quantum version of the super-FZZT (Neumann) boundary state has the boundary state wavefunction [67, 68]

$$\begin{aligned}
\Psi^{\rm NS}_{\eta}(\nu,s) &= \cos(2\pi\nu s)\Big[\frac{\Gamma(1+i\nu b)\,\Gamma(1+i\nu/b)}{(2\pi)^{1/2}\,(-i\nu)}\,(2\mu)^{-i\nu/b}\Big]\,,\\
\Psi^{RR}_{\eta=+{\rm sgn}(\mu)}(\nu,s) &= \cos(2\pi\nu s)\Big[\frac{\Gamma(\frac12+i\nu b)\,\Gamma(\frac12+i\nu/b)}{(2\pi)^{1/2}}\,(2\mu)^{-i\nu/b}\Big]\,, \qquad (7.1)\\
\Psi^{RR}_{\eta=-{\rm sgn}(\mu)}(\nu,s) &= \sin(2\pi\nu s)\Big[\frac{\Gamma(\frac12+i\nu b)\,\Gamma(\frac12+i\nu/b)}{(2\pi)^{1/2}}\,(2\mu)^{-i\nu/b}\Big]\,,
\end{aligned}$$

where μ is given by (3.36); η is the sign (5.4) in the fermion boundary condition. In the limit $b\to 0$, these wavefunctions precisely reproduce (up to phases) the τ-space wavefunctions (6.11), (6.13) of the minisuperspace approximation, provided we identify $E=\nu/b$, $\tau=2\pi sb$. Away from this limit, one cannot ignore the contribution of the gamma function $\Gamma(1+i\nu b)$, whose presence is required by the $b\to 1/b$ symmetry of quantum Liouville field theory [54]. This is the main difference between the minisuperspace approximation to the wavefunctions and those of the full 2d field theory. This factor can be absorbed in an s independent redefinition of the wave functions.

The boundary state wavefunction (7.1) determines the one-point function of closed string vertex operators on the disk with Neumann boundary conditions in the Liouville direction, when combined with the wavefunction of the matter system. With Dirichlet boundary conditions for matter, the analogous quantity in the bosonic matrix model has the interpretation of a wavefunction of a string state [71, 73], given by the correlation function of a local operator with a macroscopic loop; it is natural to propose a similar interpretation here.

R-R closed string vertex operators couple to 'unstable' D-branes such as the type 0B D0-brane only in the presence of an open string tachyon [74] (as for example in Section 11 below); the couplings vanish as the tachyon background is turned off.

The wavefunctions (7.1) also enter into the annulus partition function. In the bosonic theory this amplitude is related to the correlation function of two macroscopic loops in the matrix model [71]. The super-Liouville annulus partition function has the form

$$\mathcal{Z}^a_L(\eta,s;\eta',s') = \int d\nu\, \Psi^a_\eta(\nu,s)\Psi^a_{\eta'}(-\nu,s')\chi^{a,\eta\eta'}_\nu(q)\,, \tag{7.2}$$

where $\chi^{a,\pm}_\nu(q)$ is the character of a nondegenerate superconformal representation of Liouville momentum $\frac{Q}{2}+i\nu$; $a=NS, RR$ is the fermion boundary condition in the closed string channel, and $\eta\eta'=\pm$ is the fermion boundary condition in the open string channel, $\eta\eta'=+$ for NS and $\eta\eta'=-$ for R open strings.

Consider the $\hat{c}=1$ model with x non-compact. Combining the Liouville partition function with the corresponding partition functions for the ghosts and the non-compact boson x obeying Dirichlet boundary conditions, one finds that the oscillator contributions cancel, leaving the integral over the zero modes. We omit the details, since they are essentially identical to the calculation in Ref. [10]. The annulus amplitude for spacelike D0's is

$$\begin{aligned}\mathcal{Z}_{NS}(s,s';p) &= \int d\nu\, \frac{\cos(2\pi\nu s)\cdot\cos(2\pi\nu s')}{\sinh^2(\pi\nu)\;(\nu^2+\frac{\alpha'}{2}p^2)}\,,\\ \mathcal{Z}_{RR}(s,s';p) &= \int d\nu\, \frac{\sin(2\pi\nu s)\cdot\sin(2\pi\nu s')}{\cosh^2(\pi\nu)\;(\nu^2+\frac{\alpha'}{2}p^2)}\,,\end{aligned} \tag{7.3}$$

(implicitly there is a regulator to cut off the small ν divergence in the NS integral). We have assumed $\eta=-1$ here; for $\eta=+1$, the sines are replaced by cosines in the Ramond partition function.

These amplitudes yield off-shell information about the spacetime theory. For example, they have poles at the locations corresponding to the discrete states in the appropriate sector, see the discussion after Eq. (6.9).

7.2. *Dirichlet Boundaries*

There are also super-ZZ (Dirichlet) boundary states associated to degenerate representations of the superconformal algebra. These are labelled by a pair of integers (m,n), with $m-n$ even(odd) for NS(R) representations. Their wavefunctions are [67, 68]

$$\begin{aligned}\Psi_{NS}(\nu;m,n) &= 2\sinh(\pi m\nu/b)\,\sinh(\pi n\nu b)\Big[\frac{\Gamma(1+i\nu b)\,\Gamma(1+i\nu/b)}{(2\pi)^{1/2}\,(-i\nu)}\,(2\mu)^{-i\nu/b}\Big]\,,\\ \Psi_{RR}(\nu;m,n) &= -\,i^{\,m+n}\,2\,\sin\big[\pi m(1/2+i\nu/b)\big]\,\sin\big[\pi n(1/2+i\nu b)\big]\\ &\quad\times\Big[\frac{\Gamma(\frac12+i\nu b)\,\Gamma(\frac12+i\nu/b)}{(2\pi)^{1/2}}\,(2\mu)^{-i\nu/b}\Big]\,.\end{aligned} \tag{7.4}$$

Normalized disk one-point functions are

$$U_{NS,RR}(\nu; m, n) = \frac{\Psi_{NS,RR}(\nu; m, n)}{\Psi_{NS}(iQ/2; 1, 1)} \,. \tag{7.5}$$

In the semiclassical limit $b \to 0$, the normalized one-point function of $e^{b\Phi}$, i.e. $i\nu = \frac{1}{2}(b - b^{-1})$, reproduces the semiclassical geometry (5.10), provided $m = 1$. As in Ref. [14], we expect that the diagrammatic expansion in Liouville theory will match only the properties of the $m = n = 1$ boundary state in this limit.

A key feature of the ZZ branes is that only a finite list of (degenerate) boundary operators couple to them – the (m, n) degenerate boundary operator can create open strings stretched between the (m', n') and (m'', n'') boundary states only if the (m, n) degenerate representation appears in the fusion of (m', n') and (m'', n''), i.e. only if $m \in \{|m' - m''| + 1, \ldots, m' + m'' - 1\}$, and similarly for n. In particular, only the identity operator couples to the $(1, 1)$-brane. This means that the effective open string dynamics on a collection of such branes involves only the open string tachyon and gauge field, which depend only on the time coordinate x.

It is argued in Ref. [67] that degenerate boundary states exist only for $\eta = (-1)^{m-n}$. For $m = n = 1$, this agrees with the value determined from the semiclassical analysis of Dirichlet boundaries in Section 5. In general rational conformal field theories the sign of η for the boundary state is determined by whether the label of the Cardy state is an NS or Ramond operator. Since the $(1, 1)$ ZZ brane is labeled by an NS operator (the identity) we see that it has $\eta = 1$. A consequence of this property is that the matrix model of tachyons on N $(1, 1)$ ZZ branes involves only one type of brane.

The ZZ branes are the appropriate boundary states which describe, in quantum Liouville theory, dynamical type 0 D0-branes. Below, we will restrict our consideration to the simplest possibility, namely that all such D0-branes are built out of the $(m, n) = (1, 1)$ boundary state.

8. Spacetime Effective Dynamics of Type 0 D-Branes

D0-branes in type 0A couple to only one of the two R-R vector fields, which are governed by the action (2.9). The relevant part of that action is[a]

$$S = -\int d^2x \sqrt{G} \left(\frac{2\pi\alpha}{4} f_3(T) \left(F^{(+)}_{\mu\nu}\right)^2 + \frac{2\pi\alpha'}{4} f_3(-T) \left(F^{(-)}_{\mu\nu}\right)^2 \right) . \tag{8.1}$$

[a] Below we do not have to use the asserted form of $f_3 = e^{-2T}$, but only its expansion in powers of T. For large and negative ϕ, where T is small, $f_3 = 1 - 2T + \cdots$.

Thus, as T grows, so does the asymmetry between the two gauge fields: one of them becomes more weakly coupled while the other becomes more strongly coupled. We are interested in $T = \mu e^{\phi}$ which corresponds to the Liouville theory. In the absence of a background value of T, there are two types of stable D0-branes: those charged under $F^{(+)}_{\mu\nu}$ and those charged under $F^{(-)}_{\mu\nu}$. The branes charged under each gauge field are branes with the two possible signs for η. As we saw above, the ZZ brane with $(n,m) = (1,1)$, has $\eta = 1$ and therefore is charged only under one gauge field. In fact it is charged under $F^{(-)}$.[b]

We can make an analogous argument in the world sheet language. The origin of the doubling of D-brane spectrum in conventional type 0 theories is that one can impose fermion boundary conditions $\psi_L = \psi_R$ or $\psi_L = -\psi_R$. The two types of D-branes are related by the transformation $\psi_L \to -\psi_L$. However, after the Liouville potential is turned on, this transformation is no longer a symmetry. So, the existence of one type of D-brane no longer guarantees the existence of the other. This is a way of seeing from the spacetime point of view the fact that each ZZ boundary state comes in only one variety.

Analogous arguments applied to type 0B theory show that there is only one type of D-instanton. It corresponds in the matrix model to an eigenvalue tunnelling from the left well to the right well, while an anti-instanton corresponds to tunnelling in the opposite direction. An unstable D0-brane in 0B is obtained by starting with D0 brane-antibrane pair of the 0A theory and applying a Z_2 projection. In this way we obtain an uncharged unstable D0-brane. It should correspond to an eigenvalue at the unstable symmetric point of the double-well potential. If we have N such unstable D0-branes we have a $U(N)$ theory with the tachyon in the adjoint of $U(N)$.

In the type 0A theory we can have N D0-branes and M anti-D0-branes, and the theory on the branes is $U(N) \times U(M)$ Yang-Mills quantum mechanics with a complex tachyon in the bifundamental. Let $N - M = q > 0$. There will be q D0-branes that cannot decay. In the eigenvalue space of the matrix, the q D0-branes are localized near the origin. In the Liouville coordinate ϕ, they are near $\phi = \infty$. Since the branes are charged, they lead to q units of a constant background flux of one of the R-R vector field strengths. For positive μ it is $F^{(-)}$ in (8.1). Ignoring the NS tachyon field, the effective

[b] It is amusing to note that the first term of a coupling of the form $\int dt[e^{-\Phi}(1+T+\cdots)+C^{(\pm)}+\cdots]$ on the worldvolume of the brane would give the right mass for the ZZ brane, and also for the bosonic string ZZ brane.

Lagrangian of the system is

$$e^{-2\Phi}\big(R+4(\partial\Phi)^2+4\big)-\frac{2\pi\alpha'}{4}\left(F^{(-)}\right)^2+qF^{(-)}\,. \tag{8.2}$$

The coupling $qF^{(-)}$ is like a θ angle for $F^{(-)}$. Nevertheless, its value cannot be changed by the standard process of pair creation. The reason for that is that the mass of the D0-branes depends on ϕ. Therefore, it takes infinitely more energy to create a pair and to separate it to infinity than is being gained by screening the background charge. q is an integer and it represents the net number (branes minus antibranes) of ZZ D0-branes at $\phi=\infty$.

Integrating out the gauge field, we find the Lagrangian

$$e^{-2\Phi}\big(R+4(\partial\Phi)^2+4\big)-\frac{q^2}{4\pi\alpha'}\,. \tag{8.3}$$

This Lagrangian and its classical solutions were studied in Ref. [75].

In our problem there is also an NS tachyon field T. It is sourced both by the linear dilaton and the background R-R field strength $F^{(-)}$. For example, as in (8.1), we have a coupling of the form $\frac{1}{4}e^{2T}(F^{(-)})^2$. Near $\phi=-\infty$ the Lagrangian (8.3) can be used because the string coupling is small and the tachyon and the backreaction are exponentially suppressed. However, for finite ϕ the corrections to the leading order Lagrangian (8.3) are important.

Since in most of spacetime the Lagrangian (8.3) can be used, we can conclude that the q dependence of the torus amplitude is given by

$$\mathcal{Z}_q=\frac{q^2}{4\pi\sqrt{2\alpha'}}\ln\big(|\mu|/\Lambda\big)\,, \tag{8.4}$$

where Λ is a cutoff on the Liouville direction ϕ, and the factor of $\ln(\Lambda/|\mu|)$ is the effective length of this direction. It is cut off at $\ln|\mu|$ by the coupling to the tachyon which we neglected in (8.3). The coefficient of this term can be exactly calculable using this Lagrangian by properly normalizing the various terms. Note that this contribution is infinite and therefore the one loop contribution to the D-brane mass is infinite. Therefore, unlike ordinary D-branes in the critical string, such D0-branes are not finite energy excitations of the theory. They constitute different sectors which are separated by infinite energy.

The net charge q is a parameter in the theory, which corresponds to a background R-R field in the worldsheet action. However, since it is quantized, it cannot be changed in a continuous fashion.

9. The Matrix Model for Type 0B Strings

The type 0B version of the ZZ boundary state [14, 67, 68] contains an open string tachyon of mass $m_T^2 = -1/(2\alpha')$ .The matrix model dual to type 0B string theory should correspond to the dynamics of $N \to \infty$ such D0-branes. It is described by the double-scaled matrix quantum mechanics (1.1), with two important modifications. First, α' has to be replaced by $2\alpha'$ to obtain the curvature at the top corresponding to the NSR open string tachyon. Second, the NSR tachyon effective action must be symmetric under the Z_2 symmetry $M \to -M$ which, in the world sheet language, is the spacetime $(-1)^{\mathbf{F}_L}$. This is the operation that reverses the sign of R-R states. Hence, the Fermi sea fills the potential symmetrically on both sides. This two-cut hermitian matrix model is equivalent to the double scaling limit of the quantum mechanics of a unitary matrix whose potential $\sim \cos\lambda$; the eigenvalue distribution is automatically symmetric.

In analogy with the bosonic string case, we conjecture an exact duality between the double-well hermitian (or, equivalently, the unitary) matrix model describing open strings on unstable D0-branes of type 0B theory, and the closed type 0B strings. In the double-scaling limit, this implies that closed type 0B strings are described by $N \to \infty$ eigenvalues moving in the potential

$$V(\lambda) = -\frac{1}{4\alpha'}\lambda^2 , \tag{9.1}$$

and Fermi sea filling both sides symmetrically to Fermi level $-\mu$ as measured from the top of the potential. Thus, for $\mu > 0$ the Fermi level lies below the top, while for $\mu < 0$ it is above the top. The latter possibility is absent in the matrix model for the 2d bosonic string, but is present for the 0B theory.

This conjecture has immediate consequences for perturbative closed string dynamics of type 0B strings. The Fermi sea fluctuations, $T_L(k)$ and $T_R(k)$, of the left and right sides of the barrier are perturbatively independent. Therefore, connected correlation functions involving both $T_L(k)$ and $T_R(k)$ vanish. Correlation functions involving either only the left or only the right modes are the same as those given by the bosonic matrix model where only one side of the barrier is filled, up to the rescaling $\alpha' \to 2\alpha'$ (or equivalently, the rescaling of momenta $k \to \sqrt{2}k$).

In Section 3, we saw that the type 0B excitations exhibit precisely the same structure. In particular, the first line of Eq. (3.47) implies that the modes T_L and T_R defined in (3.45) decouple on the sphere. The second line of (3.47) shows that the tree level scattering amplitudes of T_L and T_R agree with those of the bosonic string. This provides a strong argument in favor of the conjecture.

In the matrix model, one can form symmetric and antisymmetric combinations, $T_L \pm T_R$. These combinations are related to the natural observables of the type 0B string as follows: the symmetric combination

$$T(k) = e^{i\delta_{NS}(k)} \left[T_R(k) + T_L(k)\right] \tag{9.2}$$

is related to the NS-NS tachyon (2.18) while the antisymmetric combination

$$V(k) = e^{i\delta_{RR}(k)} \left[T_R(k) - T_L(k)\right] \tag{9.3}$$

is related to the R-R scalar (2.19). The phase factors in these equations are given by

$$e^{i\delta_{NS}(k)} = \frac{\Gamma(ik\sqrt{\alpha'/2})}{\Gamma(-ik\sqrt{\alpha'/2})}\,, \qquad e^{i\delta_{RR}(k)} = \frac{\Gamma(\frac{1}{2}+ik\sqrt{\alpha'/2})}{\Gamma(\frac{1}{2}-ik\sqrt{\alpha'/2})}\,, \tag{9.4}$$

see Eq. (3.44) (T is normalized slightly differently there).

Another argument in favor of our proposal is related to the fact that, in super-Liouville theory, both signs of μ are admissible. As we showed in Section 5, the transformation $\mu \to -\mu$ is equivalent to $\psi_L \to -\psi_L$. Similarly, in the quantum mechanics of a unitary matrix, both signs of μ are admissible and the perturbative expansion of the theory is singular as $|\mu| \to 0$ in both cases.[a] In the double scaling limit, when the potential is an inverted parabola, there is a simple transformation relating theories with opposite signs of μ [1,3]: interchange of the coordinate λ with conjugate momentum $p = -\frac{i}{\beta}\frac{\partial}{\partial\lambda}$, accompanied by particle-hole conjugation. This is evident from the fact that the Fermi surface $p^2 - \lambda^2 = -2\mu$ becomes transformed into $\lambda^2 - p^2 = -2\mu$. Furthermore, the second-quantized Hamiltonian

$$\hat{H} = \int d\lambda \left\{ \frac{1}{2\beta^2}\frac{\partial\Psi^\dagger(\lambda)}{\partial\lambda}\frac{\partial\Psi(\lambda)}{\partial\lambda} - \frac{\lambda^2}{2}\Psi^\dagger(\lambda)\Psi(\lambda) + \mu\Psi^\dagger(\lambda)\Psi(\lambda) \right\}, \tag{9.5}$$

becomes in momentum space

$$\hat{H} = -\int dp \left\{ \frac{1}{2\beta^2}\frac{\partial\tilde{\Psi}^\dagger(p)}{\partial p}\frac{\partial\tilde{\Psi}(p)}{\partial p} - \frac{p^2}{2}\tilde{\Psi}^\dagger(p)\tilde{\Psi}(p) - \mu\tilde{\Psi}^\dagger(p)\tilde{\Psi}(p) \right\}. \tag{9.6}$$

In the Liouville theory the corresponding transformation $\psi_L \to -\psi_L$ changes the relative sign of the two R-R states, the self-dual scalar from the $(R+,R+)$ sector and the anti-self-dual scalar from the $(R-,R-)$ sector. This implies that the left moving part of the R-R scalar changes a sign relative to the right moving part. Thus, this is the same as a spacetime

[a] The nonperturbative answers are non-singular as $\mu \to 0$.

T-duality (electric-magnetic duality) of the massless R-R scalar C_0. This is the S-duality transformation we discussed above.

As we discussed above, filling the two sides asymmetrically corresponds to adding a constant gradient for the R-R scalar at $\phi = -\infty$, see also Ref. [7]. This gradient is in the time or ϕ direction depending on the sign of μ. This corresponds to the fact that we are either changing the Fermi levels on the two sides of the potential for $\mu > 0$, or we are changing the Fermi levels of the left and right movers for $\mu < 0$.[b]

In the remainder of this section we carry out another sensitive check comparing the matrix model at finite temperature to the compact $\hat{c} = 1$ theory coupled to supergravity.

9.1. *Matrix Quantum Mechanics at Finite Temperature*

Consider the matrix quantum mechanics at finite temperature $T = 1/(2\pi R)$ and its connection with Liouville and super-Liouville theories. Again we first review the bosonic case.

Compactified bosonic two-dimensional string is described by a Euclidean path integral for a Hermitian $N \times N$ matrix

$$\mathcal{Z}_R = \int D^{N^2} M(x) \exp\left[-\beta \int_0^{2\pi R} dx \, \mathrm{Tr}\left(\frac{1}{2}(D_x M)^2 - \frac{1}{2\alpha'} M^2\right)\right]. \tag{9.7}$$

The gauge field A acts as a lagrange multiplier that projects onto $SU(N)$ singlet wave functions. In the Hamiltonian language, (9.7) is a path integral representation for the thermal partition function

$$\mathcal{Z}_R = \mathrm{Tr}\left[e^{-2\pi R \beta H}\right], \tag{9.8}$$

where the trace runs over singlet states only. Since the singlet wave functions depend only on the matrix eigenvalues, which act as free fermions [16], evaluation of the path integral reduces to studying free fermions in the upside down harmonic oscillator potential (1.2) at finite temperature [66]. This problem is exactly solvable, and the free energy as a function of the original "cosmological constant" Δ is specified by the equations

$$\frac{\partial \mathcal{F}}{\partial \Delta} = \mu, \qquad \frac{\partial \Delta}{\partial \mu} = \tilde{\rho}(\mu), \tag{9.9}$$

[b] This statement is true perturbatively. Nonperturbatively, due to eigenvalue tunnelling (i.e. D-instantons), for $\mu > 0$ there is both an asymmetric filling of the two sides and an exponentially small flux from one side to the other; for $\mu < 0$ there is both a net flux of eigenvalues in one direction, and an exponentially small asymmetry in the filling of the two sides.

where

$$\frac{\partial\tilde{\rho}}{\partial\mu} = \frac{\sqrt{\alpha'}}{\pi\mu}\mathrm{Im}\int_0^\infty dt e^{-it}\,\frac{t/(2\beta\mu\sqrt{\alpha'})}{\sinh[t/(2\beta\mu\sqrt{\alpha'})]}\,\frac{t/(2\beta\mu R)}{\sinh[t/(2\beta\mu R)]}\,. \tag{9.10}$$

Equations (9.9) are suggestive of a Legendre transform where μ is a variable conjugate to Δ. Let us define the Legendre transform of $\mathcal{F}(\Delta)$,

$$\Gamma(\mu) = \Delta\mu - \mathcal{F}(\Delta)\,, \tag{9.11}$$

which satisfies [66]

$$\frac{\partial^2\Gamma(\mu)}{\partial\mu^2} = \tilde{\rho}(\mu)\,. \tag{9.12}$$

This Legendre transformation arises because the sum over surfaces corresponds to fixed N while Γ is the thermodynamic potential for fixed chemical potential μ. Another interpretation of this Legendre transform is that it arises in the $c=1$ matrix model with trace-squared terms [76], which was argued to describe Liouville theory perturbed by the operator $\mu e^{2\phi}$ (as opposed to the operator $\phi e^{2\phi}$). In such a model the sum over surfaces is given by $I=2\pi R\beta^2\Gamma(\mu)$, and μ is therefore naturally interpreted as the cosmological constant. The double scaling limit is taken as $\mu\to 0$, $\beta\to\infty$, with $\beta\mu$ kept fixed. This double-scaled parameter, proportional to $1/g_s$, corresponds to the parameter denoted by μ in Liouville theory. Since the double scaling limit is taken at fixed $\beta\mu$ and at infinite N, we do not need to be concerned about the consistency of adding a finite number of fermions to the system. Creation of a D-brane, strictly speaking, corresponds to exciting a fermion from the Fermi level to the top of the potential. Alternatively, we may simply add a fermion near the top of the potential with the understanding that it is borrowed from the infinite pool at large λ.

Both (9.10) and the sum over surfaces are symmetric under the T-duality

$$R\to\frac{\alpha'}{R}\,,\qquad \beta\mu\to\frac{R}{\sqrt{\alpha'}}\,\beta\mu\,. \tag{9.13}$$

In particular, if we divide the answer by 2, which corresponds to filling the potential on one side, then we find the answer matches the bosonic Liouville theory path integral calculation of Section 4. Thus the fact that in the bosonic theory the Fermi sea fills only one side of the potential is crucial for obtaining exactly the same factor as in the Liouville calculation.

9.2. *The type 0B model at finite temperature*

According to our conjecture, in order to adapt the matrix model to describe the type 0B theory, all we need to do is send $\alpha' \to 2\alpha'$ and also fill both sides of the potential with fermions up to the same Fermi level $-\mu$. Therefore, the non-perturbative sum over surfaces is given by $I_B = 2\pi R\beta^2\Gamma_B(\mu)$, where

$$\frac{\partial^2\Gamma_B(\mu)}{\partial\mu^2} = \tilde{\rho}_B(\mu)\,, \tag{9.14}$$

and

$$\frac{\partial\tilde{\rho}_B}{\partial\mu} = \frac{\sqrt{2\alpha'}}{\pi\mu}\mathrm{Im}\int_0^\infty dte^{-it}\,\frac{t/(2\beta\mu\sqrt{2\alpha'})}{\sinh[t/(2\beta\mu\sqrt{2\alpha'})]}\,\frac{t/(2\beta\mu R)}{\sinh[t/(2\beta\mu R)]}\,. \tag{9.15}$$

This is the exact non-perturbative expression. See Appendix B for a derivation.

After these modifications, the term in the matrix model free energy that should correspond to the sum over tori in the NSR string becomes

$$\mathcal{Z}_{MB} = -\frac{\ln|\mu|}{12}\left(\frac{R}{\sqrt{2\alpha'}} + \frac{\sqrt{2\alpha'}}{R}\right) \tag{9.16}$$

which agrees with (4.8). The theory has two types of excitations: those symmetric under the Z_2 will be identified with states in the NS-NS sector, and those antisymmetric under the Z_2 with states in the R-R sector.

The winding operators are related to inserting Wilson lines in the matrix quantum mechanics description. In the type 0B model we have a single $U(N)$ gauge group and correspondingly we have only one type of winding modes in the CFT (the ones coming from the NS sector). On the other hand, in the type 0A matrix model we have a $U(N)\times U(N)$ gauge group and two possible Wilson lines. The symmetric and antisymmetric combination of Wilson lines corresponds to the NS and R-R winding modes that we have in the 0A string theory.

10. The Matrix Model for Type 0A

The type 0B matrix model contains a gauge field and a scalar: (A_0, M), both of which are hermitian matrices. This is the field theory on a set of unstable zero-branes of the 0B theory. The Z_2 action $(-1)^{\mathbf{F}_L}$ on the 0B theory maps $(A_0, M) \to (A_0, -M)$. Quotienting by this symmetry, we get the stable branes of type 0A theory, which contain no tachyon. In addition we can embed the Z_2 into the gauge group [20]. Suppose that we start with

$2N+q$ D0-branes in the 0B theory. A particular embedding would lead to the identification

$$(A_0, M) \sim \tilde{\sigma}_3(A_0, -M)\tilde{\sigma}_3\,, \tag{10.1}$$

where $\tilde{\sigma}_3$ is $diag(1, ..., 1; -1, ..., -1)$, with N 1's and $(N+q)$ -1's. The states that are invariant have the form

$$A_0 = \begin{pmatrix} A_0 & 0 \\ 0 & \tilde{A}_0 \end{pmatrix}, \qquad \phi = \begin{pmatrix} 0 & t \\ t^\dagger & 0 \end{pmatrix}, \tag{10.2}$$

where A_0 , $\tilde{A}_0$ are hermitian and t is a complex matrix. In other words, the gauge group is $U(N)\times U(N+q)$ with t in the bifundamental.

This is the matrix model that gives the 0A theory. It is a $U(N)\times U(N+q)$ theory with a tachyon field t in the bifundamental. The Lagrangian is

$$L = \mathrm{Tr}\big[(D_0 t)^\dagger D_0 t + \frac{1}{2\alpha'}\, t^\dagger t\big]\,; \tag{10.3}$$

the mass of the tachyon is still $m^2 = -1/(2\alpha')$ as in all fermionic string theories. Complex square and rectangular matrix models have been extensively studied. For a recent paper and a list of earlier references see Ref. [77].

This model can be thought of as having $N+q$ D-branes and N anti-D-branes. Therefore, it describes the type 0A background with q D0-branes.

The fact that the ground ring generators $(q \pm p)e^{\mp t}$ (3.14) , (3.32) are projected out in the worldsheet description of type 0A is consistent with the fact that the eigenvalues themselves are not gauge invariant. Instead, we must form products $\big[q\bar{q} + p\bar{p} \pm (q\bar{p} + p\bar{q})\big]e^{\mp 2t}$ and $q\bar{q} - p\bar{p}$, which are independent of the phase of the eigenvalues; these correspond to the NS sector generators of the type 0A ground ring $\mathcal{O}_{1,3}$, $\mathcal{O}_{3,1}$ and $\mathcal{O}_{2,2}$ discussed in Section 3.

In order to analyze this model it is instructive to look first at the case $N=1$, $q=0$. Here only the difference of gauge fields couples to t and this just removes the phase of the field t. It is convenient however to quantize the complex field t on the plane and then impose the condition that its wavefunctions are $U(1)$ invariant. This $U(1)$ is the $U(1)$ of angular momentum on the plane. We have two harmonic oscillators of frequency $w^2 = m^2$. Suppose for a moment that w^2 were positive. In this case we can quantize the system in terms of creation and annihilation operators $a_\pm$ with definite angular momentum. States with zero angular momentum have the same number of $a_+^\dagger$ and $a_-^\dagger$. So the spectrum of $U(1)$ invariant states is given by $\epsilon_n = w(1+2n)$. The important point to notice is that this gives a result that is a factor of two bigger than the result we would have obtained in the corresponding computation for the $N=1$ hermitian matrix model.

Now let us consider the case $N>1$. For simplicity we first continue to take $q=0$. As discussed in [77–79] we can first gauge fix the matrix $t(\tau)$ to a real diagonal matrix with positive entries at every time τ. The ghosts of this gauge fixing lead to a measure factor

$$\prod_i d\lambda_i \lambda_i \prod_{i<j} \left(\lambda_i^2 - \lambda_j^2\right)^2 \tag{10.4}$$

at every time τ. This measure factor implies that we have N fermions, each of them moving on a plane with zero angular momentum. The λ_i is just the radius in each plane. The integral over the vector fields $A_0(\tau)$ almost completely cancels this measure factor. More precisely, if we discretize the time direction, the measure factor (10.4) from the ghosts appears on the sites; the gauge fields are link variables, and integrating over them cancels the measure factor (10.4) everywhere except at the end points of the time evolution. We end up with a factor of

$$\prod_i \sqrt{\lambda_i} \prod_{i<j} \left(\lambda_i^2 - \lambda_j^2\right) \tag{10.5}$$

in the wave function in the initial and final state. The factor of $\sqrt{\lambda_i}$ is characteristic of motion in two dimensions and the second product in (10.5) makes the eigenvalues fermions. In other words, the wavefunction

$$\chi = \prod_{i<j} \left(\lambda_i^2 - \lambda_j^2\right) \Psi$$

obeys the single particle equation

$$\left(-\frac{1}{2\lambda_i} \frac{\partial}{\partial \lambda_i} \lambda_i \frac{\partial}{\partial \lambda_i} - \frac{1}{4\alpha'} \lambda_i^2\right) \chi = E\chi\,. \tag{10.6}$$

Let us now consider $q>0$. As in Ref. [77–79], Equation (10.4) becomes

$$\prod_i d\lambda_i \lambda_i^{1+2q} \prod_{i<j} \left(\lambda_i^2 - \lambda_j^2\right)^2. \tag{10.7}$$

Again, integrating out the gauge fields cancels this measure factor except at the end points of the time evolution. The initial and final wave functions have a factor of

$$\lambda_i^{\frac{1}{2}+q} \prod_{i<j} \left(\lambda_i^2 - \lambda_j^2\right).$$

The product again turns the eigenvalues into fermions. The first factor $\lambda_i^{\frac{1}{2}+q}$ can be given two different physical interpretations. First, we can think of λ_i as being the radius of a motion in $2+2q$ dimensions. Alternatively, as for $q=0$, we can keep the motion in two dimensions, but state that the angular momentum is not zero but it is instead q. Mathematically, the extra factor of λ_i^q has the effect of pushing the eigenvalues away from the origin. The two different physical interpretations (higher dimensional motion or two-dimensional motion with nonzero angular momentum) have the same effect.

Now the wavefunction $\chi = \prod_{i<j} \left(\lambda_i^2 - \lambda_j^2\right) \Psi$ obeys the single particle equation

$$\begin{aligned}
&\Big(-\frac{1}{2\lambda_i^{1+2q}} \frac{\partial}{\partial \lambda_i} \lambda_i^{1+2q} \frac{\partial}{\partial \lambda_i} - \frac{1}{4\alpha'} \lambda_i^2\Big) \lambda^{-q} \chi = E \lambda^{-q} \chi \,, \\
&\Big(-\frac{1}{2\lambda_i} \frac{\partial}{\partial \lambda_i} \lambda_i \frac{\partial}{\partial \lambda_i} + \frac{q^2}{\lambda_i^2} - \frac{1}{4\alpha'} \lambda_i^2\Big) \chi = E \chi \,,
\end{aligned} \tag{10.8}$$

where the first (second) equations are more natural in the first (second) physical interpretation above.

The system of N eigenvalues moving in two dimensions with angular momentum q also arises from another matrix model. We can start with a $U(N) \times U(N)$ matrix model with bifundamentals t as in (10.3) and add a Chern-Simons term

$$q \int d\tau \, \mathrm{Tr}(A - \tilde{A}) \,, \tag{10.9}$$

where A and $\tilde{A}$ are the gauge fields of the two $U(N)$ factors. We can gauge fix t to a diagonal matrix with eigenvalues λ_i, but we do not yet gauge fix the phases of λ_i. As above, the measure factor is

$$\Delta^2 = \prod_{i<j\leq N} \left(|\lambda_i|^2 - |\lambda_j|^2\right)^2$$

and it turns the eigenvalues into fermions. We integrate out all the gauge fields but keep the diagonal elements of $A - \tilde{A}$ and denote them by a_i. This leads to the Lagrangian

$$\sum_i \Big(\big|(\partial_\tau + i a_i)\lambda_i\big|^2 + q a_i - V\big(|\lambda_i|^2\big) \Big) \,. \tag{10.10}$$

We now fix axial gauge $a_i = 0$. Gauss' law which is the equation of motion of a_i, states that the angular momentum of λ_i is q. Therefore, the resulting quantum mechanics of the eigenvalues is exactly as we found above when we started with the $U(N)\times U(N+q)$ gauge theory.

We conclude that a system of q D0-branes in the 0A theory can be described either by a $U(N)\times U(N+q)$ quiver theory or by a $U(N)\times U(N)$ quiver theory with a Chern-Simons term (10.9) with coefficient q. The latter construction makes it clear that the flux due to the q D0-branes is represented in the system by the operator $q\int d\tau\,\mathrm{Tr}\big(A-\tilde{A}\big)$. This coupling is similar to the way the flux is introduced in the spacetime Lagrangian (8.2) $q\int d\tau d\phi F^{(-)}_{\tau\phi} = q\int d\tau\big(A^{(-)}_\tau(\phi{=}\infty) - A^{(-)}_\tau(\phi{=}-\infty)\big)$.

10.1. *The Type 0A Theory at Finite Temperature*

The free energy for this case can be computed as reviewed around (9.10). We use the trick reviewed in Ref. [1], which consists in doing the computation for a right-side up harmonic oscillator and then reversing w^2. The only difference relative to the type 0B matrix model is the doubling of energies. This amounts to replacing $\beta\mu\to\beta\mu/2$ in the first fraction in the integrand of (9.15), and introducing an overall factor of $1/2$. Using this trick it is possible to compute the thermal partition function for the $q=0$ case. The general q case can be treated by computing the exact density of states as is spelled out in Appendix B. This gives

$$\frac{\partial^2\Gamma_A(\mu)}{\partial\mu^2} = \tilde{\rho}_A(\mu)\,, \tag{10.11}$$

where

$$\begin{aligned}\frac{\partial\tilde{\rho}_A}{\partial\mu} = \frac{\sqrt{\alpha'/2}}{\pi\mu}\,\mathrm{Im}\int_0^\infty dt\, e^{-it}\,\frac{t/(2\beta\mu\sqrt{\alpha'/2})}{\sinh\big[t/(2\beta\mu\sqrt{\alpha'/2})\big]}\,\frac{t/(2\beta\mu R)}{\sinh\big[t/(2\beta\mu R)\big]}\\ \times e^{-qt/(2\beta|\mu|\sqrt{\alpha'/2})}\,.\end{aligned} \tag{10.12}$$

Again, this is the exact non-perturbative expression. The one-loop term is

$$\mathcal{Z}_{MA} = \ln|\mu|\Big[-\frac{1}{24}\Big(4\,\frac{R}{\sqrt{2\alpha'}} + \frac{\sqrt{2\alpha'}}{R}\Big) + \frac{1}{2}\,q^2\frac{R}{\sqrt{2\alpha'}}\Big]\,. \tag{10.13}$$

The term proportional to q^2 agrees precisely with the contribution of the R-R field strength to the ground state energy of the system (8.4), after we multiply by $(2\pi R)$. We also see that the $q=0$ torus contribution agrees with the result of the continuum calculation (4.8). It is self-dual under $R\to\alpha'/(2R)$. This is actually a duality of the full $q=0$ answer, provided that we also change the coupling $\beta\mu\to\big(2R/\sqrt{2\alpha'}\,\big)\beta\mu$.

An important check on the matrix models is that T-duality along the compact direction relates 0A and 0B theories. Indeed, for $q=0$, the full Γ_A is obtained from Γ_B given in (9.14), (9.15) through the T-duality transformation $R\to\alpha'/R$, $\beta\mu\to\left(R/\sqrt{2\alpha'}\right)\beta\mu$.

An amusing limit that one could take is the 't Hooft limit where

$$q\to\infty\,,\quad |\mu|\to\infty\,,\qquad \lambda=\frac{q}{2\beta|\mu|\sqrt{\alpha'/2}}=\text{fixed}\,. \tag{10.14}$$

In this case the free energy (10.12) becomes

$$\partial_w^2\big(2\pi R\Gamma\big)=-\frac{R}{\sqrt{\alpha'/2}}\,q^2\frac{1}{4}\ln\big(1+w^2\big)\,, \tag{10.15}$$

where $w=1/\lambda$. It scales as q^2 as we expect in the 't Hooft limit. It would be nice to understand what the dual string background is. It would also be nice to see if one can write a decoupled matrix model in the limit (10.14). We could also rescale R, at the same time that we take (10.14) and then we get a more interesting function of R in the limit. Setting $\hat{R}=2\beta|\mu|R$ and keeping $\hat{R}$ fixed in the limit we obtain

$$\lambda\partial_\lambda\tilde{\rho}=-\frac{\sqrt{\alpha'/2}}{\pi}\,\mathrm{Im}\int_0^\infty dt\,e^{-it}\frac{t/\hat{R}}{\sinh(t/\hat{R})}\,e^{-\lambda t}\,. \tag{10.16}$$

There are various other interesting limits that will probably yield interesting relationships.

10.2. *Affine A Theory*

In Subsection 4.3 we discussed the theories that arise by acting with $(-1)^{\mathbf{F}_L}$ when we go around the circle, where $\mathbf{F}_L$ is the spacetime left-fermion number. In the 0A matrix model, this corresponds to the operation exchanging the two gauge groups. So let us set $q=0$. When we go around the circle we impose the boundary condition

$$A_0(\tau+\pi R)=\tilde{A}_0(\tau)\,,\quad \tilde{A}_0(\tau+\pi R)=A_0(\tau)\,,\quad t(\tau+\pi R)=t^\dagger(\tau)\,. \tag{10.17}$$

We can compute the partition function by imposing the $\dot{A}_0=\dot{\tilde{A}}_0=0$ gauge. Then we are left only with the zero mode of $A_0=\tilde{A}_0$ by (10.17). We then expand the t field in Fourier modes $t\sim\sum_n t_n e^{int/R}$. Then (10.17) implies that the even modes are hermitian and the odd modes are anti-hermitian, $t_n^\dagger=(-1)^n t_{-n}$. So the computation has the same structure as the computation we would do if we were doing the 0B model. So we get the same as the 0B answer. In particular the term corresponding to the one loop string amplitude agrees with (4.20).

11. D-brane Decay

In Ref. [8], it was argued that the D-brane corresponding to $(1,1)$ ZZ boundary conditions and Neumann boundary conditions in the $c=1$ direction x, with a boundary perturbation

$$\delta S = \lambda \oint d\sigma \cos x \,, \tag{11.1}$$

corresponds to an eigenvalue in the matrix model executing the motion[a]

$$q = \sqrt{\mu} \sin \pi\lambda \cos x \,. \tag{11.2}$$

The Minkowski analog of this solution (the rolling tachyon with $x = it$) corresponds to an eigenvalue coming in from $q = \infty$, reaching a turning point at $q = \sqrt{\mu} \sin \pi\lambda$, and then going back out to infinity.

Following the discussion (and conventions) of Section 3, this can now be verified in Liouville theory by computing the (normalized) expectation values of the ground ring generators $\mathcal{O}_{1,2}$ and $\mathcal{O}_{1,2}$ in this state,[b] and comparing to (3.14). Consider for example $\langle \mathcal{O}_{1,2} \rangle$. The matter part of the expectation value is [80]

$$\left\langle V_{1,2}^{(M)}(z) \right\rangle = \left\langle e^{ix(z)} \right\rangle = \frac{\sin \pi\lambda}{|z-\bar{z}|^{1/2}} \;; \tag{11.3}$$

the Liouville expectation value is [14]

$$\left\langle V_{-\frac{b}{2}}(z) \right\rangle = \frac{U(-b/2)}{|z-\bar{z}|^{2\Delta(-b/2)}} \;, \tag{11.4}$$

where

$$U(-b/2) = \mu^{1/2} \frac{\Gamma(2+b^2)\Gamma(1+\frac{1}{b^2})}{\Gamma(2+2b^2)\Gamma(2+\frac{1}{b^2})} = \frac{1}{6}\sqrt{\mu} \;; \tag{11.5}$$

the last equality holds in the limit $b \to 1$. The one point function of $\mathcal{O}_{1,2}$ has two contributions,

$$\left\langle \mathcal{O}_{1,2} \right\rangle = \left\langle c\, b\, \bar{c}\, \bar{b}\, V_{-\frac{b}{2}} e^{ix} \right\rangle + \frac{1}{b^4} \left\langle \left| L_{-1}^{(L)} - L_{-1}^{(M)} \right|^2 V_{-\frac{b}{2}} e^{ix} \right\rangle . \tag{11.6}$$

[a] The relative factor of two in the coefficient of μ with respect to the corresponding formula in Ref. [8] is due to the fact that we normalize the matrix model Hamiltonian differently; see (3.15).

[b] An alternative way of thinking about this calculation, in analogy to (3.22), is that one is computing the limit $z \to \bar{z}$ of the operator $\mathcal{O}_{1,2}$. In this limit, the operator collides with its image and we are computing the coefficient of the identity in this OPE.

Evaluating the two terms using Eqs. (11.3) − (11.5), we find

$$\langle\mathcal{O}_{1,2}\rangle = \Big(1-\frac{28}{4}\Big)U\Big(-\frac{b}{2}\Big)\sin\pi\lambda = -\sqrt{\mu}\sin\pi\lambda\,. \tag{11.7}$$

A similar calculation leads to the same result for $\langle\mathcal{O}_{2,1}\rangle$. Using the map (3.14) to relate these quantities to matrix model variables, we find

$$q = -\sqrt{\mu}\sin\pi\lambda\cosh t\,, \tag{11.8}$$

or after analytic continuation $t\to ix$, (11.2) (up to a transformation $q\to -q$). Comparing to (3.15), we see that the energy of the solution (as measured from the top of the potential) is $E=-\mu\sin^2\pi\lambda$. The filled Fermi sea corresponds to $\lambda=\frac{1}{2}$.

Thus in the bosonic string the ZZ $(1,1)$ D-brane with a marginal perturbation (11.1) corresponds to an eigenvalue rolling according to (11.2), or its Minkowski continuation. We next discuss the analogous question in the type 0B case.

11.1. *D-brane Decay in Type 0B*

We would like to compute $\langle\mathcal{O}_{1,2}\rangle$ on the disk (where $\mathcal{O}_{1,2}$ is given in Eq. (3.32)), with ZZ-like boundary conditions for Liouville, and the supersymmetric analog

$$\lambda\oint dx d\theta_t\,\cos X \tag{11.9}$$

of the "rolling tachyon" (11.1) for x. We adopt the conventions of Section 3. It is useful to think of the calculation as the limit of the operator $\mathcal{O}_{1,2}(z,\bar z)$ as $z\to\bar z$. In this limit, $\mathcal{O}_{1,2}$ collides with its image and makes a boundary perturbation. We are interested in the coefficient of the identity in this OPE.

The limit $z\to\bar z$ of $\mathcal{O}_{1,2}$ receives contributions only from the cross-terms on the first line of (3.32). One has

$$\lim_{z\to\bar z}\mathcal{O}_{1,2}(z)\simeq -2\,\frac{c}{\sqrt{2}}\,\partial\xi e^{-\frac{3\varphi}{2}-\frac{i}{2}H}\;e^{-\frac{\bar\varphi}{2}+\frac{i}{2}\bar H}\;e^{\frac{i}{2}x-\frac{1}{2}\phi}\,. \tag{11.10}$$

The OPE of the matter part of the CFT gives a factor of $sin(\pi\lambda)$ [81], while the Liouville contribution is $U^{(R)}(\alpha=-1/2)$, where [68]

$$U^{(R)}(\alpha) = \frac{(2\mu)^{-\alpha}}{\big(\Gamma(\frac{3}{2}-\alpha)\big)^2}\,. \tag{11.11}$$

Thus, we conclude that

$$\lim_{z\to\bar z}\mathcal{O}_{1,2}(z,\bar z) = \sqrt{2}c\,\partial\xi e^{-2\varphi}\sin\pi\lambda\sqrt{2\mu}\,. \tag{11.12}$$

Just like in the discussion following (3.39), the boundary operator $c\partial\xi e^{-2\varphi}$ is a picture changed version of the identity operator.[c] Finally, we have

$$\langle \mathcal{O}_{1,2} \rangle = \sqrt{\mu} \sin \pi\lambda \ . \tag{11.13}$$

Again, one finds the same answer for $\langle \mathcal{O}_{2,1} \rangle$, and by using the identification of $\mathcal{O}_{1,2}$, $\mathcal{O}_{2,1}$ with matrix model variables, one sees that the D-branes in question correspond to rolling of eigenvalues either to the left or to the right of the top of the potential,

$$q = \sqrt{\mu} \sin \pi\lambda \, \cosh t \,, \tag{11.14}$$

with the direction of rolling correlated with the sign of λ.

11.2. *Closed String Radiation from the Decay*

We can also compute the radiation of closed strings from the decaying D-brane, analogous to the computation in Ref. [8].[d] The unnormalized super-Liouville disk one point function of the observables (3.27) on $(1,1)$-branes is given by (7.4) (up to a numerical factor independent of ν and b). Consider first the NS sector; this one point function is

$$\langle N_\alpha \rangle = \frac{\text{const}}{P\,\Gamma(-iP/b)\,\Gamma(-iP/b)}\, \mu^{-iP/b} \ . \tag{11.15}$$

For $b=1$ this becomes

$$\langle N_\alpha \rangle \sim \sinh \pi P \, \frac{\Gamma(iP)}{\Gamma(-iP)}\, \mu^{-iP} \, . \tag{11.16}$$

The one point function in the time direction is given by the Fourier transform of the NS decay profile

$$\begin{aligned} \rho_{NS}(t) &= \left[\frac{1}{1+\sin^2\!\lambda\, e^t} + \frac{1}{1+\sin^2\!\lambda\, e^{-t}} - 1 \right] \\ &= \frac{1-\sin^4\!\lambda}{1+\sin^4\!\lambda + 2\sin^2\!\lambda \cosh t} \ ; \end{aligned} \tag{11.17}$$

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[c] The picture changed version of (11.12) is half of what one naively gets by looking at the coefficient of $1/(z-w)$ in the OPE of $J_{\rm BRST}$ with ξ times (11.12).

[d] Recall again our conventions (2.14), (2.15), (2.16) for super-Liouville theory. In particular, recall that $\hat{c}=1$ matter corresponds to $Q=2$, $b=1$, $\hat{c}_L=9$. The cosmological constant goes as $e^{b\phi}$ and the normalizable operators have $\alpha=\frac{Q}{2}+iP$. These conventions correspond to the standard CFT conventions of $\alpha'=2$.

the result is

$$\left\langle e^{-iEt} \right\rangle = e^{-iE \log \sin^2 \pi\lambda} \frac{\pi}{\sinh \pi E} \,, \tag{11.18}$$

where we used the so called "Hartle-Hawking" contour [82], see also Ref. [83] . When we compute the one point function we can do it in the $(-1, -1)$ picture and we get the total amplitude

$$\mathcal{A}_{NS} \sim e^{-iE \log \sin^2 \pi\lambda} \frac{\Gamma(iP)}{\Gamma(-iP)} \, \mu^{-iP} \,. \tag{11.19}$$

The result is similar to the bosonic case.

One can also compute the Ramond one point functions. For the Liouville sector we find

$$\begin{aligned} \langle R_\alpha \rangle &= \frac{\text{const}}{\Gamma(-iP + 1/2)\,\Gamma(-iP + 1/2)} \, \mu^{-iP} \\ &\sim \cosh \pi P \, \frac{\Gamma(iP + 1/2)}{\Gamma(-iP + 1/2)} \, \mu^{-iP} \,, \end{aligned} \tag{11.20}$$

where R_α is defined in (3.27). For the time direction, the Ramond sector amplitude is given by the Fourier transform of the Ramond decay profile [81]

$$\rho_R(t) = \sin \pi\lambda \left[\frac{e^{t/2}}{1 + \sin^2 \pi\lambda \, e^t} + \frac{e^{-t/2}}{1 - \sin^2 \pi\lambda \, e^{-t}} \right] . \tag{11.21}$$

This Fourier transform is given by

$$\frac{\sin \pi\lambda}{|\sin \pi\lambda|} \, e^{-iE \log \sin^2 \pi\lambda} \, \frac{1}{\cosh \pi E} \,. \tag{11.22}$$

Thus the final total amplitude is

$$\mathcal{A}_{RR} \sim \frac{\sin \pi\lambda}{|\sin \pi\lambda|} \, e^{-iE \log \sin^2 \pi\lambda} \, \frac{\Gamma(\frac{1}{2} + iP)}{\Gamma(\frac{1}{2} - iP)} \, \mu^{-iP} \,. \tag{11.23}$$

We have left undetermined the overall normalization of the amplitudes (11.19) , (11.23); these are computed in Appendix C. We see that when we express the amplitudes in terms of the fields $T_{R,L}(p)$ defined in (9.2)–(9.4), we find that if the tachyon rolls to the right ($\lambda > 0$) then only T_R radiation is produced, while if it rolls to the left ($\lambda < 0$), then only T_L radiation is produced.

11.3. *States Below the Fermi Level*[e]

We have argued that a single fermion eigenvalue whose energy lies between the Fermi surface and the top of the potential barrier can be described, in the bosonic string, in terms of a ZZ brane tensored with the matter boundary state of Refs. [80,81,84,85] with $0 \le \lambda \le 1/2$ (in the fermionic string there is an analogous construction, and $\lambda \in [-1/2\,, 1/2]$). These are fermions whose energies are[f]

$$E = \mu \cos^2 \pi\lambda\,; \tag{11.24}$$

(again, we assume $\mu > 0$). One can similarly describe fermions with energies above the oscillator barrier, $E > \mu$, by setting $\lambda = is$, and continuing $t \to t + i\pi/2$; see Refs. [84, 85]. Under this analytic continuation, the decay profiles (11.17) , (11.21) remain real, and the trajectory (11.14) given by the ground ring computation is the correct one.

Continuation of λ to complex values amounts to performing a rotation of the left and right movers in the time direction by an element of $SL(2, C)$. This operation was shown in Ref. [86] to lead to boundary states obeying the Cardy condition. For generic $SL(2, C)$ rotations, the annulus partition function of the Euclidean theory is complex; however, for $\lambda \in \left[0, \frac{1}{2}\right]$, and $\lambda \in \frac{1}{2} + i\mathbf{R}$, the annulus amplitude is real.[g] This suggests that these latter boundary states can be given a physical interpretation. Indeed, the energy (11.24) and classical motion (11.8) , (11.14) for these values of λ correspond to eigenvalue trajectories below the Fermi surface.

The D-brane states discussed above describe eigenvalues following various classical trajectories. These trajectories are sharply defined in the $g_s \to 0$ limit. At finite g_s, the phase space coordinates p and q cease to commute, and one should quantize the eigenvalue dynamics, and construct wavefunctions $\psi(q, t)$ for each energy. The wavefunctions associated to the trajectories correspond to the coefficients of creation operators of particles for $E > 0$, and annihilation operators of holes for $E < 0$, in the second quantized theory.

The conjugate wavefunctions are associated to the conjugate operators. The creation of a hole or the destruction of a particle carries the opposite sign of the energy. It is natural to mimic this effect in the classical limit by

[e] We thank A. Sen and L. Susskind for raising this question.

[f] For the discussion of this subsection, it will be convenient to measure energies relative to the Fermi surface.

[g] Again, for the fermionic string one should allow $\lambda \in \pm\frac{1}{2} + i\mathbf{R}$ to get trajectories on either side of the potential.

reversing the sign of the boundary state, which produces an extra minus sign in the expression for the energy (11.24). Thus the creation operators of holes are associated with the D-brane boundary states with $\lambda \in \pm\frac{1}{2}+i\mathbf{R}$; similarly, the annihilation operators of particles are associated with $\lambda \in [-1/2\,, 1/2]$. In both cases one puts a minus sign in the boundary state wavefunction.

The extra minus sign in the Cardy state implies that the one point functions for the tachyon and the R-R scalar will have a minus sign compared to (11.19), (11.23). This is precisely what we need since for example the branes for real λ correspond to fermions $e^{i2\sqrt{\pi}\,T_{L,R}}$ while the holes for $\lambda \in \pm\frac{1}{2}+i\mathbf{R}$ will give us $e^{-i2\sqrt{\pi}\,T_{L,R}}$.

It is amusing to note that if we consider a standard brane and the brane that results from reversing the sign of its boundary state, then the action for the open strings on the wrong sign brane has an overall minus sign and the open strings stretched between the branes become fermions, since the one loop diagram changes sign. Perhaps it means that the gauge theory on N branes with the correct sign and M branes with the wrong sign is based on the supergroup $U(N|M)$.[h] The holes indeed have negative energies and they "fill up" the region above the Fermi sea. Only holes below the Fermi sea are allowed physical excitations.

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[h] Such theories were studied in Ref. [87] in the context of topological strings, where they were interpreted as branes and anti-branes. Here we interpret them as fermions and holes. This should not be confused with the matrix model for type 0A theory introduced above, where all the kinetic terms are positive.

Appendix A. The Odd Spin Structure

In this Appendix we show how the answer (4.7) arises from an explicit path integral calculation. The odd spin structure $(r,s)=(0,0)$ is subtle due to the presence of supermoduli and fermionic and superghost zero modes. There are different ways to organize the calculation.

Appendix A.1. *Method 1*

We fix the superconformal gauge on the torus in the odd spin structure. There is a γ zero mode (and a $\bar\gamma$ zero mode) which is associated with the existence of a covariantly constant spinor on the torus and a β zero mode (and a $\bar\beta$ zero mode) which is associated to the fact that there is still a gravitino component that cannot be gauged away in the superconformal gauge.

In addition there are four fermion zero modes, two from each of the two Majorana fermions we have in the theory, χ, $\bar\chi$, ψ and $\bar\psi$.

The zero modes of the Liouville fermions ψ and $\bar\psi$ have a geometric interpretation. They are associated with the conformal Killing spinor on the torus, exactly like the γ and $\bar\gamma$ zero modes. Therefore, the "zero" due to the Liouville fermion zero modes exactly cancels the "infinity" due to the γ and $\bar\gamma$ zero modes. This is analogous to absorbing c [55, 88] and γ [89] zero modes associated with conformal Killing vectors and conformal Killing spinors in Liouville theory on the sphere.

The β and $\bar\beta$ zero modes lead to an insertion of the left and right moving supercharges

$$G(z)\bar G(\bar w) = \big(\chi\partial x(z) + \psi\partial\phi(z)\big)\big(\bar\chi\bar\partial x(\bar w) + \bar\psi\bar\partial\phi(\bar w)\big)\,, \qquad \text{(A.1)}$$

where ϕ is the Liouville field. The expectation value of the operator (A.1) is independent of z and $\bar w$ because a derivative with respect to them turns into a derivative with respect to the modular parameter τ which integrates to zero. The insertions (A.1) absorb the remaining χ and $\bar\chi$ fermion zero modes. Now that all the fermion zero modes have been absorbed, the fermion determinants can be easily computed. The remaining nontrivial part of the computation is proportional to

$$\big\langle \partial x(z)\bar\partial x(\bar w)\big\rangle_x\,, \qquad \text{(A.2)}$$

where the expectation value is in the functional integral over x only. This expectation value has two contributions. At separated points, it is $-\pi/\tau_2$, and there is a contact term proportional to $\delta^{(2)}(z-w)$. We set $z=w$ in (A.2) and integrate over z. Since the bosonic action is proportional to $\partial x\bar{\partial}x$,

$$\begin{aligned}\mathcal{Z}_{\text{odd}} &\sim \big\langle \partial x(z)\bar{\partial}x(\bar{w})\big\rangle_x = R\,\frac{\partial}{\partial R}\,\langle 1\rangle_x \\ &\sim R\,\frac{\partial}{\partial R}\,\ln|\mu|\Big(\frac{R}{\sqrt{\alpha'}}+\frac{\sqrt{\alpha'}}{R}\Big) = \ln|\mu|\Big(\frac{R}{\sqrt{\alpha'}}-\frac{\sqrt{\alpha'}}{R}\Big)\,.\end{aligned} \tag{A.3}$$

The overall normalization of $\mathcal{Z}_{\text{odd}}$ can be determined by matching its large R behavior with the value determined by the spectrum at infinite R. It follows that $\mathcal{Z}_{\text{odd}}$ has the value (4.7), as we conjectured.

Appendix A.2. *Method 2*

An alternative computation of $\mathcal{Z}_{\text{odd}}$, which relies heavily on the treatment of super-Liouville path integral in Ref. [64], is based on inserting a discrete state vertex operator in the $(-1,-1)$ picture,

$$e^{\varphi+\bar{\varphi}}\chi\bar{\chi}\,, \tag{A.4}$$

where e^{φ} can be thought of as $\delta(\gamma)$. On the one hand, this insertion will soak up the $\gamma,\bar{\gamma},\chi,\bar{\chi}$ zero modes in a natural way. On the other hand, since in the $(0,0)$ picture this operator is $R^2\partial x\bar{\partial}x$, the one-point function is simply $R^2(\partial/\partial R^2)$ acting on the torus path integral. This will allow us to extract the torus path integral in a simple way.

For the odd spin structure one cannot completely gauge away the gravitino field. The partition function may be written as an integral over the supermoduli space. Let us represent the supertorus as the quotient of the superspace (z,θ) under the supertranslations

$$(z,\theta)\sim(z{+}1\,,\,\theta)\sim(z{+}\tau{+}\lambda\theta\,,\,\theta{+}\lambda)\,, \tag{A.5}$$

where λ is the odd coordinate of the supermoduli space. A variation with respect to λ gives

$$\frac{\partial}{\partial\lambda}\,\big\langle Z\big\rangle = \big\langle\big(\sqrt{2\pi i}G_0 + 2\pi i\lambda(L_0-\hat{c}/16)\big)Z\big\rangle\,.$$

This equation can be easily integrated, and one finds that the shift into an odd direction of supermoduli space is generated by $e^{\sqrt{2\pi i}\lambda G_0}$. The matter partition function for the $(+,+)$ sector can be represented as the trace

$$\text{Tr}\; q^{L_0-\hat{c}_m/16}\,\xi^{\lambda G_0}\,\bar{q}^{\,\bar{L}_0-\hat{c}_m/16}\,\bar{\xi}^{\,\bar{\lambda}\bar{G}_0}\,(-1)^F\,,$$

where $\xi = e^{\sqrt{2\pi i}}$. Similar representations can be obtained for the super-Liouville and superghost partition functions. The bottom component of the trace ($\lambda=\bar{\lambda}=0$) is given by the Witten index. The top component determines the dependence on the supermoduli.

The dependence of the super-Liouville action on the supermodulus λ is

$$S_L = S(\Phi) - \frac{i}{2\pi\tau_2}\int d^2\sigma\big[\lambda\psi\partial\phi - \bar{\lambda}\bar{\psi}\bar{\partial}\phi\big] - \frac{\lambda\bar{\lambda}}{4\pi\tau_2^2}\int d^2\sigma\psi\bar{\psi}\,, \tag{A.6}$$

where $\psi(w)$ is the superpartner of $\phi(w)$. This is the standard super-Liouville action in the presence of constant gravitino background λ/τ_2.

Now consider a superconformal matter system coupled to the supergravity theory. We will calculate the fixed "area" genus one path integral, and subsequently Laplace transform it to obtain a function of Δ (this is another way to see the appearance of the volume factor V_L due to the Liouville zero mode). Let us first perform the path integral over the super-Liouville field Φ,

$$Z_L(A,\tau,\lambda) = \int d\phi_0 d\psi_0 d\bar{\psi}_0 (d\tilde{\Phi}) e^{-S}\delta\Big(\frac{1}{4\pi}\int d^2z d^2\theta O_{\min} e^{\beta_{\min}\Phi} - A\Big)\,,$$

where we have decomposed Φ into the zero modes and the remaining modes as

$$\Phi = \phi_0 + \theta\psi_0 + \bar{\theta}\bar{\psi}_0 + \tilde{\Phi}\,.$$

As in the bosonic Liouville model, the integration over the bosonic zero mode ϕ_0 removes the delta function that fixes the area and reduces the theory to free field path integrals,

$$Z_L\big(A,\tau,\lambda\big) = A^{-1}\beta_{\min}^{-1}\big(2\pi\sqrt{2\tau_2}\big)^{-1}\int d\psi_0 d\bar{\psi}_0 (d\tilde{\Phi}) e^{-S_L}\,.$$

Thus, $Z_L(A,\tau,\lambda) \sim W_L + \lambda\bar{\lambda}B(A,\tau)$, where W_L is the Witten index. For the super-Liouville model, W_L vanishes. In the path integral formalism, this is due to the integration over the fermion zero modes ψ_0 and $\bar{\psi}_0$ because $S(\Phi)$ does not depend on them. Therefore,

$$\begin{aligned} Z_L\big(A,\tau,\lambda\big) = {} & A^{-1}\beta_{\min}^{-1}\big(2\pi\sqrt{2\tau_2}\big)^{-1}\lambda\bar{\lambda}\int d\psi_0 d\bar{\psi}_0 (d\tilde{\Phi}) e^{-S(\tilde{\Phi})} \\ & \times \frac{1}{4\pi^2\tau_2^2}\Big(-\int d^2\sigma_1\psi\partial\phi\int d^2\sigma_2\bar{\psi}\bar{\partial}\phi + \pi\int d^2\sigma\psi\bar{\psi}\Big)\,. \end{aligned} \tag{A.7}$$

Now recall that

$$\big\langle\partial\phi(w_1)\bar{\partial}\phi(\bar{w}_2)\big\rangle = \pi\delta^2(\sigma_1-\sigma_2) - \frac{\pi}{\tau_2}\,. \tag{A.8}$$

Therefore, the contact part of the first term in Equation (A.7) completely cancels the second term in it. The remainder reduces to

$$Z_L\big(A,\tau,\lambda\big) = \frac{\lambda\bar{\lambda}}{4\pi^2 A|\beta_{\min}|(2\tau_2)^{3/2}}\int d\psi_0 d\bar{\psi}_0 \psi_0\bar{\psi}_0\int\big[d\tilde{\Phi}\big]e^{-S(\tilde{\Phi})}\,.$$

Comparing with the operator formalism, one finds that the integral over the fermionic zero mode should be normalized as $\int d\psi_0 \ \psi_0 = \sqrt{2\pi}$. Therefore, we find that

$$Z_L(A,\tau,\lambda) = -\frac{\lambda\bar{\lambda}}{|\beta_{\rm min}|A2\pi(2\tau_2)^{3/2}} \, .$$

We have soaked up the Liouville fermion zero modes. Also, since the Witten index of the super-Liouville theory is equal to zero, the Liouville path integral is proportional to $\lambda\bar{\lambda}$.

Now we couple the super-Liouville path integral to the $\hat{c}=1$ circle theory. To soak up the remaining zero modes, we insert the operator (A.4). In the resulting path integral, all the non-zero modes cancel, and the result is

$$\langle D \rangle \sim \int d\lambda d\bar{\lambda}\lambda\bar{\lambda}\, \frac{R}{\sqrt{\alpha'}}\, \frac{1}{4\pi\sqrt{2}A} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \sum_{n,m} \exp\left(-\frac{\pi R^2|n-m\tau|^2}{\alpha'\tau_2}\right) . \quad \text{(A.9)}$$

Performing the integral over supermoduli space, and integrating over the area, we find

$$\langle D \rangle = R^2\, \frac{\partial \mathcal{Z}_{\rm odd}}{\partial R^2} \sim -\frac{1}{12\sqrt{2}}\, \ln|\mu| \Big(\frac{R}{\sqrt{\alpha'}} + \frac{\sqrt{\alpha'}}{R} \Big) . \qquad \text{(A.10)}$$

It follows that $\mathcal{Z}_{\rm odd}$ is proportional to (4.7), as we conjectured.[a]

Appendix B. Density of States

In this appendix we compute directly the density of states. The approach is similar to methods developed in Refs. [6, 90, 91]. The inverted harmonic oscillator potentials that we consider throughout this paper have continuous spectra. The density of states has, therefore, an infinite volume factor and a finite piece. We are interested in the finite piece, given by

$$\rho(e) = \frac{1}{2\pi}\, \frac{\partial \delta(e)}{\partial e} \, , \qquad \text{(B.1)}$$

where δ is the phase shift for the scattering of a wave from the potential.

We now proceed to compute δ for the cases of interest. The wave equation that we need to solve is

$$-\frac{1}{\lambda^{d-1}}\partial_\lambda(\lambda^{d-1}\partial_\lambda\psi) - \frac{\lambda^2}{2\alpha'} = 2e\psi \, , \qquad \text{(B.2)}$$

[a] There could be an additive constant in $\mathcal{Z}_{\rm odd}$ since we are evaluating its derivative. However, it can be argued away because, from the perspective of the effective target space dynamics, the free energy should have the form $\mathcal{F}(T) = a + bT^2$; the additive constant would correspond to a term linear in T.

where $d=1$ for 0B and $d=2+2q$ for 0A.

Redefine variables $\lambda=(\alpha'/2)^{1/4}x$ and $a=-\sqrt{2\alpha'}e$. Then we get

$$\frac{1}{x^{d-1}}\,\partial_x(x^{d-1}\partial_x\psi)+\frac{x^2}{4}\,\psi=a\psi\,. \tag{B.3}$$

Writing $z=ix^2/2$ and $\psi=z^{-d/4}M$, one finds that M obeys the equation

$$\partial_z^2 M+\left(-\frac{1}{4}+\frac{i\frac{a}{2}}{z}+\frac{\frac{1}{4}-\kappa^2}{z^2}\right)M=0\,, \tag{B.4}$$

where

$$\kappa=\frac{1}{2}\left(\frac{d}{2}-1\right) \tag{B.5}$$

($\kappa=-1/4$ for type 0B, $\kappa=q/2$ for type 0A). Its solutions are the Whittaker functions $M_{i\frac{a}{2},\kappa}(z)$, $M_{i\frac{a}{2},-\kappa}(z)$. The phase shift for the Whittaker function is given by

$$e^{i\delta(a,\kappa)}=\frac{\Gamma\left(\kappa+\frac{1}{2}-i\frac{a}{2}\right)}{\Gamma\left(\kappa+\frac{1}{2}+i\frac{a}{2}\right)}\,, \tag{B.6}$$

where we have neglected possible a independent terms in the phase shift.

In the 0B case, the two solutions with $\kappa=\pm1/4$ correspond to even and odd wave functions. To get the total density of states we should sum the contributions from both solutions. Using gamma functions identities this gives

$$e^{i\delta_+ +i\delta_-}=\frac{\Gamma\left(\frac{1}{4}+\frac{1}{2}-i\frac{a}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}+i\frac{a}{2}\right)}\,\frac{\Gamma\left(-\frac{1}{4}+\frac{1}{2}-i\frac{a}{2}\right)}{\Gamma\left(-\frac{1}{4}+\frac{1}{2}+i\frac{a}{2}\right)}\sim\frac{\Gamma\left(\frac{1}{2}-ia\right)}{\Gamma\left(\frac{1}{2}+ia\right)}\,, \tag{B.7}$$

where in the last equality we neglected a term linear in a in the phase, which contributes just a constant to the density of states. The individual phase shifts $\delta_\pm$ for even and odd wave functions were computed in Ref. [90].

In the 0A case we find

$$e^{i\delta}=\frac{\Gamma\left(\frac{q}{2}+\frac{1}{2}-i\frac{a}{2}\right)}{\Gamma\left(\frac{q}{2}+\frac{1}{2}+i\frac{a}{2}\right)}\,. \tag{B.8}$$

Note that the 0A result for $q=0$ is the same as the 0B result (B.7) up to $a\to a/2$, which is a result we derived in an alternative way in the main text.

It is convenient to write the density of states for the 0A case as

$$\frac{1}{2\pi}\,\partial_a\delta=\frac{1}{4\pi}\int_0^\infty dt\,\sin\left(\frac{a}{2}\,t\right)\frac{t/2}{\sinh t/2}\,e^{-qt/2}\,. \tag{B.9}$$

After remembering the definitions of a and putting in the thermal density factors we can write the expressions for the density of states given in the main text. Remembering that (B.9) is an exact formula, we conclude that the formulae in the text are exact.

Appendix C. Normalization of Disk One-Point Functions

Let us review the bosonic string case first. In Ref. [8] it has been found that the emitted state has the form $|\Psi\rangle = e^{i\alpha\varphi_L(0)}|0\rangle$, where φ_L is the left moving part of a scalar field with the usual CFT normalization $\big(S = \frac{1}{8\pi}\int(\partial\varphi)^2\big)$. Our goal is to determine the value of the coefficient α .

The norm of the emitted state is

$$\langle\psi|\psi\rangle = e^{\alpha^2\langle\varphi_L(0^+)\varphi_L(0^-)\rangle} = e^{\alpha^2\log(1/\epsilon)}\,, \tag{C.1}$$

where ϵ is a short distance cutoff arising from $\varphi_L(x)\varphi_L(0)\sim-\log x$.

It was shown [82] that the log of the norm of the emitted state is equal to the one loop partition function for an array of D-instantons, where strings connect only instantons at positive Euclidean time with instantons at negative Euclidean time. The divergence comes from the D instantons closest to $t=0$. These are separated by a critical distance so that the stretched open strings are massless. This partition function is IR divergent in the open string channel:

$$\log\langle\psi|\psi\rangle = Z_1 = 2\int\frac{dt}{2t}\,\mathrm{Tr}\big[e^{-2\pi L_0}\big] = \int\frac{dt}{t}\Big(1-e^{-2\pi t}\Big)\sim\log\Big(\frac{1}{\epsilon}\Big)\,, \tag{C.2}$$

where the first factor of 2 comes from the two orientations of the stretched open strings. Note that in the open string string channel the cutoff is $t\lesssim 1/\epsilon$ since it arises from separating the D-instantons by an extra amount ϵ in Euclidean time.[b] Equating the divergence from the two points of view we get

$$\alpha^2 = 1\,. \tag{C.3}$$

Now let us consider the type 0 string. We have shown that the final state has the form

$$|\Psi_\pm\rangle = e^{i(\alpha_t\varphi_t\pm\alpha_v\varphi_v)}|0\rangle\,, \tag{C.4}$$

[b] So the total separation is $d=d_c+\epsilon$, where d_c is the critical distance. Then the exponent $-2\pi t$ in (C.2) becomes $2\pi t d^2/d_c^2\sim 2\pi(1+2\epsilon/d_c)$. Then the integral over t produces $\log(1/\epsilon)$. Note that even in 26 dimensions the one loop partition function for two D-instantons has a short distance singularity like $\log(1/\epsilon)$. This suggests that maybe even in 26 dimensions we could think of them as fermions in some sense.

where $\pm$ refers to the sign of λ, and $\varphi_{t,v}$ are the canonically normalized tachyon and R-R scalar fields. We are going to determine $\alpha_{t,v}$.

Let us start with an unstable D0-brane of 0B theory. An $SU(2)$ rotation on the boundary state yields a D(-1)-brane and an anti-D(-1)-brane separated by a critical distance. This configuration is again related to the computation of the norm of the state. More precisely, let us consider the inner product of the states $\langle\Psi_+|\Psi_-\rangle$. This is computed by considering two D(-1)-branes of the same charge. This does not give a divergence from the string theory point of view. (Note that this is a result valid for tachyon decay in 10 dimensions also). On the other hand, from (C.4), the coefficient of the divergence is of the form $\alpha_t^2 - \alpha_v^2$. This implies that $\alpha_t = \alpha_v$ (the sign is already taken into account in (C.4)).

Now let us compute the norm of $\langle\Psi_+|\Psi_+\rangle$. Using the expression in terms of free fields in (C.4) we find that it diverges with a coefficient equal to $\alpha_t^2 + \alpha_v^2$. In string theory this divergence appears when a D(-1) and an anti-D(-1) brane are separated by a critical distance. This divergence has the same form as in (C.2), since it comes from the ground state of open strings stretching between the branes (which is massless at this distance). We conclude that

$$\alpha_t = \alpha_v = \frac{1}{\sqrt{2}}\,. \tag{C.5}$$

Note that this computation also gives the coupling of the D-instanton to the Ramond-Ramond scalar. If we define the compactification radius to be the minimum allowed by the coupling to the D-instantons then we see that the scalar is at the self-dual radius. In other words, we are finding that the D-instantons contribute with a phase $e^{\pm iC/\sqrt{2}}$. This radius agrees with the claim that the zero mode of the R-R scalar corresponds to the relative phase of the left and right wavefunctions.

The above computation applies to type 0B theory. If we consider the 0A, then we find that in order to compute the norm we need to consider two unstable D(-1)-branes that are separated in the Euclidean time direction by a critical distance. Again we obtain the same divergence (C.2). On the other hand, here only one field is present, so the matching is the same as in the bosonic theory.

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