FUZZY SPACES, THE M(ATRIX) MODEL AND THE QUANTUM HALL EFFECT

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This is a short review of recent work on fuzzy spaces in their relation to the M(atrix) theory and the Quantum Hall Effect. We give an introduction to fuzzy spaces and how the limit of large matrices is obtained. The complex projective spaces $\mathbf{CP}^k$, and to a lesser extent spheres, are considered. The Quantum Hall Effect and the behavior of edge excitations of a droplet of fermions on these spaces and their relation to fuzzy spaces are also discussed.

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1. Introduction

It is a well known fact that in many quantum mechanical systems, as the occupation number becomes very large, the quantum theory can be approximated by a classical theory. Quantum observables which are linear hermitian operators on the Hilbert space can be approximated by functions on the classical phase space. Properties of classical functions on the phase space can thus be obtained as a limit of the quantum theory. This raises the possibility that one may consider the quantum Hilbert space and the algebra of operators on it as the fundamental entities for constructing a manifold, the classical (phase) space being only an approximation to it. Fuzzy spaces are a realization of this possibility [1–4]. They are defined by a sequence of triples, $(\mathcal{H}_N, Mat_N, \Delta_N)$, where Mat_N is the matrix algebra of $N \times N$ matrices which act on the N-dimensional Hilbert space $\mathcal{H}_N$, and Δ_N is a matrix analog of the Laplacian. The inner product on the matrix algebra is given by $\langle A, B\rangle = \frac{1}{N}\mathrm{Tr}(A^\dagger B)$. Such fuzzy spaces may be considered as a finite-state approximation to a smooth manifold M, which will be the classical phase space corresponding to $\mathcal{H}_N$ as $N \to \infty$. More specifically, the matrix algebra Mat_N approximates to the algebra of functions on a smooth manifold M as $N \to \infty$. The Laplacian Δ_N is needed to recover metrical and other geometrical properties of the manifold M. For example, information about the dimension of M is contained in the growth of the number of eigenvalues.

Fuzzy spaces are part of the more general framework of noncommutative geometry of A. Connes and others [1, 4, 5]. Noncommutative geometry is a generalization of ordinary geometry, motivated by the following observation. Consider the algebra of complex-valued square-integrable functions on a manifold M. The algebra of such functions with pointwise multiplication is a commutative C^*-algebra. It captures many of the geometrical features of the manifold M. Conversely, any commutative C^*-algebra can be represented by the algebra of functions on an appropriate space M. This leads to the idea that a noncommutative C^*-algebra may be considered as the analog of an "algebra of functions" on some noncommutative space. One can then develop properties of this noncommutative space in terms of the properties of the algebra. This is the basic idea.

More specifically, one introduces the notion of a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, where $\mathcal{A}$ is a noncommutative algebra with an involution, $\mathcal{H}$ is a Hilbert space on which we can realize the algebra $\mathcal{A}$ as bounded operators and $\mathcal{D}$ is a special operator which will characterize the geometry. In terms of such a spectral triple, there is a construction of the analog of differential calculus on a manifold. In particular, if $\mathcal{H}$ is the space of square-integrable spinor functions on a manifold M (technically, sections of the irreducible spinor bundle), $\mathcal{A}$ is the algebra of complex-valued smooth functions on M, and $\mathcal{D}$ is the Dirac operator on M for a particular metric and the Levi-Civita spin connection, then the usual differential calculus on M can be recovered from the spectral triple. (For further developments in physics along these lines, see [6].) In what follows, we shall be interested in fuzzy spaces where we use matrix algebras to approximate the algebra of functions on a manifold.

A number of fuzzy spaces have been constructed by now. A finite dimensional Hilbert space is obtained if one quantizes a classical phase space of finite volume. Thus, for manifolds M which have a symplectic structure, so that they can be considered as classical phase spaces, and have finite volume, we have a natural method of constructing fuzzy approximations to M. We quantize the phase space M and consider the algebra of matrices acting on the resultant Hilbert space.

A natural family of symplectic manifolds of finite volume are given by the co-adjoint orbits of a compact semisimple Lie group G. (In this case, there is no real distinction between co-adjoint and adjoint orbits. For quantization of co-adjoint orbits, see [7, 8].) One can quantize such spaces, at least when a Dirac-type quantization condition is satisfied, and the resulting Hilbert space corresponds to a unitary irreducible representation of the group G. In this way, we can construct fuzzy analogs of spaces which are the co-adjoint orbits. In the following, we will work through this strategy for the case of $\mathbf{CP}^k = SU(k+1)/U(k)$.

In this review, we will focus on fuzzy spaces, how they may appear as solutions to M(atrix) theory and their connection to generalizations of the Quantum Hall Effect. There is a considerable amount of interesting work on noncommutative spaces, particularly flat spaces, in which case one has infinite-dimensional matrices, and the properties of field theories on them. Such spaces can also arise in special limits of string theory. We will not discuss them here, since there are excellent reviews on the subject [9].

2. Quantizing $\mathbf{CP}^k$

2.1. *The action and the Hilbert space*

We start with some observations on $\mathbf{CP}^k$.[a] This is the complex projective space of complex dimension k and is given by a set of complex numbers

[a] Much of the material in this section is well known and can be found in many places; we will follow the presentation in [10, 11]. For an earlier work on coherent states on $\mathbf{CP}^k$ and related matters see [12].

u_α, $\alpha = 1, 2, ..., (k+1)$, with the identification $u_\alpha \sim \lambda u_\alpha$ for any nonzero complex number λ. We introduce a differential one-form given by

$$A(u) = -\frac{i}{2}\left[\frac{\bar{u}\cdot du - d\bar{u}\cdot u}{\bar{u}\cdot u}\right], \tag{2.1}$$

where $\bar{u}\cdot du = \bar{u}^\alpha du_\alpha$, etc. Notice that this form is not invariant under $u \rightarrow \lambda u$; in fact,

$$A(\lambda u) = A(u) \ + \ df , \tag{2.2}$$

where $f = -\frac{i}{2}\log(\lambda/\bar{\lambda})$. ($\lambda$ can, in general, be a function of the coordinates, $\bar{u}$, u.) This transformation law shows that the exterior derivative or curl of A is in fact invariant under $u \rightarrow \lambda u$; it is the Kähler two-form of $\mathbf{CP}^k$ and is given by

$$\begin{aligned}\Omega &\equiv dA \\ &= -i\left[\frac{d\bar{u}\cdot du}{\bar{u}\cdot u} - \frac{d\bar{u}\cdot u\ \bar{u}\cdot du}{(\bar{u}\cdot u)^2}\right] .\end{aligned} \tag{2.3}$$

Notice that Ω is closed, $d\Omega = 0$, but it is not exact, since the form A is not well-defined on the manifold. (We may say that Ω is an element of the second cohomology group of $\mathbf{CP}^k$.) The symplectic form we choose to quantize $\mathbf{CP}^k$ will be proportional to Ω.

The identification of u and λu shows that, by choosing λ appropriately, we may take u_α to be normalized so that $\bar{u}^\alpha u_\alpha = 1$. In this case, we can introduce local complex coordinates for the manifold by writing

$$u_\alpha = \frac{1}{\sqrt{1+\bar{z}\cdot z}}\begin{pmatrix} 1 \\ z_1 \\ .... \\ z_k \end{pmatrix} . \tag{2.4}$$

In the local coordinates $z, \bar{z}$, the two-form Ω has the form

$$\Omega = -i\left[\frac{d\bar{z}_i\ dz_i}{(1+\bar{z}\cdot z)} - \frac{d\bar{z}\cdot z\ \bar{z}\cdot dz}{(1+\bar{z}\cdot z)^2}\right] . \tag{2.5}$$

In terms of the normalized u's, a basis for functions on $\mathbf{CP}^k$ is then given by $\{\phi_l\}$, where l can take all integral values from zero to infinity, and

$$\phi_l = \bar{u}^{\alpha_1}\cdots\bar{u}^{\alpha_l}u_{\beta_1}\cdots u_{\beta_l} . \tag{2.6}$$

Notice that, for a fixed value of l, we have complete symmetry for all the upper indices corresponding to the $\bar{u}$'s and complete symmetry corresponding to the lower indices; further any contraction of indices corresponds to a lower value of l since $\bar{u} \cdot u = 1$. Thus the number of independent functions is given by

$$d(k,l) = \left[\frac{(k+l)!}{k!l!}\right]^2 - \left[\frac{(k+l-1)!}{k!(l-1)!}\right]^2 . \tag{2.7}$$

Since the traceless part of $\bar{u}^\alpha u_\beta$ transforms as the adjoint representation of $SU(k+1)$, we see that these functions are contained in the representations obtained by reduction of the products of the adjoint with itself.

As we have mentioned before, $\mathbf{CP}^k$ can also be considered as the coset space $SU(k+1)/U(k)$. The defining representation of $SU(k+1)$ is in terms of $(k+1) \times (k+1)$-matrices, which we may think of as acting on a $k+1$-dimensional vector space. Let t_A denote the generators of $SU(k+1)$ as matrices in this representation; we normalize them by $\mathrm{Tr}(t_A t_B) = \frac{1}{2}\delta_{AB}$. The generators of $SU(k) \subset U(k)$ are then given by t_j, $j = 1, 2, ..., k^2 - 1$ and are matrices which have zeros for the $k+1$-th row and column. The generator corresponding to the $U(1)$ direction of the subgroup $U(k)$ will be denoted by t_{k^2+2k}. As a matrix,

$$t_{k^2+2k} = \frac{1}{\sqrt{2k(k+1)}} \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & -k \end{bmatrix} . \tag{2.8}$$

We can use a general element of $SU(k+1)$, denoted by g, to parametrize $\mathbf{CP}^k$, by making the identification $g \sim gh$, where $h \in U(k)$. In terms of g, the one-form A is given by

$$\begin{aligned} A &= i\sqrt{\frac{2k}{k+1}}\ \mathrm{Tr}(t_{k^2+2k} g^{-1} dg) \\ &= -i g^*_{k+1,\alpha} dg_{\alpha,k+1} \ . \end{aligned} \tag{2.9}$$

If we identify the normalized u by $u_\alpha = g_{\alpha,k+1}$, we see that this agrees with (2.1). On the group element $g \in SU(k+1)$, considered as a $(k+1)\times(k+1)$ matrix, we can define left and right $SU(k+1)$ actions by

$$\hat{L}_A\, g = t_A\, g\,, \qquad \hat{R}_A\, g = g\, t_A\ . \tag{2.10}$$

If we denote the group parameters in g by φ^i, then we can write, in general,

$$g^{-1}dg = -it_A E^A_B\, d\varphi^B, \qquad dgg^{-1} = -it_A \tilde{E}^A_B\, d\varphi^B\ . \tag{2.11}$$

For functions of g, the right and left translations are represented by the differential operators

$$\hat{R}_A = i(E^{-1})^B_A \frac{\partial}{\partial\varphi^B}\ , \qquad \hat{L}_A = i(\tilde{E}^{-1})^B_A \frac{\partial}{\partial\varphi^B}\ . \tag{2.12}$$

The action which we shall quantize is given by

$$\mathcal{S} = in\sqrt{\frac{2k}{k+1}} \int dt \ \mathrm{Tr}\,(t_{k^2+2k} g^{-1} \dot{g}) \ . \tag{2.13}$$

Since $\mathrm{Tr}(t_{k^2+2k} t_j) = 0$ for generators t_j of $SU(k)$, this action is invariant under $g \to gh$, for $h \in SU(k)$. For the $U(1)$ transformations of the form $\exp(it_{k^2+2k}\theta)$, the action changes by a boundary term; the equations of motion are not affected and the classical theory is thus defined on $SU(k+1)/U(k)$, as needed. In quantizing the theory, we observe that there is no coordinate corresponding to the $SU(k)$ directions; the corresponding canonical momenta are constrained to be zero. Further, the canonical momentum corresponding to the angle θ for the $U(1)$ direction is given by $-nk/\sqrt{2k(k+1)}$. The states in the quantum theory must thus obey the conditions

$$\begin{aligned} \hat{R}_j \, \Psi &= 0\,, \qquad\qquad j = 1, \cdots, k^2-1\,, \\ \hat{R}_{k^2+2k} \, \Psi &= -nk \, \frac{1}{\sqrt{2k(k+1)}} \, \Psi \ . \end{aligned} \tag{2.14}$$

Another way to see the last condition is to notice that, under $g \to gh$, $h = \exp(it_{k^2+2k}\theta)$, the action changes by

$$\Delta\mathcal{S} = -\frac{nk}{\sqrt{2k(k+1)}} \, \Delta\theta\,, \tag{2.15}$$

leading to the requirement

$$\Psi(gh) = \Psi(g) \exp\left(-i \, \frac{nk}{\sqrt{2k(k+1)}} \, \theta \right) \tag{2.16}$$

for wave functions $\Psi(g)$. (This also shows that the wave functions are not genuine functions on $\mathbf{CP}^k$, but rather they are sections of a $U(1)$ bundle on $\mathbf{CP}^k$.)

We will now consider these wave functions in some more detail. A basis of functions on $SU(k+1)$ is given by the Wigner $\mathcal{D}$-functions which are the matrices corresponding to the group elements in a representation J

$$\mathcal{D}^{(J)}_{L,R}(g) = \langle J, L_i | \ \hat{g} \ | J, R_i \rangle\,, \tag{2.17}$$

where L_i, R_i stand for two sets of quantum numbers specifying the states on which the generators act, for left and right $SU(k+1)$ actions on g, respectively. The quantum numbers R_i in (2.17) must be constrained by the conditions (2.14). Thus the state $|J, R_i\rangle$ corresponds to an $SU(k)$ singlet with a specific $U(1)$ charge given by (2.14). In addition to these conditions, we must recall that g parametrizes the whole phase space, and so the Wigner functions depend on all phase space coordinates, not just half of them. To eliminate half of them, we first define the derivatives on the phase space.

There are $2k$ right generators of $SU(k+1)$ which are not in $U(k)$; these can be separated into t_{+i} which are of the raising type and t_{-i} which are of the lowering type. The derivatives on $\mathbf{CP}^k$ can be identified with these $\hat{R}_{\pm i}$ right rotations on g. The $\hat{R}_{-i}$ commute among themselves, as do the $\hat{R}_{+i}$'s; for the commutator between them we have

$$\begin{aligned}[\hat{R}_{+i}, \hat{R}_{-j}] &= i f_{ij}^{a}\ \hat{R}_a + \delta_{ij} \sqrt{\frac{2(k+1)}{k}}\ \ \hat{R}_{k^2+2k} \\ &= -n\ \delta_{ij}\end{aligned} \tag{2.18}$$

where $\hat{R}_a$ is a generator of $SU(k)$ transformations, f_{ij}^a are the appropriate structure constants, and in the second line, we give the values when acting on wave functions obeying (2.14). The derivatives thus split into conjugate pairs, analogously to the creation and annihilation operators. Because of this, the requirement that the wave functions should not depend on half of the phase space coordinates can be taken as

$$\hat{R}_{-i}\Psi = 0\ . \tag{2.19}$$

(In the geometric quantization approach to the action (2.13) and the construction of the Hilbert space, this is the so-called polarization condition; for general works on geometric quantization, see [7].) Based on the requirements (2.14) and (2.19), we see that a basis for the wave functions is given by the Wigner functions corresponding to irreducible $SU(k+1)$ representations J, where right state $|J, R_i\rangle$ is an $SU(k)$ singlet, has a value of $-nk/\sqrt{2k(k+1)}$ for $\hat{R}_{k^2+2k}$, and further it must be a highest weight state, so that (2.19) holds.

Representations which contain an $SU(k)$ singlet, with the appropriate value of $\hat{R}_{k^2+2k}$, can be labeled by two integers $J = (p, q)$ such that $p - q = n$. The highest weight condition requires $p = n,\quad q = 0$. These are completely symmetric representations. The dimension of this representation $J = (n, 0)$ is

$$dim J = \frac{(n+k)!}{n!k!} \equiv N\ , \tag{2.20}$$

where N expresses the number of states in the Hilbert space upon quantization. Notice that n has to be an integer for this procedure to go through; this requirement of integrality is the Dirac-type quantization condition mentioned earlier.

A basis for the wavefunctions on $\mathbf{CP}^k$ can thus be written as

$$\Psi_m^{(n)}(g) = \sqrt{N}\ \mathcal{D}_{m,-n}^{(n)}(g)\ . \tag{2.21}$$

We denote the fixed state for the right action on the Wigner function above as $-n$, indicating that the eigenvalue for $\hat{R}_{k^2+2k}$ is $-nk/\sqrt{2k(k+1)}$ as in (2.14). The index m specifies the state in this basis for the Hilbert space. The Wigner $\mathcal{D}$-functions obey the orthogonality condition

$$\int d\mu(g)\ \mathcal{D}^{*(J)}_{m,k}(g)\ \mathcal{D}^{(J')}_{m',k'}(g)\ =\ \delta^{JJ'}\ \frac{\delta_{mm'}\delta_{kk'}}{dimJ}\ , \tag{2.22}$$

where $d\mu(g)$ is the Haar measure on the group $SU(k+1)$; we normalize it so that $\int d\mu(g) = 1$. Specializing to our case, we see that the wave functions (2.21) are normalized since

$$\int d\mu(g)\ \mathcal{D}^{*(n)}_{m,-n}(g)\ \mathcal{D}^{(n)}_{m',-n}(g)\ =\ \frac{\delta_{mm'}}{N}\ . \tag{2.23}$$

Strictly speaking, the integration for the wave functions should be over the manifold $\mathbf{CP}^k = SU(k+1)/U(k)$. The measure should be the Haar measure for $SU(k+1)/U(k)$; however, we can integrate over the whole group since the integrand is $U(k)$ invariant.

We now use the notation $u_\alpha \equiv g_{\alpha,k+1}$; in terms of the u_α's, the Wigner $\mathcal{D}$-functions are of the form $\mathcal{D} \sim u_{\alpha_1}u_{\alpha_2}\cdots u_{\alpha_n}$. Using the local complex coordinates introduced in (2.4), the wave functions (2.21) are

$$\begin{aligned}\Psi^{(n)}_m &= \sqrt{N}\ \mathcal{D}^{(n)}_{m,-n}(g)\ ,\\ \mathcal{D}^{(n)}_{m,-n}(g) &= \left[\frac{n!}{i_1!i_2!\ldots i_k!(n-s)!}\right]^{\frac{1}{2}}\ \frac{z_1^{i_1}z_2^{i_2}\cdots z_k^{i_k}}{(1+\bar{z}\cdot z)^{\frac{n}{2}}}\ ,\\ s &= i_1+i_2+\cdots+i_k\ .\end{aligned} \tag{2.24}$$

Here $0 \leq i_l \leq n$, $0 \leq s \leq n$. The condition $\hat{R}_{-i}\mathcal{D}^{(n)}_{m,-n} = 0$ is a holomorphicity condition and this is reflected in the fact that the wave functions are holomorphic in z's, apart from certain overall factors. The states (2.24) are coherent states for $\mathbf{CP}^k$ [8]. The inner product for the Ψ's may be written in these coordinates as

$$\begin{aligned}\langle\Psi|\Psi'\rangle &= \int d\mu\ \Psi^*\ \Psi'\ ,\\ d\mu &= \frac{k!}{\pi^k}\frac{d^kzd^k\bar{z}}{(1+\bar{z}\cdot z)^{k+1}}\ .\end{aligned} \tag{2.25}$$

At this point, we are able to define more precisely what we mean by fuzzy $\mathbf{CP}^k$ [13–15]. Functions on fuzzy $\mathbf{CP}^k$ will correspond to matrices acting on the N-dimensional Hilbert space given by the basis (2.21). They are thus $N \times N$-matrices and the composition law is matrix multiplication.

The composition law is not commutative and so this corresponds to a non-commutative version of $\mathbf{CP}^k$. There are N^2 independent elements for an arbitrary $N \times N$ matrix; thus there are N^2 independent "functions" possible on fuzzy $\mathbf{CP}^k$ at finite N (or finite n). What we need to show is that these functions are in one-to-one correspondence with functions on the usual commutative $\mathbf{CP}^k$, as $n \to \infty$. Further, in this limit, the matrix product of two matrices tend to the ordinary commutative product of the corresponding functions. The first step is to define the symbol corresponding to any matrix; the symbol is an ordinary function on commutative $\mathbf{CP}^k$ to which the matrix approximates in the large n limit. The matrix product can then be represented in terms of symbols by a deformation of the ordinary product, known as the star product. We now turn to a discussion of these concepts and their properties.

2.2. *Star products, commutators and Poisson brackets*

Let $\hat{A}$ be a general matrix acting on the N-dimensional Hilbert space generated by the basis (2.21) with matrix elements A_{ms}. We define the symbol corresponding to A as the function

$$
\begin{aligned}
A(g) = A(z,\bar{z}) &= \sum_{ms} \mathcal{D}^{(n)}_{m,-n}(g) A_{ms} \mathcal{D}^{*(n)}_{s,-n}(g) \\
&= \langle -n | \hat{g}^T \hat{A} \hat{g}^* | -n \rangle \,,
\end{aligned}
\tag{2.26}
$$

where $|-n\rangle = |J=n, -n\rangle$. We are interested in the symbol corresponding to the product of two matrices A and B. This can be written as

$$
\begin{aligned}
(AB)(g) &= \sum_r A_{mr} B_{rs} \mathcal{D}^{(n)}_{m,-n}(g) \mathcal{D}^{*(n)}_{s,-n}(g) \\
&= \sum_{rr'p} \mathcal{D}^{(n)}_{m,-n}(g) A_{mr}\ \mathcal{D}^{*(n)}_{r,p}(g) \mathcal{D}^{(n)}_{r',p}(g)\ B_{r's} \mathcal{D}^{*(n)}_{s,-n}(g)
\end{aligned}
\tag{2.27}
$$

using $\delta_{rr'} = \sum_p \mathcal{D}^{*(n)}_{r,p}(g) \mathcal{D}^{(n)}_{r',p}(g)$. The term with $p = -n$ on the right hand side of (2.27) gives the product of the symbols for A and B. The terms with $p > -n$ may be written using raising operators as

$$
\mathcal{D}^{(n)}_{r,p}(g) = \left[\frac{(n-s)!}{n! i_1! i_2! \cdots i_k!} \right]^{\frac{1}{2}} \hat{R}^{i_1}_{+1}\ \hat{R}^{i_2}_{+2} \cdots \hat{R}^{i_k}_{+k}\ \mathcal{D}^{(n)}_{r,-n}(g)\ . \tag{2.28}
$$

Here $s = i_1 + i_2 + \cdots + i_k$ and the t_{k^2+2k}-eigenvalue for the state $|p\rangle$ is $(-nk + sk + s)/\sqrt{2k(k+1)}$. Since $\hat{R}_{+i}\mathcal{D}^{*(n)}_{s,-n} = 0$, we can also write

$$\left[\hat{R}_{+i}\mathcal{D}^{(n)}_{r',-n}(g)\right]B_{r's}\mathcal{D}^{*(n)}_{s,-n}(g) = \left[\hat{R}_{+i}\mathcal{D}^{(n)}_{r',-n}B_{r's}\mathcal{D}^{*(n)}_{s,-n}(g)\right] = \hat{R}_{+i}B(g)\ . \tag{2.29}$$

Further keeping in mind that $\hat{R}^*_+ = -\hat{R}_-$, we can combine (2.27-2.29) to get

$$\begin{aligned}(AB)(g) = \sum_s (-1)^s \left[\frac{(n-s)!}{n!s!}\right] &\sum^{n}_{i_1+i_2+\cdots+i_k=s} \frac{s!}{i_1!i_2!\cdots i_k!}\\ &\times \hat{R}^{i_1}_{-1}\hat{R}^{i_2}_{-2}\cdots\hat{R}^{i_k}_{-k}A(g)\hat{R}^{i_1}_{+1}\hat{R}^{i_2}_{+2}\cdots\hat{R}^{i_k}_{+k}B(g)\\ \equiv A(g) * B(g)\,. \end{aligned} \tag{2.30}$$

The right hand side of this equation is what is known as the star product for functions on $\mathbf{CP}^k$. It has been written down in different forms in the context of noncommutative $\mathbf{CP}^k$ and related spaces [15, 16]; our argument here follows the presentation in [17], which gives a simple and general way of constructing star products. The first term of the sum on the right hand side gives $A(g)B(g)$, successive terms involve derivatives and are down by powers of n, as $n \to \infty$. For the symbol corresponding to the commutator of A and B, we have

$$\big([A,B]\big)(g) = -\frac{1}{n}\sum_{i=1}^{k}(\hat{R}_{-i}A\ \hat{R}_{+i}B - \hat{R}_{-i}B\ \hat{R}_{+i}A)\ +\ \mathcal{O}(1/n^2)\,. \tag{2.31}$$

The Kähler two-form on $\mathbf{CP}^k$ may be written as

$$\begin{aligned}\Omega &= -i\sqrt{\frac{2k}{k+1}}\mathrm{Tr}\big(t_{k^2+2k}\ g^{-1}dg \wedge g^{-1}dg\big)\\ &= -\frac{1}{4}\sum_{i=1}^{k}\ \big(E^{+i}_C E^{-i}_D - E^{-i}_C E^{+i}_D\big)\ d\varphi^C \wedge d\varphi^D\\ &= -\frac{1}{4}\sum_{i=1}^{k}\ \epsilon_{M_iN_i}E^{M_i}_C E^{N_i}_D\ d\varphi^C \wedge d\varphi^D\\ &\equiv \frac{1}{2}\Omega_{CD}\ d\varphi^C \wedge d\varphi^D\,,\end{aligned} \tag{2.32}$$

where $\epsilon_{M_iN_i}$ is equal to 1 for $m_i = +i, N_i = -i$ and is equal to -1 for $M_i = -i, N_i = +i$.

Functions $A,\ B$ on $\mathbf{CP}^k$ obey the condition $\hat{R}_\alpha A = \hat{R}_\alpha B = 0$, where $\hat{R}_\alpha$ (with the index α) is any generator of the subgroup $U(k)$. With this condition, we find

$$i\sum_i \epsilon^{M_iN_i}(\hat{R}_{M_i}A)(E^{-1})^C_{N_i}\ \Omega_{CD} = -\frac{\partial A}{\partial\varphi^D}\,. \tag{2.33}$$

The Poisson bracket of A and B, as defined by Ω, is thus given by

$$\begin{aligned}\{A,B\} &\equiv (\Omega^{-1})^{MN}\frac{\partial A}{\partial\varphi^M}\frac{\partial B}{\partial\varphi^N}\\ &= i\sum_{i=1}^{k}\Big(\hat{R}_{-i}A\ \hat{R}_{+i}B\ -\ \hat{R}_{-i}B\ \hat{R}_{+i}A\Big)\,.\end{aligned} \tag{2.34}$$

Combining with (2.31), we find

$$([A,B])\,(g) = \frac{i}{n}\{A,B\}\ +\ \mathcal{O}(1/n^2)\,. \tag{2.35}$$

This is the general correspondence of commutators and Poisson brackets, here realized for the specific case of $\mathbf{CP}^k$. If desired, one can also write the Poisson bracket in terms of the local coordinates $z, \bar{z}$ introduced in (2.4). The relevant expressions are

$$\{A,B\} = i(1+\bar{z}\cdot z)\left(\frac{\partial A}{\partial z^i}\frac{\partial B}{\partial \bar{z}^i} - \frac{\partial A}{\partial \bar{z}^i}\frac{\partial B}{\partial z^i} + z\cdot\frac{\partial A}{\partial z}\bar{z}\cdot\frac{\partial B}{\partial \bar{z}} - \bar{z}\cdot\frac{\partial A}{\partial \bar{z}}z\cdot\frac{\partial B}{\partial z}\right). \tag{2.36}$$

The trace of an operator $\hat{A}$ may be written as

$$\begin{aligned}\mathrm{Tr}\hat{A} = \sum_m A_{mm} &= N\int d\mu(g)\mathcal{D}^{(n)}_{m,-n}\ A_{mm'}\ \mathcal{D}^{*(n)}_{m',-n}\\ &= N\int d\mu(g)\ A(g)\ .\end{aligned} \tag{2.37}$$

The trace of the product of two operators A, B is then given by

$$\mathrm{Tr}\hat{A}\hat{B}\ =\ N\int d\mu(g)\ A(g) * B(g)\ . \tag{2.38}$$

2.3. *The large n limit of matrices*

We now consider the symbol for the product $\hat{T}_B\hat{A}$ where $\hat{T}_B$ are the generators of $SU(k+1)$, viewed as linear operators on the states in the representation J. Using the formula (2.26), it can be simplified as follows,

$$\begin{aligned}(\hat{T}_B\hat{A})_{\alpha\beta} &= \langle\alpha|\ \hat{g}^T\ \hat{T}_B\ \hat{A}\ \hat{g}^*\ |\beta\rangle \\ &= S_{BC}\ \langle\alpha|\ \hat{T}_C\ \hat{g}^T\ \hat{A}\ \hat{g}^*\ |\beta\rangle \\ &= S_{Ba}(T_a)_{\alpha\gamma}\ \langle\gamma|\ \hat{g}^T\ \hat{A}\ \hat{g}^*\ |\beta\rangle + S_{B+i}\ \langle\alpha|\ \hat{T}_{-i}\ \hat{g}^T\ \hat{A}\ \hat{g}^*\ |\beta\rangle \\ &\quad + S_{B\ k^2+2k}\ \langle\alpha|\ \hat{T}_{k^2+2k}\ \hat{g}^T\ \hat{A}\ \hat{g}^*\ |\beta\rangle \\ &= \mathcal{L}_{B\alpha\gamma}\ \langle\gamma|\ \hat{g}^T\ \hat{A}\ \hat{g}^*\ |\beta\rangle \\ &= \mathcal{L}_{B\alpha\gamma}\ A(g)_{\gamma\beta}\,,\end{aligned} \tag{2.39}$$

where we have used $\hat{g}^T\hat{T}_B\hat{g}^* = S_{BC}\hat{T}_C$, $S_{BC} = 2\text{Tr}(g^T t_B g^* t_C)$. (Here t_B, t_C and the trace are in the fundamental representation of $SU(k+1)$.) $\mathcal{L}_B$ is defined as

$$\mathcal{L}_{B\alpha\gamma} = -\delta_{\alpha\gamma}\frac{nk}{\sqrt{2k(k+1)}}S_{B\ k^2+2k} \ + \ \delta_{\alpha\gamma}S_{B+i}\hat{\hat{R}}_{-i} \tag{2.40}$$

and $\hat{\hat{R}}_{-i}$ is a differential operator defined by $\hat{\hat{R}}_{-i}g^T = T_{-i}g^T$. (This can be related to $\hat{R}_{-i}$ but it is immaterial here.) We have also used the fact that the states $|\alpha\rangle$, $|\beta\rangle$ are $SU(k)$-invariant. By choosing $\hat{A}$ as a product of $\hat{T}$'s, we can extend the calculation of the symbol for any product of $\hat{T}$'s using equation (2.39). We find

$$(\hat{T}_A\hat{T}_B\cdots\hat{T}_M)_{\alpha\beta} = \mathcal{L}_{A\alpha\gamma_1}\mathcal{L}_{B\gamma_1\gamma_2}\cdots\mathcal{L}_{M\gamma_r\beta}\cdot 1\ . \tag{2.41}$$

A function on fuzzy $\mathbf{CP}^k$ is an $N\times N$ matrix. It can be written as a linear combination of products of $\hat{T}$'s and by using the above formula, we can obtain its large n limit. When n becomes very large, the term that dominates in $\mathcal{L}_A$ is $S_{A\ k^2+2k}$. We then see that for any matrix function we have the relation, $F(\hat{T}_A) \approx F(S_{A\ k^2+2k})$.

We will now define a set of "coordinates" X_A which are $N\times N$-matrices by

$$X_A = -\frac{1}{\sqrt{C_2(k+1,n)}}\,\hat{T}_A\,, \tag{2.42}$$

where

$$C_2(k+1,n) = \frac{n^2k^2}{2k(k+1)} + \frac{nk}{2} \tag{2.43}$$

is the quadratic Casimir value for the symmetric rank n representation. X_A will be considered as coordinates of fuzzy $\mathbf{CP}^k$ embedded in $\mathbf{R}^{k^2+2k}$. In the large n limit, we evidently have $X_A \approx S_{A\ k^2+2k} = 2\text{Tr}(g^T t_A g^* t_{k^2+2k})$. By its definition, $S_{A\ k^2+2k}$ obeys algebraic constraints which can be verified to be the correct ones for describing $\mathbf{CP}^k$ as embedded in $\mathbf{R}^{k^2+2k}$.

2.4. *The symbol and diagonal coherent state representation*

The states we have constructed are the coherent states for $\mathbf{CP}^k$ and we have an associated diagonal coherent state representation [8], [12]. Notice that the states have the expected holomorphicity property that coherent states have. In fact, the condition (2.19) is the statement of holomorphicity; explicitly, the wave functions in (2.24), apart from the prefactor involving $(1+\bar{z}\cdot z)^{-\frac{n}{2}}$, involve only z_i and not $\bar{z}_i$. We now show that the symbol of an operator is related to, but is not exactly, the expectation value of the operator in this coherent state representation. The first step towards this is the Wigner-Eckart theorem, which is a standard result, and can also be seen easily as follows.

Let F_α be a tensor operator belonging to the representation r. We then have

$$\hat{g} F_\alpha \hat{g}^\dagger = \mathcal{D}^{(r)}_{\beta,\alpha} F_\beta \ . \tag{2.44}$$

We can now write

$$\begin{aligned} (F_\alpha)_{km} &= \langle n,k|F_\alpha|n,m\rangle \\ &= \langle n,k|\hat{g}^\dagger \hat{g} F_\alpha \hat{g}^\dagger \hat{g}|n,m\rangle \\ &= \sum_{p,q,\beta} \mathcal{D}^{(n)}_{k,p}(g^\dagger)\ \mathcal{D}^{(n)}_{q,m}(g)\mathcal{D}^{(r)}_{\beta,\alpha}(g)\ \langle n,p|F_\beta|n,q\rangle \ . \end{aligned} \tag{2.45}$$

In this expression, we can combine the product of the representations by the Clebsch-Gordon theorem; the Clebsch-Gordon coefficients $\langle j,p|r,\beta;n,q\rangle$ for reduction of product representations are defined by

$$|r,\beta;n,q\rangle = \sum_{j,p} \langle j,p|r,\beta;n,q\rangle\ |j,p\rangle\ , \tag{2.46}$$

where the sum is over all representations which can be obtained by the product of representations r and n, and over the states within each such representation. Using this result and integrating both sides of equation (2.45) over all g, we get the Wigner-Eckart theorem

$$(F_\alpha)_{km} = \langle n,k|r,\alpha;n,m\rangle\ \langle\!\langle F\rangle\!\rangle\ , \tag{2.47}$$

the reduced matrix element $\langle\!\langle F\rangle\!\rangle$ being given by

$$\langle\!\langle F\rangle\!\rangle = \sum_{p,q,\beta} \frac{1}{N}\ \langle r,\beta;n,q|n,p\rangle\ \langle n,p|F_\beta|n,q\rangle \tag{2.48}$$

where N is the dimension of the representation labeled by n.

Now, from the definition of the Clebsch-Gordon coefficient in (2.46), we have the relation

$$\int d\mu(g)\mathcal{D}^{(n)}_{k,p}(g^\dagger)\ \mathcal{D}^{(n)}_{q,m}(g)\mathcal{D}^{(r)}_{\beta,\alpha}(g) = \frac{1}{N}\,\langle r,\beta;n,q|n,p\rangle\ \langle n,k|r,\alpha;n,m\rangle\ . \tag{2.49}$$

We can put $\beta = 0$, $p = q = -n$ in this equation and use (2.47) to obtain

$$\begin{aligned}\langle\!\langle F\rangle\!\rangle\ &\langle n,k|r,\alpha;n,m\rangle\ \langle r,0;n,-n|n,-n\rangle\\ &= N\int d\mu(g)\ \mathcal{D}^{(n)*}_{-n,k}(g^\dagger)\ \mathcal{D}^{(n)}_{-n,m}(g)\langle\!\langle F\rangle\!\rangle\mathcal{D}^{(r)}_{0,\alpha}(g)\\ &= N\int d\mu(g)\mathcal{D}^{(n)*}_{k,-n}(g^T)\mathcal{D}^{(n)}_{m,-n}(g^T)\langle\!\langle F\rangle\!\rangle\mathcal{D}^{(r)}_{\alpha,0}(g^T)\\ &= \int d\mu(g)\ \Psi^{(n)*}_k\left[\langle\!\langle F\rangle\!\rangle\mathcal{D}^{(r)}_{\alpha,0}(g)\right]\Psi^{(n)}_m\,,\end{aligned} \tag{2.50}$$

where, in the last step, we made a change of variable $g \to g^T$, and used the definition of states (2.24). We now define a function $f_\alpha(g)$ by

$$f_\alpha(g)\ \langle r,0;n,-n|n,-n\rangle = \mathcal{D}^{(r)}_{\alpha,0}(g)\langle\!\langle F\rangle\!\rangle\ . \tag{2.51}$$

We can then rewrite (2.50) as

$$\begin{aligned}\int d\mu(g)\Psi^{(n)*}_k\ f_\alpha(g)\ \Psi^{(n)}_m &= \langle n,k|r,\alpha;n,m\rangle\ \langle\!\langle F\rangle\!\rangle\\ &= (F_\alpha)_{km}\ .\end{aligned} \tag{2.52}$$

We see that the matrix element of F_α can be reproduced by the expectation value of a function $f_\alpha(g)$. This is the diagonal coherent state representation. The result is easily extended to any matrix since we can always write it as a linear combination of tensor operators which have definite transformation properties. Thus

$$\langle k|\hat{F}|m\rangle = \int d\mu(g)\ \Psi^{(n)*}_k\ f(g)\ \Psi^{(n)}_m \tag{2.53}$$

or, equivalently,

$$\hat{F} = \int d\mu(g)\ |g\rangle\ f(g)\ \langle g|\,, \tag{2.54}$$

where the states $|g\rangle$ are defined by $\langle k|g\rangle = \Psi^{(n)*}_k$, $\langle g|m\rangle = \Psi^{(n)}_m$.

The function $f(g)$ is not the same as the symbol for F. In fact, the symbol may be written as

$$\begin{aligned} F(g) &= \sum_{km} \mathcal{D}^{(n)}_{k,-n}(g)\; F_{km}\; \mathcal{D}^{(n)*}_{m,-n}(g) \\ &= N \sum_{km} \int_h \mathcal{D}^{(n)}_{k,-n}(g) \mathcal{D}^{(n)*}_{k,-n}(h)\; f(h)\; \mathcal{D}^{(n)}_{m,-n}(h)\; \mathcal{D}^{(n)*}_{m,-n}(g) \\ &= \int_h \mathcal{D}^{(n)*}_{-n,-n}(g^\dagger h)\; f(h)\; \mathcal{D}^{(n)}_{-n,-n}(g^\dagger h) \\ &= \int_u \mathcal{D}^{(n)*}_{-n,-n}(u)\; f(gu)\; \mathcal{D}^{(n)}_{-n,-n}(u)\ . \end{aligned} \tag{2.55}$$

Since F can be written as a combination of tensor operators, we have $f(g) = \sum_{r,\alpha} C^r_\alpha \mathcal{D}^{(r)}_{\alpha,0}$, for some coefficient numbers C^r_α. Using this in the above equation, we get

$$\begin{aligned} F(g) &= \sum_{r,\alpha} C^r_\alpha \mathcal{D}^{(r)}_{\alpha,0}(g)\; |\mathcal{C}|^2\ , \\ \mathcal{C} &= \langle n, -n | r, 0; n, -n \rangle\ . \end{aligned} \tag{2.56}$$

In the large n limit, $\mathcal{C} \to 1$, and the symbol and the function f coincide.

3. Special cases

It is instructive at this point to consider some special cases.

3.1. *The fuzzy two-sphere*

This is one of the best-studied cases [18]. Since $S^2 \sim \mathbf{CP}^1 = SU(2)/U(1)$, this is the special case of $k = 1$ in our analysis. In this case, the representations of $SU(2)$ are given by standard angular momentum theory. Representations are labeled by the maximal angular momentum $j = \frac{n}{2}$, with $N = 2j + 1 = n + 1$. The generators of the group are the angular momentum matrices, and one may identify the coordinates of fuzzy S^2 by $X_i = J_i/\sqrt{j(j+1)}$. At finite n, the coordinates do not commute,

$$[X_i, X_j] = \frac{i}{\sqrt{j(j+1)}}\, \epsilon_{ijk} X_k\ . \tag{3.1}$$

We can parametrize an element of $SU(2)$ as

$$g = \frac{1}{\sqrt{(1 + \bar{z} z)}} \begin{bmatrix} \bar{z} & 1 \\ -1 & z \end{bmatrix}\ . \tag{3.2}$$

For $S_{i3}(g)$ we then find

$$S_{13} = -\frac{z+\bar{z}}{(1+z\bar{z})}\ , \qquad S_{23} = -i\,\frac{z-\bar{z}}{(1+z\bar{z})}\ , \qquad S_{33} = \frac{z\bar{z}-1}{z\bar{z}+1}\ . \quad (3.3)$$

At the matrix level, we have $X_i X_i = 1$; in the large n limit, $X_i \approx S_{i3}$, which also obey the same condition. $z, \bar{z}$ are the local complex coordinates for the sphere.

A function on fuzzy S^2 is an $N \times N$ matrix, so, at the matrix level, there are $N^2 = (n+1)^2$ independent "functions". On the smooth S^2, a basis for functions is given by the spherical harmonics, labeled by the integer $l = 0, 1, 2, ...$ There are $(2l+1)$ such functions for each value of l. If we consider a truncated set of functions with a maximal value of l equal to n, the number of functions is $\sum_0^n (2l+1) = (n+1)^2$. Notice that this number coincides with the number of "functions" at the matrix level. By using the relation $X_i \approx S_{i3}$, we can see that the functions involved correspond to products of S_{i3} with up to n factors. These are in one-to-one correspondence with the spherical harmonics, for $l = 0, 1, 2$, etc., up to $l = n$, since S_{i3} has angular momentum 1. Thus we see that the set of functions at the matrix level will go over to the set of functions on the smooth S^2 as $n \to \infty$.

Fuzzy S^2 may thus be viewed as a regularized version of the smooth S^2 where we impose a cut-off on the number of modes of a function. n is the regulator or cut-off parameter.

3.2. *Fuzzy* $\mathbf{CP^2}$

This corresponds to the case $k = 2$ [13–15]. The large n limit of the coordinates X_A is $S_{A8} = 2\mathrm{Tr}(g^T t_A g^* t_8)$. It is easily checked that, in this limit, they obey the conditions

$$\begin{aligned} X_A X_A &= 1\ , \\ d_{ABC} X_B X_C &= -\frac{1}{\sqrt{3}}\, X_C\,, \end{aligned} \qquad (3.4)$$

where $d_{ABC} = 2\mathrm{Tr}t_A(t_B t_C + t_C t_B)$. These conditions are well known to be the equations representing $\mathbf{CP}^2$ as embedded in $\mathbf{R}^8$. Thus, in the large n limit, our definition of fuzzy $\mathbf{CP}^2$ does recover the smooth $\mathbf{CP}^2$. One can impose these conditions at the level of matrices to get a purely matrix-level definition of fuzzy $\mathbf{CP}^2$.

The dimension N of the Hilbert space is in this case given by $\frac{1}{2}(n+1)(n+2)$. We may also consider the matrix functions which are $N \times N$-matrices; they can be thought of as products of the $\hat{T}$'s with up to $N-1$ factors.

There are N^2 independent functions possible. On the smooth $\mathbf{CP}^2$, a basis of functions is given by (2.6). There are $d(2,l) = (l+1)^3$ such functions for each value of l. If we consider a truncated set, with values of l going up to n, the number of independent functions will be

$$\sum_0^n (l+1)^3 = \frac{1}{4}(n+1)^2(n+2)^2 = N^2 \,. \tag{3.5}$$

It is thus possible to consider the fuzzy $\mathbf{CP}^2$ as a regularization of the smooth $\mathbf{CP}^2$ with a cut-off on the number of modes of a function. Since any matrix function can be written as a sum of products of $\hat{T}$'s, the corresponding large n limit has a sum of products of S_{A8}'s. The independent basis functions are thus given by representations of $SU(3)$ obtained from reducing symmetric products of the adjoint representation with itself; they are the ϕ_l's of equation (2.6). In fact, since the states are of the form $u_{\alpha_1} u_{\alpha_2} ... u_{\alpha_n}$, a general linear transformation is of the form $M^{\alpha_1...\alpha_n}_{\beta_1...\beta_n}$. The traceless part of this forms the irreducible representation ϕ_n, the traces give lower rank irreducible representations ϕ_l, $l < n$. We see that there is complete agreement with the set of functions on $\mathbf{CP}^2$ with a cut-off on the modes at $l = n$.

Since we can regard fuzzy $\mathbf{CP}^k$ as a regularization of the smooth $\mathbf{CP}^k$ with a cut-off on the number of modes of a function, one can use these fuzzy spaces to construct regulated field theories, in much the same way that lattice regularization of field theories is carried out. There are novel features associated with such a regularization; for example, the famous (or notorious) fermion doubling problem on the lattice can be evaded in an interesting way. For these and other details, see [19].

4. Fields on fuzzy spaces

In this section we will briefly consider how one can define a field theory on a fuzzy space.

A scalar field Φ on a fuzzy space is obviously an $N \times N$ matrix which can take arbitrary values. We may write $\Phi(X)$, indicating that it is a function of the coordinate matrices X_A. For constructing an action, we need derivatives. From the general property (2.35), we see that we can write

$$\begin{aligned} [T_A, \Phi] &\approx -\frac{i}{n}\, \frac{nk}{\sqrt{2k(k+1)}} \,\{S_{A\ k^2+2k}, \Phi\} \\ &\equiv -iD_A \Phi \ . \end{aligned} \tag{4.1}$$

On the left hand side of this equation we have the matrix quantities while on the right hand side we have the corresponding symbols. D_A, as defined by this equation, are given by

$$D_A = \sqrt{\frac{k}{2(k+1)}}\,(1+\bar{z}z)\Bigg[\left(\frac{\partial}{\partial z^i} + \bar{z}_i z\cdot\frac{\partial}{\partial z}\right) S_{A\ \ k^2+2k}\frac{\partial}{\partial \bar{z}^i} - \left(\frac{\partial}{\partial \bar{z}^i} + z_i\bar{z}\cdot\frac{\partial}{\partial \bar{z}}\right) S_{A\ \ k^2+2k} t\frac{\partial}{\partial z^i}\Bigg]. \quad (4.2)$$

D_A are derivative operators appropriate to the space we are considering. For example, for the fuzzy S^2, we find

$$\begin{aligned} D_1 &= \frac{1}{2}\left(\bar{z}^2\partial_{\bar{z}} + \partial_z - z^2\partial_z - \partial_{\bar{z}}\right)\,, \\ D_2 &= -\frac{i}{2}\left(\bar{z}^2\partial_{\bar{z}} + \partial_z + z^2\partial_z + \partial_{\bar{z}}\right)\,, \\ D_3 &= \bar{z}\partial_{\bar{z}} - z\partial_z\ . \end{aligned} \quad (4.3)$$

These obey the $SU(2)$ algebra, $[D_A, D_B] = i\epsilon_{ABC}D_C$; they generate the translations on the two-sphere. They are, in fact, the three isometry transformations. This shows that we can define the derivative of Φ, at the matrix level, as the commutator $i[T_A,\Phi]$, which is the adjoint action of T_A on Φ. The Laplace operator is then given by $-\Delta\cdot\Phi = [T_A,[T_A,\Phi]]$.

An example of the Euclidean action for a scalar field is then

$$\mathcal{S} = \frac{1}{N}\mathrm{Tr}\Big[\Phi^\dagger[T_A,[T_A,\Phi]] + V(\Phi)\Big]\,, \quad (4.4)$$

where $V(\Phi)$ is a potential term which does not involve derivatives.

Another interesting class of fields is given by gauge fields. Since the derivatives are given by the adjoint action of the T_A, we can introduce a gauge field $\mathcal{A}_A$ and the covariant derivative

$$-i\mathcal{D}_A\Phi = [T_A,\Phi] + \mathcal{A}_A\Phi\,, \quad (4.5)$$

where $\mathcal{A}_A$ is a set of hermitian matrices. In the absence of the the gauge field, we have the commutation rules $[T_A,T_B] = if_{ABC}T_C$, so that the field strength tensor $\mathcal{F}_{AB}$ may be defined by

$$-i\mathcal{F}_{AB} = [T_A + \mathcal{A}_A, T_B + \mathcal{A}_B] - if_{ABC}(T_C + \mathcal{A}_C)\ . \quad (4.6)$$

One can now construct a Yang-Mills type action for a gauge theory as

$$\mathcal{S} = \frac{1}{N}\,\mathrm{Tr}\Big[\frac{1}{4}\,\mathcal{F}_{AB}\mathcal{F}_{AB}\Big]\,. \quad (4.7)$$

Starting with actions of the type (4.4) and (4.7), it is possible to develop the functional integral for the quantum theory of these fields and do perturbation theory in terms of Feynman diagrams, etc. We will not do this analysis here for two reasons. The analysis of field theories on fuzzy spaces where the nontrivial geometry plays an important role has not yet been developed to a great extent. We give some of the references which can point the reader to ongoing work [19]. Properties of field theories on flat noncommutative spaces (which we have not discussed here) has been more extensively investigated; for this there are good reviews available [9, 20].

5. Construction of spheres

We have discussed in some detail the complex projective spaces. They are the spaces which emerge most naturally in any matrix construction. The reason is simple. Matrices are linear operators on a Hilbert space and so they are related to the quantization of a classical phase space. Therefore, spaces which admit a symplectic structure are natural candidates for fuzzification. Spheres, except for S^2 and products thereof, do not fall into this category. The construction of the spheres is thus more involved. The general strategy has been to identify them as subspaces of suitable fuzzy spaces and to introduce conditions restricting the functions to be on the sphere.

There is good reason to seek fuzzification of spheres, apart from the general mathematical interest in constructing them. As we mentioned before, one of the ways fuzzy spaces can be used is that they provide a finite mode truncation of field theories. Therefore, one may think of them as an alternative to the usual lattice formulation of field theory which is necessary to formulate field theories in a finite way and ask and answer questions about whether they exist and so on. Further, they can be useful for numerical analyses of field theories. Four dimensions, of course, are the most interesting from this point of view; however, $\mathbf{CP}^2$ is not the best, since the smooth $\mathbf{CP}^2$ does not have a spin structure. (It can have a so-called spin^c structure, with an additional $U(1)$ field, which we can take to be the "monopole potential" given in (2.1) [21]. For a recent analysis of the solution of Dirac equation in the background of this field, see [22].) Fuzzification of S^4 would be very useful for this.

The method for the construction of spheres is exemplified and illustrated by the case of fuzzy S^1. We start with fuzzy $S^2 \sim \mathbf{CP}^1$; the modes at finite n are the fuzzy versions of the spherical harmonics $Y^l_m(\theta, \varphi)$ with $l = 0, 1$, etc., up to $l = n$. Take the highest spherical harmonic; this has m values $-n, -n+1$, etc., up to n. The φ-dependence of this function corresponds to modes $e^{im\varphi}$ on S^1 for the same range of values for m. If we take the large n limit, we see that this single spherical harmonic can give all the modes required for S^1. So, one strategy, advocated in [23], is to introduce a projection operator which, acting on a matrix F, which may be viewed as a function on the fuzzy space, retains only the highest mode. If we split F as $F = F_+ + F_-$, where F_+ is the part corresponding to the harmonics on S^1 and F_- the remainder, the projection operator P is defined by $P(F) = F_+$. This can be done for higher dimensional spheres as well. The difficulty with this approach is that the product of two such projected matrices will generate the other unwanted modes, so we have to define the operation of multiplication by $F * G = P(FG)$; the algebra is done before the projector is applied. This product is not associative in general, so that one cannot interpret the matrix functions on the fuzzy sphere as a linear transformation on a Hilbert space; in the large n limit, associativity is recovered. Nevertheless, the lack of associativity limits the utility of this approach.

A related idea starts with the question: what do we want to use the fuzzy sphere for? If it is for the purpose of constructing field theories on it and studying their behavior as $n \to \infty$, then a different strategy is possible [24]. It would be more practical, even for numerical simulations of theories, to include all modes, for example, all the spherical harmonics for all $l \leq n$, so that we do have the algebra of functions on the bigger space, the fuzzy two-sphere in this example. One can then choose the action, so that all the unwanted modes have a large contribution to the Euclidean action. Such modes are then suppressed and one gets a softer way of approaching the modes on the sphere we want. In the example of S^1, notice that the quantity $h[n(n+1) - T^2]$, where T_A are the angular momentum generators, is positive for all $l < n$ and is zero for $l = n$. Thus adding a term with this eigenvalue would prejudice it against all modes $l < n$, and by taking the parameter h to be large, we can get a good approximation to the circle S^1. For a scalar field, such an action is given by

$$\mathcal{S} = \frac{1}{n+1} \operatorname{Tr} \left[\frac{1}{2} \Phi^\dagger [T_3, [T_3, \Phi]] + \frac{h}{2} \Phi^\dagger [n(n+1) - T^2] \cdot \Phi \right] , \tag{5.1}$$

where $T^2 \cdot \Phi = [T_A, [T_A, \Phi]]$.

These ideas can be extended to higher dimensional spheres. We have to start with suitable co-adjoint orbits to construct the Hilbert space via quantization as we have done. For spheres, some of the useful co-adjoint orbits are [24]:

(1) $SO(3)/SO(2) \sim S^2$.

(2) $SO(4)/[SO(2)\times SO(2)] \sim S^2\times S^2$. (This can be related to $S^3/\mathbf{Z}_2$ as we show below.)

(3) $SO(5)/[SO(3)\times SO(2)] \sim \mathbf{CP}^3/\mathbf{Z}_2$. (This can be used for the fuzzy version of S^4 utilizing the fact that $\mathbf{CP}^3$ is an S^2 bundle over S^4. It can also be used to approximate S^3 by prejudicing the action against the unwanted modes as in the case of S^1.)

(4) $SO(N{+}2)/[SO(N)\times SO(2)]$. (This can be used for higher dimensional spheres in a similar way.)

A matrix version of the four-sphere, which is useful in the context of solutions to M(atrix) theory, is worthy of special mention [26]. For the four sphere, we expect to have five matrices X_μ, $\mu = 1, ..., 5$, such that $X_\mu X_\mu = 1$. This can be achieved by using the Euclidean Dirac γ-matrices. There is only one irreducible representation for the γ matrices, so to get a sequence of larger and larger matrices, one can take tensor products of these,

$$X_\mu = (\gamma_\mu \otimes 1 \otimes 1 ... \otimes 1 \ + \ 1 \otimes \gamma_\mu \otimes 1 ... \otimes 1 \ + \ \cdots)_{sym}\,, \tag{5.2}$$

where the subscript *sym* indicates symmetrization.

We now consider the fuzzy version of $S^3/\mathbf{Z}_2$ which is related to $S^2\times S^2$ [27]. The end result is not quite a sphere, but it is still an interesting example, since there is an algebra of functions which has closure under multiplication. In this case, $S^2\times S^2$ plays a role, vis-à-vis $S^3/\mathbf{Z}_2$, analogous to what $\mathbf{CP}^3$ does for S^4.

In the smooth limit, the space $S^3/\mathbf{Z}_2$ can be embedded in $S^2\times S^2$. The latter space can be described by $n^2 = 1$, $m^2 = 1$, $n = (x_1, x_2, x_3)$, $m = (y_1, y_2, y_3)$. The space $S^3/\mathbf{Z}_2$ is now obtained by imposing the further condition $n\cdot m = x_1y_1 + x_2y_2 + x_3y_3 = 0$. It is clear that any solution to these equations gives an $SO(3)$ matrix $R_{AB} = (\epsilon_{ABC} m_B n_C, m_A, n_A)$. Conversely, given any element $R_{AB} \in SO(3)$, we can identify $n_A = R_{A3}$, $m_A = R_{A2}$. There are other ways to identify (n, m) but these are equivalent to choosing different sets of values for the $SO(3)$ parameters; this statement can be easily checked using the Euler angle parametrization. What we have described is essentially the angle-axis parametrization of rotations [28]. Since the space $S^2\times S^2$ has Kähler structure, it is the simplest enlargement of space we can use to define coherent states.

We now turn to the fuzzy version of $S^3/\mathbf{Z}_2$. Consider $SU(2)\times SU(2)$, with generators T_A, T'_A and take a particular representation where $l = l'$, so that we can think of T, T' as $(N\times N)$-matrices, $N = 2l + 1$. Since the quadratic Casimirs satisfy $T^2 = T'^2 = l(l+1)$, this gives the standard realization of fuzzy $S^2 \times S^2$ [18]. As l becomes large, we can use our result (2.40)

$$T_A \approx l\ 2\,\mathrm{Tr}\,(g^\dagger t_A g t_3), \qquad T'_A \approx l\ 2\,\mathrm{Tr}\,(g'^\dagger t_A g' t_3) \tag{5.3}$$

where g, g' are (2×2)-matrices parametrizing the two $SU(2)$'s. All functions of T, T' are similarly approximated. To get to a smaller space, clearly we need to put an additional restriction which we will take as the following. An operator is considered admissible or physical if it commutes with $T \cdot T'$, or equivalently commutes with $(T - T')^2$ or $(T + T')^2$, i.e,

$$[\mathcal{O}, (T - T')^2] = 0\ . \tag{5.4}$$

It is easily seen that the product of any two operators which obey this condition will also obey the same condition, so this leads to a closed algebra. A basis for the vector space on which T, T' act is given by $|lmlm'\rangle$ in the standard angular momentum notation. We rearrange these into multiplets of $J_A = T_A + T'_A$. For all states within each irreducible representation of the J-subalgebra labeled by j, $(T - T')^2$ has the same eigenvalue $4l(l+1) - j(j+1)$. Operators which commute with it are thus block diagonal, consisting of all unitary transformations on each $(2j+1)$-dimensional subspace. There are $(2j+1)^2$ independent transformations for each j-value putting them in one-to-one correspondence with the basis functions $\mathcal{D}^j_{a,b}(U)$ for an S^3 described by the $SU(2)$ element U. By construction, we get only integral values of j, even if l can be half-odd-integral, so we certainly cannot get S^3 in the large l limit, only $S^3/\mathbf{Z}_2$.

We can go further and ask how the condition (5.4) can be implemented in the large l limit. This can be done by fixing the value of $T \cdot T'$ to be any constant. Using (5.3) we find that this leads to

$$T \cdot T' \sim 2\,\mathrm{Tr}\,(g'^\dagger g\ t_3 g^\dagger g' t_3) \sim \text{constant}\ . \tag{5.5}$$

This means that

$$g'^\dagger g = M \exp(i t_3 \theta) \tag{5.6}$$

where M is a constant $SU(2)$ matrix. θ can be absorbed into g. Since $T \cdot T' \sim 2\,\mathrm{Tr}\,(M t_3 M^\dagger t_3)$, we can take $M = \exp(i t_2 \beta_0)$ using the Euler angle parametrization. We then find

$$\begin{aligned} T_A &\sim 2\,\mathrm{Tr}\,(g^\dagger t_A g t_3)\ , \\ T'_A &\sim \cos\beta_0\ 2\,\mathrm{Tr}\,(g^\dagger t_A g t_3) + \sin\beta_0\ 2\,\mathrm{Tr}\,(g^\dagger t_A g t_1)\ . \end{aligned} \tag{5.7}$$

Thus all functions of these can be built up from the $SO(3)$ elements $R_{AB} = 2\text{Tr}(g^\dagger t_A g t_B)$. (Actually we need $B = 1, 3$, but $B = 2$ is automatically given by the cross product.) Thus, in the large l limit, the operators obeying the further condition (5.4), tend to the expected mode functions for the group manifold of $SO(3)$ which is $S^3/\mathbf{Z}_2$. We have thus obtained a fuzzy version of $S^3/\mathbf{Z}_2$ or $\mathbf{RP}^3$. The condition we have imposed, namely (5.4), is also very natural, once we realize that $(T - T')^2$ is the matrix analog of the Laplacian, and mode functions can be obtained as eigenfunctions of the Laplacian.

A similar construction is possible for fuzzy S^4, utilizing the fact that $\mathbf{CP}^3$ is an S^2-bundle over S^4. This has been shown in a recent paper by Abe [25].

6. Brane solutions in M(atrix) theory

The idea of M-theory was formulated by Witten who showed that the five superstring theories in ten dimensions could be considered as special cases of a single theory, the M-theory [29]. Witten also showed that eleven-dimensional supergravity is another limit of M-theory, corresponding to compactification on a circle. Shortly afterwards, BFSS proposed a matrix model as a version of M-theory in the lightcone formulation; as the dimension N of the matrices becomes large, the model is supposed to describe M-theory in the large lightcone momentum limit [30]. Another matrix model, which applies to the type IIB case, as opposed to the type IIA which is described by the BFSS model, has been given by IKKT [31]. These matrix versions of M-theory are often referred to as M(atrix) models and have been rather intensively investigated over the last few years [32]. It is by now clear that M(atrix) theory does capture many of the expected features of M-theory such as the eleven-dimensional supergravity regime and the existence of extended objects of appropriate dimensions. Solutions to this theory are given by special matrices; hence these theories generically have the possibility of fuzzy spaces appearing as solutions. These will correspond to brane solutions, with smooth extended objects or branes emerging in the large N-limit. Finding such solutions is clearly of some interest. The emergence of the two-brane or the standard membrane was analyzed many years ago [33]. More recently, spherically symmetric membranes have been obtained [34]. As regards the five-brane, which is the other extended object of interest, there has been no satisfactory construction or understanding of the transverse brane where all five spatial dimensions are a subset of the nine manifest dimensions of the matrix theory. The longitudinal five-branes, called $L5$-branes, which have four manifest dimensions and one along the compactified direction (either the eleventh dimension or the lightlike circle) have been obtained. These include flat branes [35] and stacks of $S^4 \times S^1$-branes [26]. With the discussion of fuzzy spaces given in the previous sections, we are now in a position to analyze the construction of brane configurations. The presentation in this section will closely follow [13].

6.1. *The ansatz for a solution*

The action for matrix theory can be written as

$$\mathcal{S} = \mathrm{Tr}\left[\frac{\dot{X}_I^2}{2R} + \frac{R}{4}\,[X_I, X_J]^2 + \theta^T\dot{\theta} + iR\,\theta^T\Gamma_I[X_I,\theta]\right] , \tag{6.1}$$

where $I, J = 1, ..., 9$ and θ is a 16-component spinor of $O(9)$ and Γ_I are the appropriate gamma matrices. The θ's represent the fermionic degrees of freedom which are needed for the supersymmetry of the model. X_I are hermitian $(N \times N)$-matrices; they are elements of the Lie algebra of $U(N)$ in the fundamental representation. The theory is defined by this Lagrangian supplemented by the Gauss law constraint

$$[X_I, \dot{X}_I] - [\theta, \theta^T] \approx 0 \; . \tag{6.2}$$

In the following we shall be concerned with bosonic solutions and the θ's will be set to zero. The relevant equations of motion are thus

$$\frac{1}{R}\ddot{X}_I \;-\; R[X_J, [X_I, X_J]] = 0 \; . \tag{6.3}$$

Even though the solutions we consider will be mostly fuzzy **CP**'s, we will start with a more general framework, analyzing general features of solutions for the model (6.1). We shall look for solutions which carry some amount of symmetry. In this case, we can formulate a simple ansatz in terms of the group coset space G/H where $H \subset G \subset U(N)$. The ansatz we consider will have spacetime symmetries, a spacetime transformation being compensated by an H transformation. Along the lines of the discussions in the previous sections, in the large N-limit, the matrices X_I will go over to continuous brane-like solutions with the geometry of G/H.

As before, let t_A, $A = 1, ..., dimG$ denote a basis of the Lie algebra of G. We split this set of generators into two groups, t_α, $\alpha = 1, ..., dimH$, which form the Lie algebra of H, and t_i, $i = 1, ..., (dimG - dimH)$, which form the complementary set. Our ansatz will be to take X_I's to lie along the entire algebra of $\underline{G}$ or to be linear combinations of the t_i's. In the latter case, in order to satisfy the equation of motion (78), we shall then need the double commutator $[X_J, [X_I, X_J]]$ to be combinations of the t_i's themselves.

This is guaranteed if $[t_i, t_j] \subset \underline{H}$, since the t_i's themselves transform as representations of H. In this case the commutation rules are of the form

$$[t_\alpha, t_\beta] = if_{\alpha\beta\gamma} t_\gamma \ ,$$
$$[t_\alpha, t_i] = if_{\alpha ij} t_j \ ,$$
$$[t_i, t_j] = ic_{ij\alpha} t_\alpha \tag{6.4}$$

and G/H is a symmetric space. If $H = 1$, the t_i's will belong to the full algebra $\underline{G}$. In this case, c^α_{ij} of (6.4) will be the structure constants of $\underline{G}$. In such a case, even though the ansatz involves the full algebra $\underline{G}$, it can satisfy further algebraic constraints. These additional constraints may have only a smaller invariance group H; the solution will then again reduce to the G/H-type. These are the cases we analyze.

Since X_I are elements of the Lie algebra of $U(N)$, having chosen a G and an H, we must consider the embedding of G in $U(N)$. This is done as follows. We consider a value of N which corresponds to the dimension of a unitary irreducible representation (UIR) of G. The embedding is then specified by identifying the fundamental N-dimensional representation of $U(N)$ with the N-dimensional UIR of G. Eventually we need to consider the limit $N \to \infty$ as well. Thus we must have an infinite sequence of UIR's of G, so that we have a true fuzzy version of G/H. Generally, different choices of such sequences are possible, corresponding to different ways of defining the $N \to \infty$ limit. Following previous sections, we will focus on the symmetric rank n-tensors of G, of dimension, say, $d(n)$. Thus we choose $N = d(n)$, defining the large N-limit by $n \to \infty$.

The ansatz we take is of the form

$$X_i = r(t)\, \frac{T_i}{N^a} \tag{6.5}$$

for a subset $i = 1, ..., p$ of the nine X's. T_i are generators in $\underline{G} - \underline{H}$, in the symmetric rank n-tensor representation of G. The eigenvalues for any of the T_i's in the n-tensor representation will range from cn to $-cn$, where c is a constant. The eigenvalues thus become dense with a finite range of the variation of the X_i's as $n \to \infty$ if $N^a \sim n$. In this case, as $n \to \infty$, the X_i's will tend to a smooth brane-like configuration. Our choice of the index a will be fixed by this requirement, viz., $N^a \sim n$. (For fuzzy $\mathbf{CP}^k$, we defined X by equation (2.42). they correspond to a particular choice of a. We will return to that choice shortly.) Notice also that this ansatz is consistent with the Gauss law (6.2). $r(t)$ represents the radius of the brane; it can vary with time and is the only collective coordinate in our ansatz.

The ansatz (6.5) has a symmetry of the form

$$R_{ij}\; UX_jU^{-1} = X_i\,, \tag{6.6}$$

where R_{ij} is a spatial rotation of the X_i's and U is an H transformation for the G/H case or more generally it can be in $U(N)$. R_{ij} is determined by the choice of the X_i's involved in (6.5). Further, the ansatz (6.5) is to be interpreted as being given in a specific gauge. A $U(N)$ transformation, common to all the X_i's, is a gauge transformation and does not bring in new degrees of freedom. We may alternatively say that the meaning of (6.6) is that X_j is invariant under rotations R_{ij} up to a gauge transformation.

For the ansatz (6.5), the action simplifies to

$$\mathcal{S} = A_n \left[\frac{\dot{r}^2}{2RN^{2a}} \;-\; \frac{c_{ij\alpha}c_{ij\alpha}}{4N^{4a}}\, Rr^4 \right] , \tag{6.7}$$

where A_n is defined by $\mathrm{Tr}(T_AT_B) = A_n\delta_{AB}$ and the matrices and trace are in the n-tensor of G. From its definition, $A_n = d(n)C_2(n)/dimG$, where $C_2(n)$ is the quadratic Casimir of the n-tensor representation, which goes like n^2 for large n. Thus, with $N^a \sim n$, the kinetic term in (6.7) will always go like N/R for large n.

We now turn to some specific cases. Consider first $G = SU(2)$. In this case, $d(n) = n+1,\; A_n = n(n+1)(n+2)/12$. The smooth brane limit thus requires $a = 1$ or $N^a \sim N \sim n$. The kinetic energy term in (6.7) goes like N/R, while the potential term goes like R/N. Thus both these terms would have a finite limit if we take $N \to \infty,\; R \to \infty$, keeping (N/R) fixed. In fact, this particular property holds only for $G = SU(2)$ or products thereof, such as $G = O(4)$. For this reason some of the branes which are realized as cosets of products of the $SU(2)$ group can perhaps be regarded as being transverse as their energy will not depend on R in the large R limit. We can use this case to obtain a slightly different description of the spherical membrane of [34] as well as a "squashed" S^2 or $\mathbf{CP}^1$. The round $\mathbf{CP}^1$ corresponds to the case where the three generators of $SU(2)$ lie along three of the nine $X_I's$.

As another example, consider $G = O(6) \sim SU(4)$. In this case, $d(n) = (n+1)(n+2)(n+3)/6, A_n = (1/240)(n+4)!/(n-1)!$ and we need $a = \frac{1}{3}$. The kinetic energy term goes like N/R while the potential energy goes like nR. This corresponds to the longitudinal five-brane with S^4 geometry discussed in [26]. $SU(4)$ has an $O(5)$ subgroup under which the 15-dimensional adjoint representation of $SU(4)$ splits into the adjoint of $O(5)$ and the 5-dimensional vector representation. The coset generators, corresponding to the 5-dimensional representation, may be represented by the (4×4)-gamma matrices, γ_μ, $\mu = 1, ..., 5$. The ansatz (5) thus takes the form $X_\mu = r\gamma_\mu / N^{\frac{1}{3}}$. For the N-dimensional representation, we use the symmetrized tensor product form given in (5.2). The sum of the squares of the X_μ's is then proportional to the identity, thus giving effectively a four-dimensional brane. This is interpreted as n copies of a longitudinal five-brane, one of the directions being along the compactified eleventh or lightcone coordinate, of extent R, an interpretation consistent with the potential energy $\sim nR$.

6.2. *Fuzzy* $\mathbf{CP}^2$ *as a brane solution*

We now turn to an especially interesting case of a static solution to the matrix theory, which corresponds to fuzzy $\mathbf{CP}^2$ with $G = SU(3)$ obtained in [13]. In this case, $d(n) = (n+1)(n+2)/2 \ \sim n^2$ and $A_n = (1/48)(n+3)!/(n-1)!$. We choose $a = \frac{1}{2}$. The kinetic energy again goes like N/R for large N while the potential energy goes like R. The potential energy is independent of n and thus, for r independent of t, we have a single static smooth five-brane wrapped around the compactified dimension in the $n \to \infty$ limit. In other words, the tension defined by the static energy per unit volume remains finite as $n \to \infty$. An appropriate choice of H in this case is $H = U(2) \sim SU(2) \times U(1)$. The world volume geometry is then $\mathbf{CP}^2 \times S^1$, where S^1 corresponds to the compactified eleventh dimension. The immersion of this solution in $\mathbf{R}^9$ can be complicated and will depend on the details of the ansatz. As we shall see in the next section the coset embedding will produce a singular surface in a nine-dimensional Euclidean space, while when the eight generators of $SU(3)$ are set parallel to eight of nine X_I's we shall obtain the standard $\mathbf{CP}^2$ embedded in an S^7 contained in $\mathbf{R}^9$.

Notice also that since the effective mass for the degree of freedom corresponding to r goes like N/R, oscillations in r are suppressed as $N \to \infty$ with R fixed; in this limit this configuration becomes a static *solution*. We can fix r to any value and time-evolution does not change this.

Using the wave functions given earlier, it is straightforward to see that, in matrix elements, we may use

$$\begin{aligned} T_{+i} \equiv T_i &= (n+3)\,\frac{\bar{z}_i}{(1+\bar{z}\cdot z)} \ , \\ T_{-i} \equiv T_i^\dagger &= (n+3)\,\frac{z_i}{(1+\bar{z}\cdot z)} \ , \end{aligned} \tag{6.8}$$

$$[T_{+i}, T_{-j}] \equiv h_{ij} = (n+3)\,\frac{(\delta_{ij} - \bar{z}_i z_j)}{(1+\bar{z}\cdot z)} \ . \tag{6.9}$$

The ansatz for the five-brane may now be stated as follows. The simplest case to consider is the following. We define the complex combinations $Z_i = \frac{1}{\sqrt{2}}(X_i + iX_{i+2})$, $i = 1, 2$. The symmetric ansatz is then given by

$$\begin{aligned} Z_i &= r(t)\,\frac{T_i}{\sqrt{N}}\;, \qquad && i = 1, 2\,, \\ X_i &= 0\,,\,, && i = 5, ..., 9\;. \end{aligned} \tag{6.10}$$

In the large n-limit,

$$Z_i \approx r\,\frac{n}{\sqrt{N}}\,\frac{\bar{z}_i}{(1 + \bar{z}\cdot z)} \approx \sqrt{2}\,r\,\frac{\bar{z}_i}{(1 + \bar{z}\cdot z)} \tag{6.11}$$

realizing a continuous map from $\mathbf{CP}^2$ to the space $\mathbf{R}^9$. This map is not one-to-one; the region $\bar{z}\cdot z < 1$ and the region $\bar{z}\cdot z > 1$ are mapped into the same spatial region $|Z| < r\sqrt{2}$, corresponding to a somewhat squashed $\mathbf{CP}^2$.

The standard $\mathbf{CP}^2$ is obtained by considering an $\mathbf{R}^8$-subspace of $\mathbf{R}^9$, whose coordinates can be identified with the $SU(3)$ generators as in (6.5), i.e., $X_A = rT_A/\sqrt{N}$, $A = 1, 2, ..., 8$, and $X_9 = 0$. Specifically, using the expression for the $SU(3)$ generators in terms of the z's, this ansatz is given by (recall that in the large N limit $n \approx \sqrt{2N}$)

$$\begin{aligned} X_i &= \frac{r}{\sqrt{2}}\,\frac{\bar{z}\sigma^i z}{1 + \bar{z}z}\;, \\ X_4 + iX_5 &= \frac{r}{\sqrt{2}}\,\frac{2z_1}{1 + \bar{z}z}\;, \\ X_6 + iX_7 &= \frac{r}{\sqrt{2}}\,\frac{2z_2}{1 + \bar{z}z}\;, \\ X_8 &= \frac{r}{\sqrt{6}}\,\frac{2 - z\bar{z}}{1 + \bar{z}z}\;, \\ X_9 &= 0\,, \end{aligned} \tag{6.12}$$

where σ^i, $i = 1, 2, 3$ are Pauli matrices. Notice that the singularity of the squashed $\mathbf{CP}^2$ is removed by this ansatz since the regions $\bar{z}\cdot z < 1$ and $\bar{z}\cdot z > 1$ are mapped to different regions of the nine-dimensional space. In fact it is easy to see by direct inspection that the map is actually one-to-one. Furthermore it is easily seen that $\Sigma_{A=1}^{8} X_A X_A = \frac{2r^2}{3}$. Thus our surface is embedded in an S^7. In fact, we can use the above relations and express X^1, X^2 and X^3 in terms of $X^4, ..., X^8$ as

$$X^a = \frac{3}{\sqrt{2}}\,\frac{\bar{\zeta}\sigma^a\zeta}{r + \sqrt{6}X_8}\,, \tag{6.13}$$

where ζ is a two-component vector defined by $\zeta^1 = \frac{1}{\sqrt{2}}(X_4 + iX_5)$ and $\zeta^2 = \frac{1}{\sqrt{2}}(X_6 + iX_7)$.

The coset structure is clearer directly in terms of the ansatz for (6.10), which is why we started with this squashed $\mathbf{CP}^2$. The smooth configuration (6.12) may be regarded as a relaxation of (6.10) along some of the $\mathbf{R}^9$-directions.

The action for ansatz (6.10), (6.12) becomes

$$\begin{aligned}\mathcal{S} &= \frac{(n+3)!}{(n-1)!}\left[\frac{\dot{r}^2}{12NR} - \frac{Rr^4}{8N^2}\right] \\ &\approx \left[\frac{N}{R}\frac{\dot{r}^2}{3} - \frac{Rr^4}{2}\right](1+\mathcal{O}(1/n)) \ . \end{aligned} \tag{6.14}$$

In terms of the world volume coordinates, we can also write

$$\mathcal{S} \approx \int \frac{2\ d^4z}{\pi^2(1+\bar{z}\cdot z)^3}\left[\frac{n^4}{2NR}\,\dot{r}^2\,\frac{\bar{z}\cdot z}{(1+\bar{z}\cdot z)^2} - Rr^4\frac{n^4}{4N^2}\,\frac{2-2\bar{z}\cdot z+(\bar{z}\cdot z)^2}{(1+\bar{z}\cdot z)^2}\right] \tag{6.15}$$

$$\approx \int \frac{2\ d^4z}{\pi^2(1+\bar{z}\cdot z)^3}\left[\frac{N}{R}\,\frac{\dot{r}^2}{3} - \frac{Rr^4}{2}\right] . \tag{6.16}$$

Expression (6.15) applies to ansatz (6.10), expression (6.16) to ansatz (6.12). The energy densities are uniformly distributed over the world volume for (6.12), but not for (6.10).

The equation of motion for r becomes

$$\frac{N}{R}\,\ddot{f} + 6f^3 = 0\,, \tag{6.17}$$

where $r = f/\sqrt{R}$. The effective mass for the degree of freedom corresponding to r is $\frac{2}{3}(N/R)$. Thus in the limit $N \to \infty$ with R fixed, any solution with finite energy would have to have a constant r or f. In this limit, we thus get a five-brane which is a *static solution* of the matrix theory. Alternatively, if we consider $N, R \to \infty$ with (N/R) fixed, f can have a finite value. However, in this limit, the physical dimension of the brane as given by r would vanish.

Explicit solutions to (6.17) may be written in terms of the sine-lemniscate function as

$$f = A \sin\mathrm{lemn}\left(\sqrt{\frac{3R}{N}}\ A(t-t_0)\right) . \tag{6.18}$$

6.3. *M-theory properties of the fuzzy* $\mathbf{CP^2}$

There are some properties of the solution we found which are of interest from the matrix theory point of view. We will briefly mention these for completeness.

First, regarding the spacetime properties of the solution, we note that the matrix action (6.1) is in terms of lightcone coordinates. As a result, the Hamiltonian corresponding to this action gives the lightcone component T^{+-} of the energy-momentum tensor. Other components of the energy-momentum tensor can be evaluated, following the general formula of [32,36]; they will act as a source for gravitons, again along the general lines of [32,36]. The current $\mathcal{T}^{IJK}$ which is the source for the antisymmetric tensor field of eleven-dimensional supergravity is another quantity of interest. This vanishes for the spherically symmetric configurations considered in [26], since there are no invariant $O(9)$-tensors of the appropriate rank and symmetry. However, for the $\mathbf{CP}^2$ geometry, there is the Kähler form and the possibility that $\mathcal{T}^{-ij}$ could be proportional to the Kähler form has to be checked explicitly. The kinetic terms of $\mathcal{T}^{-ij}$ which depend on $\dot{r}$ are easily seen to vanish for the solution (6.12), essentially because of the symmetric nature of the ansatz. For solution (6.10), we find, by direct evaluation,

$$\begin{aligned} \mathcal{T}^{-1\bar{1}} &= \mathcal{T}^{-2\bar{2}} = -\frac{i}{60}\,\frac{n^5}{N^2}\,\dot{r}^2 r^2 \ + \dots \ , \\ \mathcal{T}^{-1\bar{2}} &= \mathcal{T}^{-2\bar{1}} = 0 \ . \end{aligned} \tag{6.19}$$

Naively, this diverges as $n \to \infty$. However, as we have noticed before, in this limit the solution becomes static, $\dot{r} = 0$, and hence this vanishes. This holds for other components of $\mathcal{T}^{IJK}$ as well. The nonkinetic terms in $\mathcal{T}^{IJK}$ are of the form $R^2 r^6 (n^5/N^3)$ and also vanish as $n \to \infty$. Thus the source for the antisymmetric tensor field is zero in the n (or N) $\to \infty$ limit.

Another important spacetime property has to do with supersymmetry. The Lagrangian of the matrix theory is invariant under supersymmetry transformations. The supersymmetry variation of θ is given by

$$\delta\theta \equiv K\epsilon + \rho = \frac{1}{2}\left[\dot{X}_I \Gamma_I + [X_I, X_J]\Gamma_{IJ}\right]\epsilon + \rho\,, \tag{6.20}$$

where ϵ and ρ are 16-component spinors of $O(9)$.

Since $\delta\theta$ has n-dependent terms, the question of supersymmetry is best understood by considering fermionic collective coordinates. These are introduced by using the supersymmetry variation (6.20) with the parameters ϵ, ρ taken to be time-dependent. Upon substitution in the Lagrangian, the term $\mathrm{Tr}[\theta^T\dot{\theta}]$ generates the symplectic structure for (ϵ, ρ). We can then construct the supersymmetry generators for fluctuations around our solution. If the starting configuration is supersymmetric, there will be zero modes in the symplectic form so constructed and we will have only a smaller number of fermionic parameters appearing in $\mathrm{Tr}[\theta^T\dot{\theta}]$. Now, in the large n-limit, we have $\mathrm{Tr}[\theta^T\dot{\theta}] \sim n^2[\epsilon^T K^T K\dot{\epsilon} + \rho^T\dot{\rho}]$ which goes to zero as $n \to \infty$ if $\delta\theta \sim n^{-1-\eta}$, $\eta > 0$. Thus if $\delta\theta$ vanishes faster than $1/n$, we can conclude that the starting bosonic configuration is supersymmetric.

Consider the squashed $\mathbf{CP}^2$ first. The finiteness of the kinetic energy in the large N limit requires that the leading term in r must be a constant, which is how we obtained a static solution. The equation of motion then shows that $\dot{r}$ must go like $\frac{1}{N} \sim \frac{1}{n^2}$. In other words, we can write $r = r_0 + \frac{1}{N}r_1 + ...$ The $\dot{X}_I$ term of $\delta\theta$ thus vanishes to the order required. The vanishing of $\delta\theta$ (or the Bogomol'nyi-Prasad-Sommerfield-like condition) then becomes, to leading order,

$$-\frac{8r_0^2}{N}(\lambda_a L_a + \sqrt{3}\lambda_8 R_1)\epsilon + \rho = 0\,, \tag{6.21}$$

where the set λ_a, λ_8, $a = 1, 2, 3$, generate an $SU(2) \times U(1)$ subgroup of $SU(3)$ while the operators L_a, R_1 generate an $SU_L(2) \times U_R(1)$ subgroup of $O(4) \sim SU_L(2) \times SU_R(2)$. In terms of (16×16) Γ-matrices they are given by

$$\begin{aligned} L_1 &= \frac{i}{4}\left(\Gamma_1\Gamma_3 + \Gamma_4\Gamma_2\right)\,, \\ L_2 &= \frac{i}{4}\left(\Gamma_1\Gamma_2 + \Gamma_3\Gamma_4\right)\,, \qquad (6.22) \\ L_3 &= \frac{i}{4}\left(\Gamma_2\Gamma_3 + \Gamma_1\Gamma_4\right)\,, \end{aligned}$$

$$R_1 = \frac{i}{4}\left(\Gamma_1\Gamma_3 - \Gamma_4\Gamma_2\right)\,. \tag{6.23}$$

Notice that the ϵ term is of order $1/n$ since λ_a, λ_8 have eigenvalues of order n and in the large n limit $N \approx \frac{1}{2}n^2$; the ρ term is of order one. Thus condition (6.21) is required for supersymmetry as explained above. Further, in our problem the $O(9)$ group is broken to $O(4) \times O(5)$. With respect to this breaking, the 16-component spinor of $O(9)$ decomposes according to $\underline{16} = ((1,2),4) + ((2,1),4)$, where 4 denotes the spinor of $O(5)$. In terms of the $SU_L(2) \times U_R(1)$ subgroup generated by the L_a and R_1 we have $(1,2) = 1_1 + 1_{-1}$ and $(2,1) = 2_0$ where the subscripts denote the $U(1)$ charges. Clearly we have no singlets under $SU_L(2) \times U_R(1)$; $SU_L(2)$ singlets necessarily carry

$U(1)$ charges. Therefore the operator $(\lambda_a L_a + \sqrt{3}\lambda_8 R_1)$ can neither annihilate ϵ nor can it be a multiple of the unit operator in the $SU(3)$ space. Hence a nontrivial ϵ supersymmetry cannot be compensated by a ρ transformation. Thus we have no supersymmetry.

The supersymmetry variation produces a θ of the form $(\lambda_a L_a + \sqrt{3}\lambda_8 R_1)\epsilon$, where we set $\rho = 0$ for the moment. The contribution of this θ to the Hamiltonian via the term $\mathrm{Tr}(\theta^T[X_i, \theta])$ is zero due to the orthogonality of the $SU(N)$ generators. In other words, the configurations $(X_i, 0)$ and $(X_i, \delta\theta)$ have the same energy. This gives a supersymmetric set of degenerate configurations, or supermultiplets upon quantization. The starting bosonic configuration is not supersymmetric but is part of a set of degenerate configurations related by supersymmetry.

Consider now the solution (6.12). In this case also, a ρ transformation cannot compensate for an ϵ transformation and the condition for supersymmetry becomes $f_{IJK}\Gamma_I\Gamma_J\epsilon = 0$, where f_{IJK} are the structure constants of $SU(3)$. $L_K = f_{IJK}\Gamma_I\Gamma_J$ obey the commutation rules for $SU(3)$ and, indeed, this defines an $SU(3)$ subgroup of $O(8)$. The spinors of $O(8)$ do not contain singlets under this $SU(3)$ and hence there is again no supersymmetry.

There is, perhaps, no surprise in this lack of supersymmetry, since $\mathbf{CP}^2$ does not admit a spin structure; nevertheless, it is interesting to see how it works out at the matrix level.

6.4. *Other Solutions*

So far we have focused mainly on fuzzy $\mathbf{CP}^2$ as a solution to M(atrix) theory. There are other interesting configurations possible, some of which we have already mentioned. One could consider $\mathbf{CP}^k$ in general; in this case, the required exponent a is given by $1/n$ and the potential energy goes like $n^{k-2}Rr^4$. Thus it is only for $k = 2$ that the potential energy becomes independent of n.

The required condition on the double commutators is satisfied if the X_I span the entire Lie algebra of G as well. The equations of motion for all these cases are generically of the form

$$\left(\frac{N}{R}\right)^{2a} \ddot{f} + C_2(adj) f^3 = 0\,, \tag{6.24}$$

where $C_2(adj)$ represents the quadratic Casimir of G in the adjoint representation and $r = fR^{1-a}$. Since there are only nine X_I, if G is not a product group, its dimension for this type of solutions cannot exceed 9. The case of $G = SU(2)$ reproduces the spherical membrane [34]. In this case $a = 1$, and, as noted before, both kinetic and potential energies have well-defined limits as $N,\ R \to \infty$ with their ratio fixed.

In the large n limit, this solution has the form,

$$\begin{aligned} Z &= \frac{X_1 + iX_2}{\sqrt{2}} = r(t)\,\frac{(n+2)}{(n+1)}\,\frac{\bar{z}}{(1+\bar{z}\cdot z)} \approx r(t)\,\frac{\bar{z}}{(1+\bar{z}\cdot z)}\ , \\ \zeta &= X_3 = \tfrac{1}{2}\,r(t)\,\frac{(n+2)}{(n+1)}\,\frac{(1-\bar{z}\cdot z)}{(1+\bar{z}\cdot z)} \approx \tfrac{1}{2}\,r(t)\,\frac{(1-\bar{z}\cdot z)}{(1+\bar{z}\cdot z)}\ . \end{aligned} \tag{6.25}$$

Clearly we have a two-sphere defined by

$$Z\bar{Z} + \zeta^2 \approx \frac{1}{4}\,r(t)^2\ . \tag{6.26}$$

The radius of the sphere remains finite as $n \to \infty$. Even though the ansatz has the full $SU(2)$-symmetry, there is a further algebraic constraint, viz., (6.26), and this reduces the space of free parameters to $SU(2)/U(1)$.

Another interesting case which was briefly mentioned is that of S^4 geometry which is related to the coset $O(6)/O(5)$, with a further algebraic condition which reduces the dimension to four [26]. There are other cases which can be considered along these lines; for example, for $SU(2)\times SU(2)$, we can set six of the X_i's proportional to the generators and the energies depend only on N/R as $N, R \to \infty$. This can be embedded in $U(N)$ in a block-diagonal way by choosing the representation $(N_1, 1) + (1, N_2)$ with $N = N_1 + N_2$. Presumably this can give two copies of the two-brane in some involved geometrical arrangement in $\mathbf{R}^9$.

7. Fuzzy spaces and the Quantum Hall Effect

There is an interesting connection between the Quantum Hall Effect and fuzzy spaces which we shall briefly discuss now.

7.1. *The Landau problem and* $\mathcal{H}_N$

In the classic Landau problem of a charged particle in a magnetic field $\vec{B}$, one has a number of equally spaced Landau levels. Other than the translational degree of freedom along the magnetic field, the dynamics is confined to a two-dimensional plane transverse to $\vec{B}$. In many physical situations, based on energy considerations, the dynamics is often confined to one Landau level, say the lowest. In this case, the observables are hermitian operators on this subspace of the Hilbert space and are obtained by projecting the full operators to this subspace. The operators representing coordinates, for example, when projected to the lowest Landau level (or any other level), are no longer mutually commuting. The dynamics restricted to the lowest Landau level is thus dynamics on a noncommutative two-plane. This has been known for a long time. One can generalize such considerations to a two-sphere, for example. (The Landau problem on the two-sphere was considered by Haldane [37]; we follow [10].) One can have a uniform magnetic field on the two-sphere which is normal to it; this would be the radial field of a magnetic monopole sitting at the origin if we think of the two-sphere as embedded in the usual way in $\mathbf{R}^3$. (The potential for a uniform magnetic field is given by (2.1).) Since $S^2 = SU(2)/U(1)$, we may think of the wave functions for a particle on S^2 as functions of $g \in SU(2)$ with the condition that they are invariant under $g \to gh$, $h \in U(1)$, so that they actually project down to S^2. We may take the $U(1)$ direction to be along the T_3 direction in the $SU(2)$ algebra. Since a basis of functions for $SU(2)$ is given by the Wigner $\mathcal{D}$-functions, a basis for functions on S^2 is given by the $SU(2)$ Wigner functions $\mathcal{D}^{(j)}_{mk}(g)$, with trivial right action of $U(1)$, in other words, the $U_R(1)$ charge $k = 0$. In this language, derivatives on S^2 can be identified as $SU(2)$ right rotations on g (denoted by $SU_R(2)$) satisfying an $SU(2)$ algebra

$$\big[R_+,\ R_-\big] = 2\ R_3\,, \tag{7.1}$$

where $R_\pm = R_1 \pm iR_2$. $R_\pm$ are dimensionless quantities. The standard covariant derivatives, with the correct dimensions, are

$$D_\pm = i\ \frac{R_\pm}{r}\,, \tag{7.2}$$

where r is the radius of the sphere. In the presence of the magnetic monopole, the commutator of the covariant derivatives is related to the magnetic field, in other words, we need $[D_+, D_-] = -2B$. With the identification (7.2), and the commutation rule (7.1), we see that this fixes R_3 to be half the monopole number n, with $n = 2Br^2$. Therefore the wave functions on S^2 with the magnetic field background are of the form $\mathcal{D}^{(j)}_{m,\frac{n}{2}}(g)$. The Dirac quantization rule is seen, from this point of view, as related to the quantization of angular momentum, as first noted by Saha [39]. For a detailed description of the formalism presented here and analysis of fields of various spin on S^2 on the monopole background see [38].

We can now write down the one-particle Hamiltonian

$$\begin{aligned} H &= -\frac{1}{4\mu}\big(D_+D_- + D_-D_+\big) \\ &= \frac{1}{2\mu r^2}\left(\sum_{A=1}^{3} R_A^2 - R_3^2\right), \end{aligned} \tag{7.3}$$

where μ is the particle mass. For the eigenvalue $\frac{1}{2}n$ to occur as one of the possible values for R_3, so that we can form $\mathcal{D}^{(j)}_{m,\frac{n}{2}}(g)$, we need $j = \frac{1}{2}n + q$, $q = 0, 1, ...$ Since $R^2 = j(j+1)$, the energy eigenvalues are

$$
\begin{aligned}
E_q &= \frac{1}{2\mu r^2}\left[(\tfrac{1}{2}n + q)(\tfrac{1}{2}n + q + 1) - \frac{n^2}{4}\right] \\
&= \frac{B}{2\mu}(2q+1) + \frac{q(q+1)}{2\mu r^2}\ . \qquad (7.4)
\end{aligned}
$$

The integer q plays the role of the Landau level index. The lowest Landau level ($q = 0$), or the ground state, has energy $B/2\mu$, and the states $q > 0$ are separated by a finite energy gap. The degeneracy of the q-th Landau level is $2j + 1 = n + 1 + 2q$. (Notice that, in the limit $r \to \infty$, the planar image of the sphere under the stereographic map becomes flat and so this corresponds to the standard planar Landau problem.) We see that, as $r \to \infty$, (7.4) reproduces the known planar result for the energy eigenvalues and the degeneracy.

In the limit of large magnetic fields, the separation of the levels is large, and it is meaningful to restrict dynamics to one level, say the lowest, if the available excitation energies are small compared to $B/2\mu$. In this case, $j = \frac{1}{2}n$, $R_3 = \frac{1}{2}n$, so that we have the highest weight state for the right action of $SU(2)$. The condition for the lowest Landau level is $R_+\Psi = 0$ and this level has degeneracy $n + 1$.

We now see the connection to fuzzy S^2. The Hilbert space of the lowest Landau level corresponds exactly to the symmetric rank n representation of $SU(2)$. The condition $R_+\Psi = 0$, which was used as the condition restricting the wave functions to depend on only half of the phase space coordinates in the quantization procedure outlined in section 2, is obtained for the Landau problem as well, but as a condition choosing the lowest Landau level. The Hilbert subspace spanned by $\mathcal{D}^{(\frac{n}{2})}_{m,\frac{n}{2}}$ is the same and hence all observables for the lowest Landau level correspond to the observables of the fuzzy S^2.

This correspondence can be extended to the Landau problem on other spaces. For all $\mathbf{CP}^k$ with a $U(1)$ background field, we have an exact correspondence between the lowest Landau level and fuzzy $\mathbf{CP}^k$. The background field specifies the choice of the eigenvalues R_i in the Wigner $\mathcal{D}$-functions; the lowest Landau level condition becomes the polarization condition for the wave functions [10]. The wave functions for the lowest Landau level are exactly those given in (2.21); they are characterized by the integer n, which gives the rank of the symmetric $SU(k+1)$ representation and corresponds to a uniform magnetic field along the direction t_{k^2+2k}, as seen from (2.18).

An especially interesting case is that of $\mathbf{CP}^3$. Because this is an S^2 bundle over S^4, the Landau problem on $\mathbf{CP}^3$ is equivalent to a similar problem on S^4 with an $SU(2)$ background field [40]. We will come back to this briefly.

7.2. *A quantum Hall droplet and the edge excitations*

In discussing the physics of the Quantum Hall Effect, we need to go beyond just the construction of the states. Typically one has a number of states occupied by electrons, which are fermions, and so there is no double occupancy for any state. Generally these electrons cluster into a droplet. Dynamically this is due to an additional potential $\hat{V}$; electrons tend to localize near the minimum of the potential. The excitations of this droplet are of interest in quantum Hall systems. Since there cannot be double occupancy and there is conservation of the number of electrons, the excitations are deformations of the droplet which preserve the total volume of occupied states. In the large n limit, these are surface deformations of an almost continuous droplet; they are hence called the edge excitations.

We can specify the droplet by a diagonal density matrix $\hat{\rho}_0$ which is equal to 1 for occupied states and zero for unoccupied states. The dynamical modes are then fluctuations which keep the number of occupied states, or the rank of $\hat{\rho}_0$, fixed. They are thus given by a unitary transformation of $\hat{\rho}_0$, $\hat{\rho}_0 \to \hat{U}\hat{\rho}_0\hat{U}^\dagger$. One can write an action for these modes as

$$S = \int dt \left[i\mathrm{Tr}\,(\hat{\rho}_0 \hat{U}^\dagger \partial_t \hat{U}) - \mathrm{Tr}\,(\hat{\rho}_0 \hat{U}^\dagger \hat{H} \hat{U}) \right] , \tag{7.5}$$

where $\hat{H}$ is the Hamiltonian. Since we are in the lowest Landau level of fixed energy, we can take the Hamiltonian to be just the potential $\hat{V}$. Variation of $\hat{U}$ leads to the extremization condition for S as

$$i\,\frac{\partial \hat{\rho}}{\partial t} = [\hat{H}, \hat{\rho}] \tag{7.6}$$

which is the expected evolution equation for the density matrix. In the large n limit, we can simplify this action by writing $\hat{U} = \exp(i\hat{\Phi})$, and replacing operators by their symbols, matrix products by $*$-products and the trace by $\mathbf{CP}^k$-integration, as discussed in section 2. We also have to write $\hat{U} \to 1 + i\Phi - \frac{1}{2!}\Phi * \Phi + \cdots$, where Φ is the symbol for $\hat{\Phi}$. We will consider a droplet with M occupied states, with M very large. The large n limit of (7.5) can then be obtained, for a simple spherical droplet, as [10]

$$S_{\mathbf{CP}^k} \approx -\frac{1}{4\pi^k} M^{k-1} \int d\Omega_{S^{2k-1}} \left[\frac{\partial \Phi}{\partial t} \left(\mathcal{L}\Phi \right) + \omega \left(\mathcal{L}\Phi \right)^2 \right] , \tag{7.7}$$

where $d\Omega_{S^{2k-1}}$ denotes the volume element on the sphere S^{2k-1}, which is the boundary of the droplet; the factor M^{k-1} is as expected for a droplet of radius $\sim \sqrt{M}$. The operator $\mathcal{L}$ is identified in terms of the coordinates $\bar{z}, z$ as

$$\mathcal{L} = i \left(z \cdot \frac{\partial}{\partial z} - \bar{z} \cdot \frac{\partial}{\partial \bar{z}} \right) . \tag{7.8}$$

Terms which vanish as $n \to \infty$ have been dropped. As an example, we have taken a potential

$$\hat{V} = \sqrt{\frac{2k}{k+1}}\, \omega \left(T_{k^2+2k} + \frac{nk}{\sqrt{2k(k+1)}} \right) \tag{7.9}$$

with ω as a constant parameter. (The form of the action is not sensitive to the specifics of the potential; more generally one has $\omega = \frac{1}{n} \frac{\partial V}{\partial(\bar{z}\cdot z)}$.) The action (7.7) is a generalization of a chiral Abelian Wess-Zumino-Witten theory. (The calculations leading to (7.7) are not complicated, after the discussion of the large n limit in section 2; but they are still quite involved and we refer the reader to the original articles.)

One can also consider non-Abelian background fields, say, constant $SU(k)$ backgrounds for $\mathbf{CP}^k$, since the latter is $SU(k+1)/U(k)$. In this case, the wave functions must obey the conditions

$$\begin{aligned} \hat{R}_a \; \Psi^J_{m,\alpha}(g) &= (T_a^{\tilde{J}})_{\alpha\beta} \Psi^J_{m,\beta}(g) \; , \\ \hat{R}_{k^2+2k} \; \Psi^J_{m,\alpha}(g) &= -\frac{nk}{\sqrt{2k(k+1)}} \; \Psi^J_{m,\alpha}(g) \end{aligned} \tag{7.10}$$

since there is a background $SU(k)$-field. The wave functions must transform under right rotations as a representation of $SU(k)$, $(T_a^{\tilde{J}})_{\alpha\beta}$ being the representation matrices for the generators of $SU(k)$ in the representation $\tilde{J}$. n is an integer characterizing the Abelian part of the background field. α, β label states within the $SU(k)$ representation $\tilde{J}$ (which is itself contained in the representation J of $SU(k+1)$). The index α carried by the wavefunctions (7.10) is basically the gauge index. The wave functions are sections of a $U(k)$ bundle on $\mathbf{CP}^k$. The wave functions for the lowest Landau level are thus given by

$$\begin{aligned} \Psi^J_{m,\alpha}(g) &= \sqrt{N} \; \langle J, L | \; \hat{g} \; | J, (\tilde{J}, \alpha), -n \rangle \\ &= \sqrt{N} \; \mathcal{D}^J_{m;\alpha}(g) \; . \end{aligned} \tag{7.11}$$

The symbol corresponding to an operator $\hat{F}$ is now a matrix, defined as

$$F_{\alpha\beta}(g) = \sum_{km} \mathcal{D}_{k,\alpha}(g)\ F_{km}\mathcal{D}^*_{m,\beta}(g)\ . \tag{7.12}$$

The simplification of the action (7.5) will now involve a field G which is a unitary matrix, an element of $U(dim\tilde{J})$. The large n limit can be calculated as in the Abelian case and gives the action [11]

$$\begin{aligned}\mathcal{S}(G) = \frac{1}{4\pi^k} M^{k-1} \int_{\partial\mathcal{D}} dt\ \mathrm{tr}\left[\left(G^\dagger \dot{G}\ +\ \omega\ G^\dagger \mathcal{L} G\right) G^\dagger \mathcal{L} G\right] \\ + (-1)^{\frac{k(k-1)}{2}} \frac{i}{4\pi} \frac{M^{k-1}}{(k-1)!} \int_{\mathcal{D}} dt\ 2\,\mathrm{tr}\left[G^\dagger \dot{G} (G^{-1} DG)^2\right] \wedge \left(\frac{i\Omega}{\pi}\right)^{k-1} .\end{aligned} \tag{7.13}$$

This is a chiral, gauged Wess-Zumino-Witten (WZW) action generalized to higher dimensions. Here the first term is on the boundary $\partial\mathcal{D}$ of the droplet and it is precisely the gauged, non-Abelian analog of (7.7). The operator $\mathcal{L}$ in (7.13) is the gauged version of (7.8),

$$\mathcal{L} = i\left(z^i D_i - \bar{z}^i D_{\bar{i}}\right)\ . \tag{7.14}$$

The gauge covariant derivative is given by $D = \partial + [\mathcal{A},\quad]$, where $\mathcal{A}$ is the $SU(k)$ gauge potential, given by $\mathcal{A}^a_i = 2i\mathrm{Tr}(t^a g^{-1} dg)$. (This potential corresponds to the spin connection on $\mathbf{CP}^k$; the corresponding Riemann curvature is constant in the tangent frame basis. The gauge field we have chosen is proportional to this.) The second term in (7.13), written as a differential form, is a higher dimensional Wess-Zumino term; it is an integral over the droplet $\mathcal{D}$ itself, with the radial variable playing the role of the extra dimension. As expected, since we have an $SU(k)$ background, the action has an $SU(k)$ gauge symmetry.

We shall discuss the fuzzy space point of view regarding the edge excitations in the Quantum Hall Effect shortly; before we do that, we shall briefly consider the Hall Effect on spheres.

7.3. *Quantum Hall Effect on spheres*

The most interesting cases of the Quantum Hall Effect on spheres pertain to S^4 and S^3.

The edge excitations for the droplet on S^4 was recently suggested by Zhang and Hu as a model for higher spin gapless states, including the graviton [40]. (In fact, this is what started many investigations into the higher dimensional Quantum Hall Effect [41,42].) This is a very nice idea, although it has not yet worked out as hoped. The action for this case can be obtained from (7.7) by utilizing the fact that $\mathbf{CP}^3$ is an S^2-bundle over S^4. We can describe $\mathbf{CP}^3$ by the four complex coordinates Z_α, $\alpha = 1, ..., 4$, with the identification $Z_\alpha \sim \lambda Z_\alpha$ where λ is any complex number except zero, $\lambda \in \mathbf{C} - \{0\}$. Explicitly, we may write Z_α as $\sqrt{\bar{Z} \cdot Z}\; u_\alpha = \sqrt{\bar{Z} \cdot Z}\; g_{\alpha 4}$, but for the present purpose, it is more convenient to write it in terms of two two-component spinors w, π as

$$(Z_1, Z_2, Z_3, Z_4) = (w_1, w_2, \pi_1, \pi_2) \; . \tag{7.15}$$

Coordinates x_μ on S^4 are then defined by

$$w = (x_4 - i\sigma \cdot x)\; \pi \; . \tag{7.16}$$

The scale invariance $Z \sim \lambda Z$ can be realized as the scale invariance $\pi \sim \lambda\pi$; the π's then describe a $\mathbf{CP}^1 = S^2$. This will be the fiber space. The coordinates x_μ are the standard stereographic coordinates for S^4; one can in fact write

$$y_0 = \frac{1 - x^2}{1 + x^2} \; , \qquad y_\mu = \frac{2x_\mu}{1 + x^2} \tag{7.17}$$

to realize the S^4 as embedded in $\mathbf{R}^5$. The definition of x_μ in terms of w may be solved as

$$\begin{aligned} x_4 &= \frac{1}{2}\, \frac{\bar{\pi} w + \bar{w} \pi}{\bar{\pi}\pi} \; , \\ x_i &= \frac{i}{2}\, \frac{\bar{\pi}\sigma_i w - \bar{w}\sigma_i \pi}{\bar{\pi}\pi} \; . \end{aligned} \tag{7.18}$$

There is a natural subgroup, $SU_L(2) \times SU_R(2)$, of $SU(4)$, with π transforming as the fundamental representation of $SU_L(2)$ and w transforming as the fundamental representation of $SU_R(2)$.

The Kähler two-form on $\mathbf{CP}^3$ is given, as in (2.3), by

$$\Omega = -i \left[\frac{d\bar{Z} \cdot dZ}{\bar{Z} \cdot Z} - \frac{d\bar{Z} \cdot Z \; \bar{Z} \cdot dZ}{(\bar{Z} \cdot Z)^2} \right] . \tag{7.19}$$

This is invariant under $Z \to \lambda Z$, and $\bar{Z} \to \lambda \bar{Z}$. We can reduce this using

(7.16), (7.18) to get

$$\begin{aligned} \Omega_{\mathbf{CP}^3} &= \Omega_{\mathbf{CP}^1} - i\ F\ , \\ F &= dA + A\ A\ , \\ A &= i\,\frac{N^a \eta^a_{\mu\nu} x^\mu dx^\nu}{(1+x^2)}\ , \end{aligned} \tag{7.20}$$

where

$$\begin{aligned} \eta^a_{\mu\nu} &= \epsilon_{a\mu\nu 4} + \delta_{a\mu}\delta_{4\nu} - \delta_{a\nu}\delta_{4\mu}\ , \\ N^a &= \bar{\pi}\sigma^a\pi/\bar{\pi}\pi\ . \end{aligned} \tag{7.21}$$

$\eta^a_{\mu\nu}$ is the 't Hooft tensor and N^a is a unit three-vector, which may be taken as parametrizing the fiber $\mathbf{CP}^1 \sim S^2$. The field F is the instanton field. We see that we can get an instanton background on S^4 by taking a $U(1)$ background field on $\mathbf{CP}^3$ which is proportional to the Kähler form.

The action (7.7) may now be used with the separation of variables indicated; it simplifies to

$$S = -\frac{M}{4\pi^2}\ n \int d\mu_{\mathbf{CP}^1} \ \int d\Omega_3 \left[\frac{\partial \Phi}{\partial t}(\mathcal{L}\Phi) + \omega(\mathcal{L}\Phi)^2 \right]\ , \tag{7.22}$$

where $\mathcal{L}\Phi = 2x^\nu K^{\mu\nu}\partial_\mu \Phi$. Φ's are to be expanded in terms of harmonics on $\mathbf{CP}^1$ which correspond to the representations of $SU_L(2)$ in $SU_L(2)\times SU_R(2)$. (This is the subgroup corresponding to the instanton gauge group.) Since Φ is a function on $\mathbf{CP}^1$, we must have invariance under the scaling $\pi \longrightarrow \lambda\pi$. The mode expansion for Φ is thus given by [10]

$$\begin{aligned} \Phi = \sum_{l\geq m}\ & C^{(\dot{A})_m(\dot{B})_l(C)_{l-m+k}(D)_k}\ f_{(\dot{A})_m(\dot{B})_l(C)_{l-m+k}(D)_k} \\ & + \sum_{l<m}\ \tilde{C}^{(\dot{A})_m(\dot{B})_l(C)_k(D)_{m-l+k}}\ \tilde{f}_{(\dot{A})_m(\dot{B})_l(C)_k(D)_{m-l+k}}\ , \end{aligned} \tag{7.23}$$

where the mode functions have the form

$$\begin{aligned} f_{(\dot{A})_m(\dot{B})_l(C)_{l-m+k}(D)_k} &= \frac{1}{(\bar{\pi}\cdot\pi)^{l+k}}\ \tilde{w}_{\dot{A}_1}\cdots\tilde{w}_{\dot{A}_m} w_{\dot{B}_1}\cdots w_{\dot{B}_l} \\ &\qquad \times\tilde{\pi}_{C_1}\cdots\tilde{\pi}_{C_{l-m+k}}\pi_{D_1}\cdots\pi_{D_k}\ , \\ \tilde{f}_{(\dot{A})_m(\dot{B})_l(C)_k(D)_{m-l+k}} &= \frac{1}{(\bar{\pi}\cdot\pi)^{m+k}}\ \tilde{w}_{\dot{A}_1}\cdots\tilde{w}_{\dot{A}_m} w_{\dot{B}_1}\cdots w_{\dot{B}_l} \\ &\qquad \times\tilde{\pi}_{C_1}\cdots\tilde{\pi}_{C_k}\pi_{D_1}\cdots\pi_{D_{m-l+k}}\ , \end{aligned} \tag{7.24}$$

and $(\dot{A})_m = \dot{A}_1 \cdots \dot{A}_m$, $(C)_k = C_1 \cdots C_k$ and similarly for the other indices. Each function f $(\tilde{f})$ transforms as an irreducible representation of $SU_L(2) \times SU_R(2)$, with the j-values $\frac{1}{2}|l - m| + k$ and $\frac{1}{2}(l + m)$ respectively. The action (7.22) and the mode expansion (7.23) show clearly the emergence of the higher spin gapless modes. In particular, it is possible to obtain massless spin-2 excitations. The difficulty, however, is that there are many other modes which do not all combine into a relativistically invariant theory [10, 42]. Perhaps, some clever projection, such as the GSO projection in string theory, may be possible.

One can formulate the Quantum Hall Effect on other spheres as well. For S^3, for example, we can use the fact that $S^3 \sim SU(2) \times SU(2)/[SU(2)]$; this shows that it is possible to have a constant $SU(2)$ gauge field on S^3 (which will be proportional to the Riemann curvature of S^3). Taking this gauge field one can obtain Landau level states and an edge action [27]. If we denote the generators of the Lie algebra of the two $SU(2)$'s by L_a and R_a, we can take the derivatives on S^3 to be proportional to $L_a - R_a$, with the $SU(2)$ being divided out defined by $J_a = L_a + R_a$. The Landau levels will correspond to the Wigner $\mathcal{D}$-functions of $SU(2) \times SU(2)$, with the representation under J_a's specifying the background field.

In the case of the $\mathbf{CP}^k$'s discussed earlier, the lowest Landau level corresponds to the Hilbert space $\mathcal{H}_N$ of the fuzzy version of the space. This suggests that one can utilize the construction of Landau levels on S^3, and more generally on other spheres, to get a definition of fuzzy spheres. Actually, for the three-sphere, the lowest Landau level corresponds, not to a fuzzy S^3, but a fuzzy $S^3/\mathbf{Z}_2$ [27]; the realization of $S^3/\mathbf{Z}_2$ is essentially identical to our discussion in section 5. Spheres of other dimensions can be considered using the fact that $S^k = SO(k+1)/SO(k)$; constant fields which correspond to $SO(k)$ gauge fields are then possible and one can carry out a similar analysis for the Quantum Hall Effect.

7.4. *The fuzzy space – Quantum Hall Effect connection*

We now return to the question of what Quantum Hall Effect has to do with fuzzy spaces.

Fuzzy spaces are based on the trio $(\mathcal{H}_N, Mat_N, \Delta_N)$. The Hilbert space $\mathcal{H}_N$ is obtained by quantization of the action (2.13); the wave functions are sections of an appropriate $U(1)$ bundle on the space M whose fuzzy version we are constructing. Mat_N is then the matrix algebra of linear transformations of this Hilbert space. The lowest Landau level for Quantum Hall Effect on a compact manifold M, as we mentioned before, defines a finite dimensional Hilbert space which is identical to $\mathcal{H}_N$. This is clear, since, with a background magnetic field, the wave functions are sections of a $U(1)$ bundle on M. Thus observables of the quantum Hall system are elements of Mat_N.

We can go further and ask how we may characterize subspaces of fuzzy spaces. A region, which is topologically a disk, may be specified by a projection operator. We assign a value 1 to the projection operator for states inside the region and zero for states outside the region. Notice that this is precisely what the droplet density operator does. The fluctuations of the projection operator preserving its rank are the analogs of volume-preserving transformations. In the large n limit, they correspond to the field Φ. We may thus regard $\hat{U}$ as specifying the modes corresponding to different embeddings of a fuzzy disk in the full fuzzy space [43]. Clearly this is of geometrical interest. In fact, we can go a bit further with this analogy. The action for $\hat{U}$, namely (7.5), is the same as the action (2.13), except that the group is now $U(N)$ and the invariant subgroup is chosen by the projection operator or density matrix. The quantization of this action will thus lead to another fuzzy space, which will correspond to the set of scalar fields (corresponding to Φ) on the boundary of the chosen region. (For these arguments, we may even set $\hat{V} = 0$.)

The case of the non-Abelian background is presumably related to vector bundles rather than functions on the fuzzy space, in a way which is not yet completely clarified. The fact that this leads to an action of the WZW type in the large n limit is also quite interesting. From what we have said so far, it is clear that there is a set of concepts linking fuzzy spaces and the Quantum Hall Effect, with the possibility of a more fruitful interplay of ideas.

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References

1. A. Connes, *Nocommutative Geometry* (Academic Press, 1994).
2. J. Madore, *An Introduction to Noncommutative Geometry and its Physical Applications*, LMS Lecture Notes 206 (1995).
3. G. Landi, *An Introduction to Noncommutative Spaces and their Geometry*, Lecture Notes in Physics, Monographs m51 (Springer-Verlag, 1997).
4. J. Fröhlich and K. Gawedzki, *Conformal Field Theory and the Geometry of Strings*, in *Mathematical Quantum Theory*, Proceedings of the conference on Mathematical Quantum Theory, Vancouver, 1993 [hep-th/9310187].
5. A. Connes, Publ. I.H.E.S. **62**, 257 (1986); Commun. Math. Phys. **117**, 673 (1988).
6. A.H. Chamseddine and A. Connes, Phys. Rev. Lett. **77**, 4868 (1996); Commun. Math. Phys. **186**, 731 (1997); A.H. Chamseddine, G. Felder and J. Fröhlich, Commun. Math. Phys. **155**, 205 (1993); A. Connes and J. Lott, Nucl. Phys. B (Proc. Suppl.) **18**, 29 (1990; S. Doplicher, K. Fredenhagen and J.E. Roberts, Phys. Lett. B **331**, 39 (1994); D. Kastler, Rev. Math. Phys. **5**, 477 (1993); *ibid.*, **8**, 103 (1996); G. Landi and C. Rovelli, Phys. Rev. Lett. **78**, 3051 (1997).
7. J. Sniatycki, *Geometric Quantization and Quantum Mechanics* (Springer-Verlag, 1980); N.M.J. Woodhouse, *Geometric Quantization* (Clarendon Press, 1992).
8. A.M. Perelomov, *Generalized Coherent States and Their Applications* (Springer-Verlag, 1996).
9. M.R. Douglas and N.A. Nekrasov, Rev. Mod. Phys. **73**, 977 (2001); J.A. Harvey, Lectures at Komaba 2000: *Nonperturbative Dynamics in String Theory*, hep-th/0102076; A.P. Balachandran, Pramana, **59**, 359 (2002) [hep-th/0203259], and references therein.
10. D. Karabali and V.P. Nair, Nucl. Phys. B **641**, 533 (2002); *ibid.* **679**, 427 (2004).
11. D. Karabali and V.P. Nair, *Edge states for quantum Hall droplets in higher dimensions and a generalized WZW model*, hep-th/0403111, to be published in Nucl. Phys. B.
12. S. Randjbar-Daemi, A. Salam and J. Strathdee, Phys. Rev. B **48**, 3190 (1993) [hep-th/9210145].
13. V.P. Nair and S. Randjbar-Daemi, Nucl. Phys. B **533**, 333 (1998).
14. H. Grosse and A. Strohmaier, Lett. Math. Phys. **48**, 163 (1999).
15. A.P. Balachandran *et al*, J. Geom. Phys. **43**, 184 (2002); G. Alexanian *et al* J. Geom. Phys. **42**, 28 (2002).
16. G. Alexanian, A. Pinzul and A. Stern, Nucl. Phys. B **600**, 531 (2001).
17. V.P. Nair, Nucl. Phys. B **651**, 313 (2003).
18. J. Madore, Class. Quant. Grav. **9**, 69 (1992).
19. In addition to the general references, for a sample of issues of recent interest, see also: U. Carow-Watamura and S. Watamura, Commun. Math. Phys. **212**, 395 (2000); S. Baez, A.P. Balachandran, S. Vaidya and B. Ydri, Commun. Math. Phys. **208**, 787 (2000); A.P. Balachandran, T.R. Govindarajan and B. Ydri, Mod. Phys. Lett. A **15**, 1279 (2000); *The fermion doubling problem and noncommutative geometry. II*, hep-th/0006216; G. Landi, Rev. Math. Phys. **12**, 1367 (2000); A.P. Balachandran and S. Vaidya, Int. J. Mod. Phys. A **16**, 17 (2001); H. Grosse, C.W. Rupp and A. Strohmaier, J. Geom. Phys. **42**, 54 (2002); B.P. Dolan, D. O'Connor and P. Prešnajder, JHEP **0203**, 013 (2002); S. Iso *et al*, Nucl. Phys. B **604**, 121 (2001); P. Valtancoli, Mod. Phys. Lett. A **16**, 639 (2001); A.P. Balachandran and G. Immirzi, Phys. Rev. D **68**, 065023 (2003); Int. J. Mod. Phys. A **18**, 5931 (2003); S. Vaidya and B. Ydri, Nucl. Phys. B **671**, 401 (2003); B. Ydri, Nucl. Phys. B **690**, 230 (2004) [hep-th/0403233]; H. Aoki, S. Iso and K. Nagao, Phys. Rev. D**67**, 065018 (2003); *ibid.* D**67**, 085005 (2003); Nucl. Phys. B**684**, 162 (2004).

20. H. Grosse, C. Klimčik and P. Prešnajder, Int. J. Theor. Phys. **35**, 231 (1996); Commun. Math. Phys. **178**, 507 (1996); H. Grosse and P. Prešnajder, Lett. Math. Phys. **46**, 61 (1998); P. Prešnajder, J. Math. Phys. **41**, 2789 (2000).
21. S.W. Hawking and C.N. Pope, Phys. Lett. B**73**, 42 (1978); I.M. Benn, B.P. Dolan and R.W. Tucker, Phys. Lett. Bbf 150, 100 (1985).
22. G. R. Dvali, S. Randjbar-Daemi and R. Tabbash, extra Phys. Rev. D **65**, 064021 (2002) [hep-ph/0102307].
23. S. Ramgoolam, Nucl. Phys. B **610**, 461 (2001); JHEP **0210**, 064 (2002); P.M. Ho and S. Ramgoolam, Nucl. Phys. **B627**, 266 (2002); Y. Kimura, Nucl. Phys. B **664**, 512 (2003).
24. B.P. Dolan and D. O'Connor, JHEP **0310**, 060 (2003); B.P. Dolan, D. O'Connor and P. Prešnajder, JHEP **0402**, 055 (2004).
25. Yasuhiro Abe, *A construction of fuzzy S**4*, hep-th/0406135.
26. J. Castelino, S-M. Lee and W. Taylor IV, Nucl. Phys. B **526**, 334 (1998).
27. V.P. Nair and S. Randjbar-Daemi, Nucl. Phys. B **679**, 447 (2004).
28. This is in many books, see, for example, Wu-Ki Tung, *Group Theory in Physics* (World Scientific, 1985).
29. E. Witten, Nucl. Phys. B **443**, 85 (1995).
30. T. Banks, W. Fischler, S. Shenker and L. Susskind, Phys. Rev. D **55**, 5112 (1997).
31. N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, Nucl. Phys. B **498**, 467 (1997).
32. For a review of M(atrix) theory, see W. Taylor IV, Rev. Mod. Phys. **73**, 419 (2001).
33. B. de Wit, J. Hoppe and H. Nicolai, Nucl. Phys. B **305** (1988) 545.
34. D. Kabat and W. Taylor IV, *Adv. Theor. Math. Phys.* **2**, 181 (1998);
S-J. Rey, *Gravitating M(atrix) Q-balls*, hep-th/9711081;
35. T. Banks, N. Seiberg and S. Shenker, Nucl. Phys. B **490**, 91 (1997);
O.J. Ganor, S. Ramgoolam and W. Taylor IV, Nucl. Phys. B **492**, 191 (1997).
36. D. Kabat and W. Taylor IV, Phys. Lett. B **426**, 297 (1998).
37. F.D. Haldane, Phys. Rev. Lett. **51**, 605 (1983).
38. S. Randjbar-Daemi, A. Salam and J. Strathdee, Nucl. Phys. B **214**, 491 (1983).
39. M.N. Saha, Indian J. Phys. **10**, 141 (1936).
40. S.C. Zhang and J.P. Hu, Science **294**, 823 (2001); J.P. Hu and S.C. Zhang, *Collective excitations at the boundary of a 4D quantum Hall droplet*, cond-mat/0112432.
41. M. Fabinger, JHEP **0205** (2002) 037; Y.X. Chen, B.Y. Hou and B.Y. Hou, Nucl. Phys. B **638** (2002) 220; Y. Kimura, Nucl. Phys. B **637**, 177 (2002); Y.X. Chen, *Matrix models of 4-dimensional quantum Hall fluids*, hep-th/0209182; B. Dolan, gauge JHEP **0305**, 018 (2003) [hep-th/0304037]. B. A. Bernevig, J.P. Hu, N. Toumbas and S.C. Zhang, Phys. Rev. Lett. **91**, 236803 (2003) [cond-mat/0306045]. G. Meng, Phys. Rev. Lett. **91**, 236803 (2003); B.A. Bernevig, C.H. Chern, J.P. Hu, N. Toumbas and S.C. Zhang, Ann. Phys. **300**, 185 (2002); S.C. Zhang, Phys. Rev. Lett. **90**, 196801; G. Sparling, *Twistor theory and the four-dimensional Quantum Hall Effect of Zhang and Hu*, cond-mat/0211679; B. Dolan, JHEP **0305**, 018 (2003); G. Meng, J. Phys. A **36**, 9415 (2003); S. Bellucci, P. Casteill and A. Nersessian, Phys. Lett. B **574**, 121 (2003); K. Hasebe and Y. Kimura, *Dimensional Hierarchy in Quantum Hall Effects on Fuzzy Spheres*, hep-th/0310274.
42. H. Elvang and J. Polchinski, *The Quantum Hall Effect on R**4*, hep-th/0209104.
43. For more on fuzzy disks and edge states, see F. Lizzi, P. Vitale and A. Zampini, Mod. Phys. Lett. A **18**, 2381 (2003); JHEP **0308**, 057 (2003).