3D GEORGI–GLASHOW MODEL AND CONFINING STRINGS AT ZERO AND FINITE TEMPERATURES

DMITRI ANTONOV*

Institute of Physics, Humboldt University of Berlin, Newtonstr. 15, 12489 Berlin, Germany

MARIA CRISTINA DIAMANTINI

Dipartimento di Fisica dell'Università di Perugia and I.N.F.N. sezione di Perugia, Via A. Pascoli, 06123 Perugia, Italy

In this review, we discuss the confining and finite-temperature properties of the 3D SU(N) Georgi–Glashow model, and of 4D compact QED. At zero temperature, we derive string representations of both theories, thus constructing the SU(N) version of Polyakov's theory of confining strings. We discuss the geometric properties of confining strings, as well as the appearance of the string θ term from the field-theoretical one in 4D, and k-string tensions at $N > 2$. In particular, we point out the relevance of negative stiffness for stabilizing confining strings, an effect recently re-discovered in material science. At finite temperature, we present a derivation of the confining-string free energy and show that, at the one-loop level and for a certain class of string models in the large-D limit, it matches that of QCD at large N. This crucial matching is again a consequence of the negative stiffness. In the discussion of the finite-temperature properties of the 3D Georgi–Glashow model, in order to be closer to QCD, we mostly concentrate at the effects produced by some extensions of the model by external matter fields, such as dynamical fundamental quarks or photinos, in the supersymmetric generalization of the model.

*Permanent address: ITEP, B. Cheremushkinskaya 25, RU-117 218, Moscow, Russia.

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Many papers related to this review would have never been written without Ian Kogan's seminal ideas on the finite-temperature phase transitions in string theory and in the 3D Georgi–Glashow model. The sudden death of Ian Kogan was a loss for physics and a personal tragedy for everybody who had known him, in particular for the authors of this manuscript. In the personal sense, Ian was a very bright man with a warm heart, and he will be deeply missed as a good friend. We nevertheless strongly believe that Ian's ideas will survive over decades, inspiring many new generations of physicists.

1. Introduction

During the last 30 years, the problem of quark confinement remains as a great challenge not only to QCD theorists, but to the whole theoretical high-energy physics community. In general, confinement can be defined as the absence in the spectrum of a certain field theory of physical $|\mathrm{in}\rangle$ and $|\mathrm{out}\rangle$ states of some particles, whose fields are nevertheless present in the Lagrangian of that theory (see e.g. [1–3] for reviews and books on confinement). With regard to QCD, this definition means the absence of asymptotic quark and gluon states, i.e. states which carry color.[a] This fact is reflected in the linear growth with the distance of the potential between two color particles, as well as in the logarithmic growth of the strong-coupling constant. The latter makes the standard perturbative diagrammatic techniques inapplicable at distances larger than $\mathcal{O}\left(\Lambda_{\mathrm{QCD}}^{-1}\right)$. However, such distances are of primary interest, since only there physically observable colorless states of hadrons are formed, whereas at smaller distances one deals with the unphysical colored states – quarks, gluons, and ghosts. A question which may naturally be posed at this point, is whether the QCD Lagrangian, operating with these colored states only, eventually yields the correct description of colorless degrees of freedom as well. A reliable indication that this is really the case comes from the simulations of the QCD Lagrangian on a lattice (for recent reviews see e.g. [2, 5, 6]), which yield a reasonably good description of hadronic spectra. However, as of today, a systematic *analytic* way of describing large-distance effects in QCD in terms of microscopic (colored) degrees of freedom is unfortunately lacking. The breakdown of the perturbative expansion at the large distances under discussion naturally introduces the notion *nonperturbative* for these effects, as well as for the techniques

[a] An attempt to construct a momentum-space interpretation of this phenomenon has been done (see e.g. Ref. [4]) In this review, we will, however, prefer to deal with the conventional space-time picture, which enables one to directly operate with such notions as the vacuum correlation length or potential between color particles.

which physicists attempt to invent for their description.

It is actually not a great fault that the perturbative expansion is inapplicable at large distances, since, being formulated in terms of colored states, it is anyway not suitable to keep gauge invariance under control. Indeed, an individual diagram is always gauge-dependent, merely because a certain gauge fixing should be performed before one starts to compute diagrams. Contrary to that, hadronic states, being colorless, are gauge-invariant. Moreover, since the QCD vacuum possesses an unbroken gauge symmetry.[b], only gauge-invariant amplitudes, being averaged over the vacuum, yield a nonvanishing result.[c] Although the gauge dependence disappears when one sums up contributions of individual diagrams to a certain gauge-invariant quantity, it would be more natural to have an expansion already operating with such quantities at any intermediate stage. Such an expansion, which is the expansion in the inverse number of colors, has been proposed by 't Hooft [9] and further developed in the framework of loop equations [10] (see e.g. [11] for reviews). This is a classic example of a nonperturbative approach to QCD.

Another nonperturbative phenomenon in QCD, which is of the same fundamental importance as confinement, and whose complete analytic understanding is also still lacking, is spontaneous chiral symmetry breaking (SCSB). This is the symmetry of the QCD Lagrangian with N_f massless flavors under the global transformations, which are the $\mathrm{U}(N_f)\times\mathrm{U}(N_f)$ independent rotations of left- and right-handed quark fields. These are equivalent to the independent vector and axial $\mathrm{U}(N_f)$ rotations of the full four-component Dirac spinors, under which the QCD Lagrangian remains invariant as well. At the same time, the axial transformations mix states with different P-parities. Therefore, were the chiral symmetry unbroken, one would observe parity degeneracy of all the states whose other quantum numbers

[b] We do not discuss in this review the modern line of research devoted to so-called color superconductivity [7] In the color superconducting phase of QCD, which takes place at high baryonic densities corresponding to the values of the quark chemical potential $\geq \mathcal{O}\,(400\ \mathrm{MeV})$, the gauge symmetry is spontaneously broken.

[c] To get a description of large-distance effects in terms of microscopic degrees of freedom, it looks natural to integrate over the high-momentum modes in the same way as is usually done in a derivation of renormalization-group (RG) equations in statistical mechanics. For instance, this approach gives the correct RG behavior of the running strong-coupling constant (at the one-loop level) [3]. However, it typically suffers from the above-mentioned problem of violation of gauge invariance as far as RG equations for Green functions are concerned. Recent progress in this direction of research is concentrated around the concept of the so-called exact RG flow (see e.g. Ref. [8] for a recent review).

are the same. The observed splittings between such states are, however, too large to be explained by the small quark masses. Namely, this splitting is of the order of hundreds of MeV, whereas the current masses of light u- and d-quarks are of the order of a few MeV.[d] This observation tells us that the chiral symmetry of the QCD Lagrangian is broken down spontaneously. The phenomenon of SCSB naturally leads to the appearance of light pseudoscalar Goldstone bosons, whose role is played by pions, which are indeed the lightest of all the hadrons. The order parameter of SCSB is the chiral quark condensate $\left\langle\bar\psi\psi\right\rangle \simeq -(250\,\mathrm{MeV})^3$. Its appearance is quite natural in QCD, where, due to the strong attraction between quarks and antiquarks, it costs very little energy to create a massless quark-antiquark pair. Having zero total momentum and angular momentum, such pairs carry net chiral charge, hence the notion "chiral condensate".

Since the early days of QCD, when it was realized that nonperturbative phenomena, confinement and SCSB, are of fundamental importance, numerous nonperturbative approaches have been proposed in an attempt to describe one or other of these two phenomena in a controllable way.[e] On the side of confinement, these approaches include e.g. the already mentioned expansion in the large number of colors [9], loop equations [10], the stochastic vacuum model [1], and the method of Abelian projections [14][f]. On the side of SCSB, the classic approach is the one based on the Nambu–Jona-Lasinio models [17], as well as on the related nonlinear chiral meson Lagrangians [18] (the latter approach has further been developed in [19]; for a review see [20]). There also exist microscopic models of SCSB in QCD, based on instantons [21] or dyons [22], which have been put forward in Refs. [23] and [24], respectively.

Although a lot of physical effects have been captured by the above-mentioned methods, unfortunately none of them provides the full solution of QCD. The concept of "full solution" has several facets. According to the

[d] The current mass of the s-quark, which is around 150 MeV, is still smaller than the typical splitting values by at least a factor of three.

[e] Unfortunately, practically no approach exists, which describes both phenomena. Partially, that is because it is not yet fully accepted that these phenomena are related to each other microscopically, i.e. that the same vacuum configurations are responsible for both phenomena. For instance, as has been shown in Ref. [12], this is not so in the case of the so-called Faddeev-Niemi effective action [13]. An example of a nonperturbative approach, where the two phenomena are related to each other, is the stochastic vacuum model of QCD [1], which does not refer to a particular microscopic vacuum configuration.

[f] A comprehensive lattice analysis of the latter has recently been performed [15] (see Ref. [16] for reviews).

conventional understanding, the solution of a certain field theory means a prescription of how to compute an arbitrary gauge-invariant vacuum amplitude. In QCD, the results should be in agreement with the experimental and lattice data on the properties of hadrons. The gluonic and quark averages should then give a correct quantitative description of confinement and SCSB, respectively. The values of local averages (condensates) and relations between these quantities should also agree with those known from QCD sum rules [25]. Furthermore, the standard diagrammatic expansion should be reproducible (giving, in particular, asymptotic freedom), and the nonperturbative dimensionful quantities should be made expressible in terms of the only dimensionful QCD parameter, $\Lambda_{\rm QCD}$. Moreover, some constraints, which are accepted to be rigorous in QCD, should be obeyed. For instance, the above-mentioned expansion in the large number of colors should be correctly reproducible and should respect the large-N loop equation. The solution of QCD should also accommodate classical vacuum configurations of the gluodynamics action, such as instantons.

In this review, we are going to discuss the 3D Georgi–Glashow (GG) model, which is a QCD-related model possessing the property of confinement [26]. Our primary goal will therefore be the study of confinement (and not of SCSB), as well as of the deconfining phase transition at finite temperatures. The advantage of the 3D GG model with respect to QCD is that confinement in it takes place in the weak-coupling regime. It turns out that, already in this regime, the vacuum (i.e. the ground state) of this model is nonperturbative, being populated by 't Hooft–Polyakov monopoles [27], which provide the permanent confinement of external fundamental charges. As a guiding principle of our analysis we will use the string picture of confinement, therefore let us briefly discuss it.

In QCD, the linearly rising confining interquark potential is associated to a string-like configuration of the gluonic field between quarks, usually called the QCD string. Indeed, the energy of a string grows linearly with its length, $E(R) = \sigma R$.[g] According to the Regge phenomenology, the string energy density σ, called the string tension, is approximately $(440\,{\rm MeV})^2$. The string can naturally be called confining (the notion, which is always used in confining gauge theories other than QCD, such as the 3D GG model), since with the increase of the distance R, it stretches and prevents a quark

[g] In this review, we only briefly discuss the phenomenon of string breaking (see the end of subsection 4.2). String breaking always happens at a certain distance if dynamical matter fields, transforming by the same representation of the gauge group as the confined external ones, are present.

and an antiquark from separating to macroscopic distances. As for any dimensionful quantity in QCD, the string tension should be proportional to the respective power of $\Lambda_{\rm QCD}$. Namely,

$$\sigma \propto \Lambda_{\rm QCD}^2 = a^{-2} \exp\left[-\int^{g^2(a^{-2})} \frac{dg'^2}{g'^2\beta(g'^2)}\right] \simeq$$
$$\simeq a^{-2}\exp\left[-\frac{16\pi^2}{\left(\frac{11}{3}N_c - \frac{2}{3}N_f\right) g^2(a^{-2})}\right], \tag{1.1}$$

where $a \to 0$ stands for the short distance UV cutoff (e.g. the lattice spacing). Furthermore, "$\simeq$" means "at the one-loop level", at which

$$\beta(g^2) \simeq -\left(\frac{11}{3}N_c - \frac{2}{3}N_f\right)\frac{g^2}{16\pi^2} .$$

As is seen explicitly from Eq. (1.1), all the coefficients in the expansion of σ in (positive) powers of g^2 vanish, which means that the QCD string is indeed an essentially nonperturbative object. The string should therefore be produced by some nonperturbative background fields. On top of these, however, one expects to have some quantum fluctuations of the gauge field, which give rise to the string excitations.

The confining quark-antiquark potential corresponds to the so-called area law of the Wilson loop, [h]

$$\langle W(C)\rangle \equiv \frac{1}{N_c}\left\langle {\rm tr}\,\mathcal{P}\,\exp\left(ig\oint_C A_\mu^a T^a dx_\mu\right)\right\rangle \stackrel{|C|\to\infty}{\longrightarrow} {\rm e}^{-\sigma|\Sigma_{\rm min}(C)|} .$$

Here, $\Sigma_{\rm min}(C)$ is the surface of the minimal area, bounded by the trajectory C of the quark-antiquark pair, and $|\ldots|$ means either a length or an area. The confining string, which sweeps out the minimal surface, can naturally be viewed as a product of the above-mentioned strong background fields. Instead, quantum fluctuations around these enable the string to sweep out with a nonvanishing probability any other surface $\Sigma(C)$, different from $\Sigma_{\rm min}(C)$. To derive the string representation of QCD would be to give sense to the

[h] It should be compared with the perimeter law,

$$\langle W(C)\rangle \stackrel{|C|\to\infty}{\longrightarrow} {\rm e}^{-{\rm const}\cdot|C|} ,$$

which corresponds to the Coulomb potential and is found in non-confining theories, e.g. (noncompact) QED.

formula

$$\langle W(C)\rangle \equiv \langle W(\Sigma_{\min}(C))\rangle = \sum_{\Sigma(C)} \mathrm{e}^{-S[\Sigma(C)]}\ . \tag{1.2}$$

Here, $\sum\limits_{\Sigma(C)}$ and $S[\Sigma(C)]$ stand for a certain sum over string world sheets and a string effective action, both of which are yet unknown in QCD.

The first problem one encounters in trying to determine $S[\Sigma(C)]$ is that fundamental strings [28] can be quantized only in critical dimensions: a consistent quantum theory describing strings out of the critical dimensions has not yet been found. The simplest model, the Nambu–Goto string, can be quantized only in space-time dimension $D = 26$ or $D \leq 1$ because of the conformal anomaly. It is appropriate to describe an effective string picture for confinement in QCD, but is inappropriate to describe fundamental smooth strings dual to QCD [29], since large Euclidean world-sheets are crumpled. The picture of a fundamental string theory dual to QCD is strongly supported by a recent lattice calculation by Lüscher and Weisz [30], where evidence of a string behavior in the static quark-antiquark potential has been found down to distances of 0.5 fm.

In the rigid-string action [31, 32], the marginal term proportional to the square of the extrinsic curvature, introduced to avoid crumpling, turns out to be infrared irrelevant and, thus, unable to provide smooth surfaces.

Recent progress in this field is based on a new type of action. In its local formulation [33, 34], the string action is induced by an antisymmetric tensor field. This action realizes explicitly the necessary zig-zag invariance of confining strings [34,35]. It can be derived without extra assumptions [36] for the confining phase of compact U(1) gauge theories [3]. An alternative approach to the induced string action was originally proposed in [37], and is based on a five-dimensional, curved space-time string action with the quarks living on a four-dimensional horizon [38]. The formulation of the string theory in the five-dimensional curved space-time is closely related to the AdS/CFT (Anti de Sitter/Conformal Field Theory) correspondence [39]. In fact, with a special choice of the metric in the curved space, one recovers the AdS_5 space, thereby providing a string theory description of a conformal gauge theory [39].

The main characteristic of the effective string action obtained by integrating out the tensor field is a non-local interaction with *negative stiffness*, that can be expressed as a derivative expansion of the interaction between surface elements. To perform an analytic analysis of the geometric properties of these strings, this expansion must be truncated; this clearly makes the model non-unitary, but in a spurious way. Moreover, since the stiffness is negative, a stable truncation must, at least, include a sixth-order term in the derivatives. The role of negative stiffness, as first pointed out in [36, 40] is crucial. It is in fact the sixth-order term, forced by the negative stiffness, that suppresses the formation of spikes on the surfaces and leads to a smooth surface in the large-D approximation. In fact, in [40, 41] it has been shown that, in the large-D approximation, this model has an infrared fixed point at zero stiffness, corresponding to a tensionless smooth string whose world sheet has Hausdorff dimension 2, exactly the desired properties to describe QCD flux tubes. As first noticed in [42, 43], the long-range orientational order in this model is due to an antiferromagnetic interaction between normals to the surface, a mechanism confirmed by numerical simulations [44]. The presence of the infrared fixed point does not depend on the truncation [41] and it is present for all ghost- and tachyon- free truncations. Moreover, the effective theory describing the infrared behavior of the confining string is a conformal field theory (CFT) with central charge $c = 1$.

Another important feature of the negative-stiffness model is its high-temperature behavior. Contrary to all previous string models for QCD, it is able to reproduce the large-N QCD behavior, found by Polchinski and Yang in [89], in both *sign and reality properties* [45].

It is remarkable that the role of negative stiffness, while first discovered in the context of string [36, 40] and membranes [47], has been rediscovered and actually experimentally tested in material science [48]. In fact, it has been found that composites with negative stiffness inclusions have higher overall stiffness than that of their constituents. Such composites find applications in which high stiffness and damping are needed, permitting extreme properties not previously anticipated.

Equation (1.2) would clearly be a generalization of a path-integral representation for a propagator of (for example) a point-like boson, $\langle\phi(x)\phi(x')\rangle = \sum\limits_{P_{xx'}} \mathrm{e}^{-S[P_{xx'}]}$, where $P_{xx'}$ is a path connecting the points x and x'. In particular, within this analogy, the role of the classical trajectory of a particle would be played by $\Sigma_{\min}(C)$. For a point-like particle, the measure in the sum over paths is known and depends only on the dimensionality of the space-time. The world-line action $S[P_{xx'}]$ can also be evaluated, either analytically (for certain potentials $V(\phi)$ or external gauge fields if ϕ is charged), or using Feynman's variational method. On the string side, a derivation of the measure in the sum over world sheets has been discussed in Ref. [49] for the case of the Abelian Higgs model in the London limit. As will be seen below, in a certain case the string effective action derivable in the 3D GG model is the same as that of the London limit of the dual Abelian Higgs model. The summation over string world sheets is, however, realized in these two models in different ways. Namely, in the dual Abelian Higgs model it stems from the integration over singularities of the dual Higgs field (vortex cores), which is already present in the original partition function. Instead, the 3D GG model does not contain any dual Abelian Higgs field, and the summation over world sheets in this model is realized by means of the resulting string effective action itself [34]. Numerous investigations of this action have further been performed (see e.g. Refs. [36, 42, 43]), and some of these will be discussed in this review. A separate section will be devoted to the finite-temperature properties of confining strings [45].

For the purpose of the study of the so-called k-strings and merely to be closer to real QCD, we will deal with the SU(N)-generalization of the standard SU(2) GG model, which will be introduced in the next section. As for the ($N = 2$) 3D GG model, it is a classic example [26] of a theory which allows for an analytic description of confinement. As has already been mentioned, confinement in this model is due to the plasma of point-like magnetic monopoles, which produce random magnetic fluxes through the contour of the Wilson loop. In the weak-coupling regime of the model, this plasma is dilute, and the interaction between monopoles is Coulombic, being induced by the dual-photon exchanges. Since the energy of a single monopole is a quadratic function of its flux, it is energetically favorable for the vacuum to support a configuration of two monopoles of unit charge (in the units of the magnetic coupling constant, g_m), rather than a single, doubly-charged monopole. Owing to this fact, monopoles of unit charge dominate in the vacuum, whereas monopoles of higher charges tend to dissociate to them. Summing over the grand canonical ensemble of monopoles of unit charge, interacting with each other by the Coulomb law, one arrives at an effective low-energy theory, which is a 3D sine-Gordon theory of a dual photon. The latter acquires a mass (visible upon the expansion of the cosine potential) by means of Debye screening in the Coulomb plasma. The appearance of this (exponentially small) mass and, hence, of a finite (albeit exponentially large) vacuum correlation length is crucial for the generation of the fundamental string tension, i.e. for the confinement of external fundamental matter. It is worth noting that a physically important interpretation of these ideas in terms of spontaneous breaking of magnetic Z_2 symmetry has been presented in reviews [50] and Refs. therein.

While the confining properties of the 3D GG model have been well known since Polyakov's pioneering paper [26], the finite-temperature properties of this model have been addressed only recently, starting with the papers [51, 52]. It turned out [52] that charged matter fields of W bosons play the crucial role for the dynamics of the phase transition. Below we will review this issue and also discuss the influence of other matter fields on the finite-temperature properties of the model. Such fields are either already present in the original Lagrangian (e.g. Higgs [53], or dual photinos in the supersymmetric generalization of the model [54]), or can be included in the framework of a certain extension of the model (e.g. massless fundamental quarks [55] or heavy fundamental bosonic matter [56]).

The outline of this review is the following. In the next section, we will introduce the 3D GG model in the general SU(N) case. In Section 3, we will find the string tension of the fundamental Wilson loop defined at a flat contour (henceforce called for shortness "flat Wilson loop"). In Section 4, we will develop a theory of confining strings based on the Kalb–Ramond field, which enables one to deal with non-flat Wilson loops. Using for concreteness matter in the fundamental representation, we will present in subsection 4.1 two methods by means of which the theory of confining strings can be derived. In subsection 4.2, the case of the adjoint Wilson loop will be considered, and the corresponding theory of confining strings will be constructed in the large-N limit. In subsection 4.3, we will study the spectrum of k-strings, i.e. strings between sources in (higher) representations with a nonvanishing N-ality. These sources carry a charge k with respect to the center of the gauge group Z_N and can be seen as a superposition of k fundamental charges. Clearly, the spectrum of such strings is an important ingredient for the complete description of the confining dynamics of the 3D GG model. In Section 5, the SU(N) theory of confining strings will be generalized to the SU(N) version of 4D compact QED (in the continuum limit) with the field-theoretical θ term. As has been found in Refs. [33, 43], for the usual compact QED, this term leads to the appearance of the string θ term. The latter, being proportional to the number of self-intersections of the world sheet, might help in the solution of the problem of crumpling of large world sheets [3,31]. The critical values of θ at which this happens will be derived in the general SU(N) case for fundamental and adjoint representations, as well as for k-strings. In Section 6, various geometric features of confining strings will be studied. In Section 7, the thermodynamics of confining strings will be discussed, and a derivation of the one-loop free energy of a string in the large-D limit will be presented. Again, we will show that it is the presence of negative stiffness that allows one to reproduce the large-N behavior of high-temperature QCD.

In Section 8, we will pass from the thermodynamics of confining strings to the thermodynamics of the 3D GG model itself. After an introduction to this subject in subsection 8.1, we will pay particular attention to the influence of various matter fields on the dynamics of the deconfining phase transition. In subsections 8.2 and 8.3, we will consider an approximation where W bosons are disregarded. Subsection 8.2 will be devoted to the influence of the Higgs field (when it is not infinitely heavy) on the RG flow, while in subsection 8.3 we will consider the model in the presence of external dynamical fundamental quarks. In subsection 8.4, we will first discuss the crucial role of W bosons in the dynamics of the phase transition in the finite-temperature 3D GG model and then consider the supersymmetric generalization of the model. In Section 9, the main points discussed in this review will be emphasized once again.

2. The SU(N) 3D GG model

The Euclidean action of the 3-d Georgi–Glashow model reads [26]

$$S = \int d^3x \left[\frac{1}{4g^2}\left(F^a_{\mu\nu}\right)^2 + \frac{1}{2}\left(D_\mu\Phi^a\right)^2 + \frac{\lambda}{4}\left((\Phi^a)^2 - \eta^2\right)^2\right], \tag{2.1}$$

where the Higgs field Φ^a transforms by the adjoint representation, i.e. $D_\mu\Phi^a \equiv \partial_\mu\Phi^a + \varepsilon^{abc}A^b_\mu\Phi^c$. The weak-coupling regime $g^2 \ll m_W$, which will be assumed henceforth, parallels the requirement that η should be large enough to ensure spontaneous symmetry breaking from SU(2) to U(1). At the perturbative level, the spectrum of the model in the Higgs phase consists of a massless photon, two heavy, charged W bosons with mass $m_W = g\eta$, and a neutral Higgs field with mass $m_H = \eta\sqrt{2\lambda}$.

What is, however, more important is the nonperturbative content of the model, represented by the famous 't Hooft–Polyakov monopole [27]. It is a solution to the classical equations of motion, which has the following Higgs- and vector-field parts:

- $\Phi^a = \delta^{a3}u(r)$, $u(0) = 0$, $u(r) \stackrel{r\to\infty}{\longrightarrow} \eta - \exp(-m_H r)/(gr)$;
- $A^{1,2}_\mu(\vec{x}) \stackrel{r\to\infty}{\longrightarrow} \mathcal{O}\left(\mathrm{e}^{-m_W r}\right)$,
$$H_\mu \equiv \varepsilon_{\mu\nu\lambda}\partial_\nu A^3_\lambda = \frac{x_\mu}{r^3} - 4\pi\delta(x_1)\delta(x_2)\theta(x_3)\delta_{\mu 3};$$
- as well as the following action $S_0 = \frac{4\pi\epsilon}{\kappa}$. Here, $\kappa \equiv g^2/m_W$ is the weak-coupling parameter, $\epsilon = \epsilon\left(m_H/m_W\right)$ is a certain monotonic, slowly varying function, $\epsilon \geq 1$, $\epsilon(0) = 1$ (BPS-limit) [57], $\epsilon(\infty) \simeq 1.787$ [58].

The approximate saddle-point solution (which becomes exact in the BPS-limit) was found in Ref. [26] to be

$$S = \mathcal{N} S_0 + \frac{g_m^2}{8\pi} \sum_{\substack{a,b=1 \\ a \neq b}}^{\mathcal{N}} \left(\frac{q_a q_b}{|\vec{z}_a - \vec{z}_b|} - \frac{\mathrm{e}^{-m_H |\vec{z}_a - \vec{z}_b|}}{|\vec{z}_a - \vec{z}_b|} \right) + \mathcal{O}\left(g_m^2 m_H \mathrm{e}^{-2m_H |\vec{z}_a - \vec{z}_b|} \right) + \mathcal{O}\left(\frac{1}{m_W R} \right),$$

where $m_W^{-1} \ll R \ll |\vec{z}_a - \vec{z}_b|$, $gg_m = 4\pi$, $[g_m] = [\mathrm{mass}]^{-1/2}$. Therefore, while at $m_H \to \infty$, the usual compact QED action is recovered, in the BPS-limit one has

$$S \simeq \mathcal{N} S_0 + \frac{g_m^2}{8\pi} \sum_{\substack{a,b=1 \\ a \neq b}}^{\mathcal{N}} \frac{q_a q_b - 1}{|\vec{z}_a - \vec{z}_b|} ,$$

i.e. the interaction of two monopoles doubles for opposite and vanishes for equal charges. Therefore, in this limit, the standard monopole-anti-monopole Coulomb plasma recombines itself into two mutually noninteracting subsystems, consisting of monopoles and anti-monopoles. The interaction between objects inside each of these subsystems has double the strength with respect to the interaction in the initial plasma.

When $m_H < \infty$, the summation over the grand canonical ensemble of monopoles has been performed in Ref. [59] and reads

$$\mathcal{Z}_{\mathrm{mon}} = 1 + \sum_{\mathcal{N}=1}^{\infty} \frac{\zeta^{\mathcal{N}}}{\mathcal{N}!} \prod_{a=1}^{\mathcal{N}} \int d^3 z_a \sum_{q_a = \pm 1} \mathrm{e}^{-S} = \int \mathcal{D}\chi \mathcal{D}\psi \times$$
$$\times \exp\left\{ - \int d^3 x \left[\frac{1}{2} (\partial_\mu \chi)^2 + \frac{1}{2} (\partial_\mu \psi)^2 + \frac{m_H^2}{2} \psi^2 - 2\zeta \mathrm{e}^{g_m \psi} \cos(g_m \chi) \right] \right\} . \tag{2.2}$$

Here, χ is the dual-photon field and ψ is the additional field which describes the Higgs boson. Furthermore, the monopole fugacity (i.e. the statistical weight of a single monopole), ζ, has the form [26],

$$\zeta = \delta \, \frac{m_W^{7/2}}{g} \, \mathrm{e}^{-S_0} \; . \tag{2.3}$$

The function $\delta = \delta\,(m_H/m_W)$ is determined by the loop corrections. It is known [60] that this function grows in the vicinity of the origin (i.e. in the BPS limit). However, the speed of this growth is such that it does not spoil the exponential smallness of ζ in the weak-coupling regime under study.

Our next aim will be to construct the SU(N) generalization of the partition function (2.2). Let us first introduce the $(N-1)$-dimensional vector $\vec{H}$ of the mutually commuting diagonal generators of the group SU(N). Together with certain pairwise linear combinations of the off-diagonal generators which, in analogy with the SU(2) group, are called step (rising and lowering) generators $E_{\pm i}$, $i = 1, \ldots, \frac{N(N-1)}{2}$, the diagonal generators form the algebra,

$$\left[\vec{H}, E_{\pm i}\right] = \vec{q}_{\pm i} E_{\pm i}\ , \quad \left[\vec{E}_i, E_{-i}\right] = \vec{q}_i \vec{H}\ .$$

Vectors $\vec{q}_i$'s here are called root vectors of the group SU(N). The vector potential, A_μ^a, $a = 1, \ldots, N^2 - 1$, can be respectively decomposed into photons and W bosons as

$$A_\mu^a = \sum_i \left[\left(W_\mu^+\right)^i E_{-i} + \left(W_\mu^-\right)^i E_i\right] + \vec{A}_\mu \vec{H}\ ,$$

where from now on $\sum\limits_i \equiv \sum\limits_{i=1}^{N(N-1)/2}$. Next, embeddings of the SU(2) monopole into the maximal Abelian subgroup $U(1)^{N-1}$ of the group SU(N) can be characterized by the space-time variations of the Higgs field Φ^a. Outside the monopole core, Φ^a can be chosen along the z-axis: $\Phi^a(0,0,z) \stackrel{r\to\infty}{\longrightarrow} \vec{\eta}_a \vec{H}$. Here, $\vec{\eta}_a$ is the $(N-1)$-dimensional vector of v.e.v.'s of the Higgs field Φ^a. For an arbitrary space-time direction,

$$\Phi^a(r,\theta,\varphi) = \Omega(\theta,\varphi)\Phi^a(0,0,z)\Omega^{-1}(\theta,\varphi).$$

The matrix Ω can be chosen such that

$$\Phi^a(r,\theta,\varphi) = Y^a + X_i^a(r)\vec{T}_i \frac{\vec{r}}{r}. \tag{2.4}$$

Here, the 3-component matrix-valued vector

$$\vec{T}_i = \left(\frac{E_i + E_{-i}}{\sqrt{2}}, \frac{E_i - E_{-i}}{\sqrt{2}\imath}, \vec{q}_i \vec{H}\right)$$

characterizes the embedding of the SU(2) Lie algebra into the root space of the group SU(N), associated with the root $\vec{q}_i$. There exist therefore $\frac{1}{2}N(N-1)$ embeddings, corresponding to the same number of monopoles. The constant $Y^a = (\vec{\eta}_a - (\vec{\eta}_a \vec{q}_i)\vec{q}_i)\,\vec{H}$, which parametrically depends on i, $Y^a \equiv Y_{(i)}^a$, is the hypercharge associated with the i-th embedding. Since the vector $\vec{\eta}_a - (\vec{\eta}_a \vec{q}_i)\vec{q}_i$ belongs to the plane containing $\vec{q}_i$ and $\vec{\eta}_a$ (and is perpendicular to $\vec{q}_i$), the first term on the r.h.s. of Eq. (2.4) breaks SU(N) down to SU(2)×U(1)$^{N-2}$, whereas the second term, with $X_i^a \stackrel{r\to\infty}{\longrightarrow} \vec{\eta}_a \vec{q}_i$, further breaks SU(2) down to U(1). Again, as in the SU(2) case, the only part of the monopole vector potential which does not vanish exponentially at large distances is the diagonal (photonic) one. It is given by

$$A_\mu^i = \frac{1}{g}\varepsilon_{\mu\nu\lambda}T_\nu^i\frac{x_\lambda}{r},$$

which corresponds to the magnetic field

$$H_\mu^i = g_m\frac{x_\mu}{4\pi r^3}\vec{q}_i\Omega(\theta,\varphi)\vec{H}\Omega^{-1}(\theta,\varphi),$$

where again $g_m = 4\pi/g$. Therefore, in the SU(N) case, monopoles also interact by means of the long-ranged Coulomb forces. This inter-monopole interaction, mediated by dual photons, results in the SU(N) analogue of the partition function (2.2),

$$\mathcal{Z}_{\rm mon}^N = \int \mathcal{D}\vec{\chi}\mathcal{D}\psi \exp\left[-\int d^3x\times \right.$$
$$\left.\times\left(\frac{1}{2}\left(\partial_\mu\vec{\chi}\right)^2+\frac{1}{2}\left(\partial_\mu\psi\right)^2+\frac{m_H^2}{2}\psi^2-2\zeta{\rm e}^{g_m\psi}\sum_i\cos\left(g_m\vec{q}_i\vec{\chi}\right)\right)\right]. \qquad (2.5)$$

Here, the dual-photon field is described by the ($N-1$)-dimensional vector $\vec{\chi}$.

Averaging in Eq. (2.5) over the Higgs field by means of the cumulant expansion one gets in the second order of this expansion [64]

$$\mathcal{Z}_{\rm mon}^N \simeq \int\mathcal{D}\vec{\chi}\exp\left[-\int d^3x\left(\frac{1}{2}(\partial_\mu\vec{\chi})^2-2\xi\sum_i\cos\left(g_m\vec{q}_i\vec{\chi}\right)\right)\right.$$
$$\left.+2\xi^2\int d^3xd^3y\sum_{i,j}\cos\left(g_m\vec{q}_i\vec{\chi}(\vec{x})\right)\mathcal{K}(\vec{x}-\vec{y})\cos\left(g_m\vec{q}_j\vec{\chi}(\vec{y})\right)\right]. \qquad (2.6)$$

In this equation, $\xi\equiv\zeta\exp\left[\frac{g_m^2}{2}D_{m_H}\left(m_W^{-1}\right)\right]$ is the modified fugacity (which can be shown to remain exponentially small as long as the cumulant expansion is convergent) and $\mathcal{K}(\vec{x})\equiv{\rm e}^{g_m^2D_{m_H}(\vec{x})}-1$. The Debye mass of the dual photon, stemming from Eq. (2.6) by virtue of the formula $\sum\limits_i q_i^\alpha q_i^\beta=\frac{N}{2}\delta^{\alpha\beta}$, $\alpha,\beta=1,\ldots,N-1$, reads

$$m_D = g_m\sqrt{N\xi}\left[1+\xi I\,\frac{N(N-1)}{2}\right], \qquad (2.7)$$

where $I\equiv\int d^3x\mathcal{K}(\vec{x})$. At $m_H\sim m_W$, I is given by [64]

$$I\simeq\frac{4\pi}{m_Hm_W^2}\exp\left(\frac{4\pi}{\kappa}{\rm e}^{-m_H/m_W}\right). \qquad (2.8)$$

The parameter of the cumulant expansion is $\mathcal{O}\left(\xi I N^2\right)$. By virtue of Eqs. (2.3) and (2.8), one can readily see that the condition for this parameter to be (exponentially) small reads $N < \exp\left[\frac{2\pi}{\kappa}\left(\epsilon - \frac{1}{\mathrm{e}}\right)\right]$. Approximating ϵ by its value at infinity, we find $N < \mathrm{e}^{8.9/\kappa}$. Therefore, when one takes into account the propagation of the heavy Higgs boson, the necessary condition for the convergence of the cumulant expansion is that the number of colors may grow not arbitrarily fast, but should rather be bounded from above by some parameter, which is nevertheless exponentially large.[i] A similar analysis can be performed in the BPS limit, $m_H \ll g^2$. There, one readily finds $I \simeq \left(g_m/m_H\right)^2$, and

$$\xi I N^2 \propto N^2 \exp\left[-\frac{4\pi}{\kappa}\left(\epsilon - \frac{1}{2}\right)\right].$$

Approximating ϵ by its value at the origin, we see that the upper bound for N in this limit is smaller than in the vicinity of the compact QED limit and reads $N < \mathrm{e}^{\pi/\kappa}$.

3. String tension of the flat Wilson loop in the fundamental representation

At zero temperature, the 3D GG model is a clear analogy of the 2D XY-model in its continuum limit (see e.g. Refs. [26, 62]). The analogy is due to the fact that vortices of the 2D XY-model correspond to monopoles of the 3D GG model, whereas spin waves correspond to free (non-dual) photons. Disorder in both theories is produced by topological defects, i.e. by vortices or monopoles [j]. Instead, spin waves or free photons cannot disorder correlation functions at large distances and affect them only at the distances smaller than the vacuum correlation length. As a result, monopoles lead to the area law of the Wilson loop, whereas free photons lead to its perimeter law.

Since the confinement and string tension we are interested in are generated by monopoles, photons will be omitted in this section. The partition function of the grand canonical ensemble of monopoles in the Euclidean

[i] As for the convergence of the cumulant expansion itself, it is a natural requirement, which should be obeyed by any field theory with a normal stochastic, rather than the coherent, vacuum.

[j] At finite temperatures, disorder is primarily generated by W bosons [52], which, at the zero temperature under discussion, are practically irrelevant due to their heaviness. The quantitative discussion of the role of W bosons at finite temperature will be presented in subsection 8.4.

space-time reads

$$\mathcal{Z}_{\rm mon}^{N} = \sum_{\mathcal{N}=0}^{\infty} \frac{\zeta^{\mathcal{N}}}{\mathcal{N}!} \left\langle \exp\left[-\frac{g_m^2}{2}\int d^3x d^3y \vec{\rho}^{\mathcal{N}}(\vec{x}) D_0(\vec{x}-\vec{y}) \vec{\rho}^{\mathcal{N}}(\vec{y})\right]\right\rangle_{\rm mon} . \quad (3.1)$$

Here, $D_0(\vec{x}) = 1/(4\pi|\vec{x}|)$ is the 3D Coulomb propagator, and the monopole density is defined as $\vec{\rho}^{\mathcal{N}}(\vec{x}) = \sum\limits_{k=1}^{\mathcal{N}} \vec{q}_{i_k}\delta(\vec{x}-\vec{z}_k)$ at $\mathcal{N} \geq 1$ and $\vec{\rho}^{\mathcal{N}=0}(\vec{x}) = 0$. Furthermore, the average is defined as

$$\langle \mathcal{O} \rangle_{\rm mon} = \prod_{n=0}^{\mathcal{N}} \int d^3 z_n \sum_{i_n=\pm 1,\ldots,\pm\frac{N(N-1)}{2}} \mathcal{O} \, . \quad (3.2)$$

Upon explicit summation, the partition function (3.1) can be represented in the sine-Gordon–type form, which is the large-m_H limit of the partition function (2.5),

$$\mathcal{Z}_{\rm mon}^{N} = \int \mathcal{D}\vec{\chi} \exp\left[-\int d^3x \left(\frac{1}{2}\left(\partial_\mu \vec{\chi}\right)^2 - 2\zeta \sum_i \cos\left(g_m \vec{q}_i \vec{\chi}\right)\right)\right] . \quad (3.3)$$

The Debye mass (2.7) becomes reduced to [64]: $m_D = g_m\sqrt{N\zeta}$.

The following comment is worth making at this point. The description of the grand canonical ensemble of monopoles in terms of the dual-photon field essentially implies the validity of the mean-field approximation. This approximation, which enables one to disregard fluctuations of fields of individual monopoles, is only valid if the number of monopoles contained in the Debye volume, m_D^{-3}, is large. Up to exponentially small corrections, the mean value of the monopole density, evaluated according to the formula $\rho_{\rm mean} = \frac{1}{V^{(3)}}\frac{\partial \ln \mathcal{Z}_{\rm mon}}{\partial \ln \zeta}$, reads $\rho_{\rm mean} \simeq \zeta N(N-1)$, where $V^{(3)}$ is the three-volume occupied by the system. Therefore, the number of monopoles contained in the Debye volume is

$$\rho_{\rm mean} m_D^{-3} \simeq \frac{N-1}{g_m^3\sqrt{N\zeta}}.$$

This is indeed an exponentially large quantity, even at $N \sim 1$. With the increase of N, the accuracy of the mean-field approximation is being further enhanced.

Next, one can introduce the monopole field-strength tensor $\vec{F}_{\mu\nu}^{\mathcal{N}}$ which violates the Bianchi identity as $\frac{1}{2}\varepsilon_{\mu\nu\lambda}\partial_\mu \vec{F}_{\nu\lambda}^{\mathcal{N}} = g_m \vec{\rho}^{\mathcal{N}}$. Recalling that photons are omitted throughout this section, we obtain

$$\vec{F}^{\mathcal{N}}_{\mu\nu}(\vec{x}) = -g_m\varepsilon_{\mu\nu\lambda}\partial_\lambda \int d^3y D_0(\vec{x}-\vec{y})\,\vec{\rho}^{\mathcal{N}}(\vec{y}) \,. \tag{3.4}$$

Then, by virtue of Stokes' theorem and the formula

$$\mathrm{tr}\,\exp\left(i\vec{\mathcal{O}}\vec{H}\right) = \sum_{a=1}^{N} \exp\left(i\vec{\mathcal{O}}\vec{\mu}_a\right) ,$$

where $\vec{\mathcal{O}}$ is an arbitrary $(N-1)$-component vector and $\vec{\mu}_a$'s are the weight vectors of the group SU(N), $a = 1,\ldots,N$, the Wilson loop in the $\mathcal{N}$-monopole configuration is

$$W(C)^{\mathcal{N}}_{\mathrm{mon}} = \frac{1}{N}\,\mathrm{tr}\,\exp\left(\frac{ig}{2}\vec{H}\int d^3x\vec{F}^{\mathcal{N}}_{\mu\nu}\Sigma_{\mu\nu}\right) = \frac{1}{N}\sum_{a=1}^{N} W^{\mathcal{N}}_a \,. \tag{3.5}$$

Here,

$$W^{\mathcal{N}}_a \equiv \exp\left(\frac{ig}{2}\vec{\mu}_a\int d^3x\vec{F}^{\mathcal{N}}_{\mu\nu}\Sigma_{\mu\nu}\right) \tag{3.6}$$

and $\Sigma_{\mu\nu}(\vec{x}) = \int\limits_{\Sigma(C)} d\sigma_{\mu\nu}(\vec{x}(\xi))\,\delta(\vec{x}-\vec{x}(\xi))$ is the vorticity tensor current defined at a certain surface $\Sigma(C)$ bounded by the contour C and parametrized by the vector $\vec{x}(\xi)$ with $\xi = (\xi_0,\xi_1)$ standing for the 2D coordinate. A straightforward calculation with the use of Eq. (3.4) yields

$$W^{\mathcal{N}}_a = \exp\left(i\vec{\mu}_a\int d^3x\vec{\rho}^{\mathcal{N}}\eta\right), \tag{3.7}$$

where $\eta(\vec{x};C) = \int\limits_{\Sigma(C)} d\sigma_\mu(\vec{x}(\xi))\,\partial_\mu\frac{1}{|\vec{x}-\vec{x}(\xi)|}$ is the solid angle under which the surface $\Sigma(C)$ is seen by an observer located at the point $\vec{x}$, $d\sigma_\mu = \frac{1}{2}\varepsilon_{\mu\nu\lambda}d\sigma_{\nu\lambda}$. (Note that owing to the Gauss' law, $\eta \equiv 4\pi$ for a closed surface, i.e. when C is shrunk to a point.) The ratio of two $W^{\mathcal{N}}_a$'s defined at different surfaces, Σ_1 and Σ_2, bounded by the contour C, reads

$$\frac{W^{\mathcal{N}}_a(\Sigma_1)}{W^{\mathcal{N}}_a(\Sigma_2)} = \exp\left(i\vec{\mu}_a\int\limits_{\Sigma_1\cup\Sigma_2} d\sigma_\mu(\vec{x}(\xi))\int d^3x\vec{\rho}^{\mathcal{N}}(\vec{x})\,\partial_\mu\frac{1}{|\vec{x}-\vec{x}(\xi)|}\right) =$$

$$= \prod_{k=1}^{\mathcal{N}} \exp\left(-i\vec{\mu}_a\vec{q}_{i_k}\int\limits_{\Sigma_1\cup\Sigma_2} d\sigma_\mu(\vec{x}(\xi))\,\partial^{\vec{x}(\xi)}_\mu\frac{1}{|\vec{z}_k-\vec{x}(\xi)|}\right). \tag{3.8}$$

Due to Gauss's law, the last integral in this equation is equal either to -4π or to 0, depending on whether the point $\vec{z}_k$ lies inside or outside the volume bounded by the surface $\Sigma_1 \cup \Sigma_2$. Since the product $\vec{\mu}_a \vec{q}_{i_k}$ is equal either to $\pm\frac{1}{2}$ or to 0, we conclude that $\frac{W_a^{\mathcal{N}}(\Sigma_1)}{W_a^{\mathcal{N}}(\Sigma_2)} = 1$. This fact proves the independence of $W_a^{\mathcal{N}}$ of the choice of the surface Σ in the definition (3.6). Clearly, this is the consequence of the quantization condition $gg_m = 4\pi$ used in the derivation of Eq. (3.7).

A representation of the partition function (3.1), alternative to Eq. (3.3) and more appropriate for the investigation of the Wilson loop, is the one in terms of the dynamical monopole densities [64]. It can be obtained by multiplying Eq. (3.1) by the identity

$$\int \mathcal{D}\vec{\rho}\,\delta\left(\vec{\rho} - \vec{\rho}^{\mathcal{N}}\right) = \int \mathcal{D}\vec{\rho}\mathcal{D}\vec{\chi}\exp\left[ig_m \int d^3x \vec{\chi}\left(\vec{\rho} - \vec{\rho}^{\mathcal{N}}\right)\right],$$

so that the field $\vec{\chi}$ plays the role of the Lagrange multiplier. We obtain for the partition function,

$$\mathcal{Z}_{\text{mon}}^N = \int \mathcal{D}\vec{\rho}\mathcal{D}\vec{\chi}\exp\left[-\frac{g_m^2}{2}\int d^3x d^3y \vec{\rho}(\vec{x}) D_0(\vec{x}-\vec{y})\vec{\rho}(\vec{y}) + ig_m \int d^3x \vec{\chi}\vec{\rho}\right] \times$$
$$\times \sum_{\mathcal{N}=0}^{\infty} \frac{\zeta^{\mathcal{N}}}{\mathcal{N}!}\left\langle \exp\left(-ig_m \int d^3x \vec{\chi}\vec{\rho}^{\mathcal{N}}\right)\right\rangle_{\text{mon}}, \tag{3.9}$$

where the last sum is equal to

$$\exp\left[2\zeta \int d^3x \sum_i \cos\left(g_m \vec{q}_i \vec{\chi}\right)\right]. \tag{3.10}$$

Accordingly, Eq. (3.7) becomes replaced by

$$W_a^{\mathcal{N}} \to W_a = \frac{1}{\mathcal{Z}_{\text{mon}}^N}\int \mathcal{D}\vec{\rho}\mathcal{D}\vec{\chi}\exp\left\{ -\frac{g_m^2}{2}\int d^3x d^3y \vec{\rho}(\vec{x}) D_0(\vec{x}-\vec{y})\vec{\rho}(\vec{y}) + \right.$$
$$\left. + \int d^3x \left[ig_m \vec{\chi}\vec{\rho} + 2\zeta\sum_i \cos\left(g_m \vec{q}_i \vec{\chi}\right) + i\vec{\mu}_a \vec{\rho}\eta\right]\right\},$$

and the full expression for the monopole contribution to the Wilson loop (instead of Eq. (3.5)) reads $W(C)_{\text{mon}} = \frac{1}{N}\sum\limits_{a=1}^{N} W_a$.

Let us next introduce the magnetic field according to the formulae $\partial_\mu \vec{B}_\mu = \vec{\rho}$, $\varepsilon_{\mu\nu\lambda}\partial_\nu \vec{B}_\lambda = 0$. This yields

$$W_a = \frac{1}{\mathcal{Z}_{\text{mon}}^N}\int \mathcal{D}\vec{B}_\mu \delta\left(\varepsilon_{\mu\nu\lambda}\partial_\nu \vec{B}_\lambda\right)\int \mathcal{D}\vec{\chi}\exp\left\{\int d^3x\Big[-\frac{g_m^2}{2}\vec{B}_\mu^2 + \right.$$

$$+ig_m\vec{\chi}\partial_\mu\vec{B}_\mu + 2\zeta\sum_i \cos\left(g_m\vec{q}_i\vec{\chi}\right)\Big] + 4\pi i\vec{\mu}_a\int\limits_{\Sigma(C)} d\sigma_\mu\vec{B}_\mu\Bigg\}, \tag{3.11}$$

where, as has been discussed, the surface $\Sigma(C)$ is arbitrary. The magnetic and the dual-photon fields can be integrated out of Eq. (3.11) by solving the respective saddle-point equations. From now on in this section, we will consider the contour C located in the (x,y)-plane. This naturally leads to the following Ansätze for the fields to be inserted into the saddle-point equations: $\vec{B}_\mu = \delta_{\mu 3}\vec{B}(z)$, $\vec{\chi} = \vec{\chi}(z)$. For the points (x,y) lying inside the contour C, these equations then read

$$ig_m\vec{\chi}' + g_m^2\vec{B} - 4\pi i\vec{\mu}_a\delta(z) = 0, \tag{3.12}$$

$$i\vec{B}' - 2\zeta\sum_i \vec{q}_i\sin\left(g_m\vec{q}_i\vec{\chi}\right) = 0, \tag{3.13}$$

where $' \equiv d/dz$. (Note that differentiating Eq. (3.12) and substituting the result into Eq. (3.13), we obtain the equation

$$\vec{\chi}'' - 2g_m\zeta\sum_i \vec{q}_i\sin\left(g_m\vec{q}_i\vec{\chi}\right) = g\vec{\mu}_a\delta'(z),$$

which is the SU(N) generalization of the saddle-point equation obtained in Ref. [26].) These equations can be solved by using one more natural Ansatz for the saddle points of the fields, $\vec{B}(z) = \vec{\mu}_a B(z)$, $\vec{\chi}(z) = \vec{\mu}_a\chi(z)$, which, due to the formula

$$\vec{\mu}_a\vec{\mu}_b = \frac{1}{2}\left(\delta_{ab} - \frac{1}{N}\right), \tag{3.14}$$

makes W_a a-independent. Let us next multiply Eq. (3.13) by $\vec{\mu}_a$, taking into account that for any a, $(N-1)$ positive roots yield the scalar product with $\vec{\mu}_a$ equal to $1/2$, while the others are orthogonal to $\vec{\mu}_a$. Equations (3.12), (3.13) then go over to

$$2i\phi' + g_m^2 B = 4\pi i\delta(z), \quad B' + 2i\zeta N\sin\phi = 0, \tag{3.15}$$

where $\phi \equiv g_m\chi/2$. One can readily check that the solution to this system of equations has the form

$$B(z) = i\frac{8m_D}{g_m^2}\frac{\mathrm{e}^{-m_D|z|}}{1+\mathrm{e}^{-2m_D|z|}}, \quad \phi(z) = 4\,\mathrm{sgn}\,z\cdot\arctan\left(\mathrm{e}^{-m_D|z|}\right). \tag{3.16}$$

Taking the value $B(0)$ for the evaluation of the Wilson loop, we obtain for the string tension,

$$\sigma = 4\pi \cdot \frac{N-1}{2N}\frac{4m_D}{g_m^2} = \frac{8\pi}{g_m}\frac{N-1}{\sqrt{N}}\sqrt{\zeta}. \tag{3.17}$$

This result demonstrates explicitly the nonanalytic dependence of σ on g, but does not yield the correct overall numerical factor. That is because $B(z)$, necessary for this calculation, is defined ambiguously. The ambiguity originates from the exponentially large thickness of the string, $|z| \le d \equiv m_D^{-1}$. It is therefore unclear which value of $B(z)$ we should take; either $B(0)$, as was done, or the field averaged over some range of z. It turns out that, in the weak-field (low-density) limit, the problem can be solved for an arbitrarily-shaped surface, by using the representation in terms of the Kalb–Ramond field which will be described in the next section.

4. SU(N) confining strings

4.1. *Fundamental representation*

The dual photon can alternatively be described in terms of the Kalb–Ramond field. One of the ways to introduce this field is to consider it as the field-strength tensor corresponding to the field $\vec{B}_\mu$, namely $\vec{B}_\mu = \frac{1}{2g_m}\varepsilon_{\mu\nu\lambda}\vec{h}_{\nu\lambda}$. The Wilson loop (3.11) then takes the form [k]

$$W_a = \frac{1}{\mathcal{Z}_{\rm mon}^N}\int \mathcal{D}\vec{h}_{\mu\nu}\delta\left(\partial_\mu \vec{h}_{\mu\nu}\right)\int \mathcal{D}\vec{\chi}\exp\left\{\int d^3x\left[-\frac{1}{4}\vec{h}_{\mu\nu}^2 + \right.\right.$$
$$\left.\left. + \frac{i}{2}\vec{\chi}\,\varepsilon_{\mu\nu\lambda}\partial_\mu\vec{h}_{\nu\lambda} + 2\zeta\sum_i \cos\left(g_m\vec{q}_i\vec{\chi}\right)\right] + \frac{ig}{2}\vec{\mu}_a\int\limits_{\Sigma(C)} d\sigma_{\mu\nu}\vec{h}_{\mu\nu}\right\}. \tag{4.1}$$

The field $\vec{\chi}$ can further be integrated out by solving the saddle-point equation of the form (3.13), where $\vec{B}'$ is replaced by $\frac{1}{2g_m}\varepsilon_{\mu\nu\lambda}\partial_\mu\vec{h}_{\nu\lambda}$ (and $\vec{\chi}$ depends on all three coordinates). Using the Ansatz $\vec{h}_{\mu\nu} = \vec{\mu}_a h_{\mu\nu}$, we arrive at the

[k] Note the correspondence between the Coulomb interaction of monopole densities and the actions of the magnetic and Kalb–Ramond fields,

$$\frac{1}{4}\int d^3x\vec{h}_{\mu\nu}^2 = \frac{g_m^2}{2}\int d^3x\vec{B}_\mu^2 = \frac{g_m^2}{2}\int d^3xd^3y\vec{\rho}(\vec{x})D_0(\vec{x}-\vec{y})\vec{\rho}(\vec{y})\,.$$

substitution in Eq. (4.1)

$$\int d^3x\Big[\frac{i}{2}\,\vec{\chi}\,\varepsilon_{\mu\nu\lambda}\partial_\mu\vec{h}_{\nu\lambda}+2\zeta\sum_i\cos\left(g_m\vec{q}_i\vec{\chi}\right)\Big]\Longrightarrow$$

$$\Longrightarrow -V\left[\vec{h}_{\mu\nu}\right]\equiv -2(N-1)\zeta\int d^3x\left[H_a\text{arcsinh}\,H_a-\sqrt{1+H_a^2}\right]. \quad (4.2)$$

Here,

$$H_a\equiv\frac{1}{2g_m\zeta(N-1)}\,\vec{\mu}_a\varepsilon_{\mu\nu\lambda}\partial_\mu\vec{h}_{\nu\lambda}\ , \quad (4.3)$$

and $V\left[\vec{h}_{\mu\nu}\right]$ is the multi-valued potential of the Kalb–Ramond field (or of monopole densities). Similarly to the case of compact QED [34], it is the summation over branches of this potential, which yields the summation over world sheets $\Sigma(C)$ in Eq. (4.1).

Another way to discuss the connection of the Kalb–Ramond field with the string world sheets is based on the semi-classical analysis of the saddle-point equations stemming from Eq. (4.1) which are

$$2i\partial_\mu\phi+\frac{g_m}{2}\varepsilon_{\mu\nu\lambda}h_{\nu\lambda}=4\pi i\Sigma_\mu\ , \quad (4.4)$$

$$\frac{1}{2g_m}\varepsilon_{\mu\nu\lambda}\partial_\mu h_{\nu\lambda}+2i\zeta N\sin\phi=0\ . \quad (4.5)$$

Here, $\Sigma_\mu\equiv\frac{1}{2}\varepsilon_{\mu\nu\lambda}\Sigma_{\nu\lambda}$, and the field ϕ is defined by the relation $\vec{\chi}=\frac{2}{g_m}\vec{\mu}_a\phi$. It is this auxiliary field ϕ, which establishes the correspondence between the Kalb–Ramond field and the stationary surface in the present approach. The summation over branches of the potential $V\left[\vec{h}_{\mu\nu}\right]$ is now replaced by the following procedure. One should restrict oneself to the domain $|\phi|\le\pi$ and solve Eqs. (4.4), (4.5) with the conditions

$$\lim_{|\vec{x}|\to\infty}\phi(\vec{x})=0, \quad (4.6)$$

$$\lim_{\varepsilon\to 0}\left\{\phi\left[\vec{x}(\xi)+\varepsilon\vec{n}(\xi)\right]-\phi\left[\vec{x}(\xi)-\varepsilon\vec{n}(\xi)\right]\right\}=2\pi, \quad (4.7)$$

where $\vec{n}(\xi)$ is the normal vector to Σ at an arbitrary point $\vec{x}(\xi)$. After that, one should get rid of the Σ-dependence of W_a by extremizing the latter with respect to $\vec{x}(\xi)$. Owing to the formula

$$\delta\int d\sigma_{\mu\nu}h_{\mu\nu}\left[\vec{x}(\xi)\right]=\int d\sigma_{\mu\nu}H_{\mu\nu\lambda}\left[\vec{x}(\xi)\right]\delta x_\lambda(\xi)\ ,$$

such an extremization is equivalent to the definition of the extremal surface through the equation $H_{\mu\nu\lambda}\left[\vec{x}(\xi)\right]=0$. Due to Eqs. (4.5), (4.7), $H_{\mu\nu\lambda}$ indeed vanishes on both sides of the stationary surface. The reason for that is the distinguishing property of the extremal surface that ϕ is equal to π on one of its sides and to $-\pi$ on the other; for any other surface, $|\phi|$ is smaller than π on one side and larger than π on the other side. Therefore, according to Eq. (4.5), on both sides of the extremal surface, $\varepsilon_{\mu\nu\lambda}\partial_\mu h_{\nu\lambda} = 0$. This is equivalent to the equation $\varepsilon_{\mu\nu\lambda}H_{\mu\nu\lambda} = 0$ and hence (upon the multiplication of both sides by $\varepsilon_{\mu'\nu'\lambda'}$) to $H_{\mu\nu\lambda} = 0$. (In particular, for a flat contour the extremal surface is the flat surface inside this contour. This can explicitly be seen from the solutions (3.16) of Eqs. (3.15). Namely, at any z such that $|z| \ll d$, $\phi(z) \simeq \pi \cdot \mathrm{sgn}z$, $B'(z) \simeq 0$, where both equalities hold with the exponential accuracy. The equation $B'(z) \simeq 0$ is equivalent to the condition of vanishing of $H_{\mu\nu\lambda}$ on both sides of the flat surface.)

Next, the general form of an antisymmetric rank-2 tensor field $\vec{h}_{\mu\nu}$ is

$$\vec{h}_{\mu\nu} = \partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu + \varepsilon_{\mu\nu\lambda}\partial_\lambda \vec{C}\ . \tag{4.8}$$

The constraint $\partial_\mu \vec{h}_{\mu\nu} = 0$, imposed in Eq. (4.1), is equivalent to setting $\vec{A}_\mu$ to zero. From now on, we will promote $\vec{h}_{\mu\nu}$ to include also the fields of free photons, $\vec{A}_\mu$, by abolishing this constraint. Let us further go into the weak-field limit, $|H_a| \ll 1$. Using the Cauchy inequality, we have

$$|H_a| \le \frac{|\vec{\mu}_a|\,|\vec{\rho}|}{\zeta(N-1)} = \frac{|\vec{\rho}|}{\zeta\sqrt{2N(N-1)}}\ , \tag{4.9}$$

so that the weak-field limit is equivalent to the low-density approximation $|\vec{\rho}| \ll \zeta\sqrt{2N(N-1)}$. To understand in which sense this inequality implies the low-density approximation, recall the mean monopole density we had from Eq. (3.3),

$$|\vec{\rho}|_{\mathrm{mean}} \simeq \zeta N(N-1)\ . \tag{4.10}$$

Therefore, at large-N, the low-density approximation implies that $|\vec{\rho}|$ should be of the order N times smaller than its mean value.

In the weak-field limit, where the constraint $\partial_\mu \vec{h}_{\mu\nu} = 0$ is removed, we obtain for the total Wilson loop,

$$W\,(C,\Sigma)^{\mathrm{tot},\ a}_{\mathrm{weak-field}} = \frac{1}{\mathcal{Z}^{\mathrm{tot}}}\int \mathcal{D}\vec{h}_{\mu\nu}\times$$
$$\times \exp\left\{-\int d^3x\left[\frac{1}{12m_D^2}\vec{H}^2_{\mu\nu\lambda} + \frac{1}{4}\vec{h}^2_{\mu\nu} - \frac{ig}{2}\vec{\mu}_a\vec{h}_{\mu\nu}\Sigma_{\mu\nu}\right]\right\}\ . \tag{4.11}$$

Here, $\mathcal{Z}^{\mathrm{tot}}$ is given by the same integral over $\vec{h}_{\mu\nu}$, but with $\Sigma_{\mu\nu}$ set to zero, and $\vec{H}_{\mu\nu\lambda} = \partial_\mu \vec{h}_{\nu\lambda} + \partial_\lambda \vec{h}_{\mu\nu} + \partial_\nu \vec{h}_{\lambda\mu}$ is the Kalb–Ramond field-strength tensor. (The apparent Σ-dependence of the r.h.s. of Eq. (4.11) is due to the fact that in the $\vec{h}_{\mu\nu}$-expansion of $V\left[\vec{h}_{\mu\nu}\right]$, only one branch of this potential has been taken into account. As discussed above, the Σ-dependence disappears upon the summation over all the branches.) Owing to Eq. (4.8), Eq. (4.11) is obviously factorized as $W(C,\Sigma)^{\text{tot},\, a}_{\text{weak-field}} = W_a W_a^{\text{phot}}$, where the free photonic contribution to the Wilson loop reads

$$W_a^{\text{phot}} = \exp\left[-g^2\frac{N-1}{4N}\oint\limits_C dx_\mu \oint\limits_C dx'_\mu D_0\left(\vec{x}-\vec{x}'\right)\right]. \tag{4.12}$$

Doing the integration over $\vec{h}_{\mu\nu}$ in Eq. (4.11), we obtain

$$W(C,\Sigma)^{\text{tot},\, a}_{\text{weak-field}} = \exp\left\{-g^2\frac{N-1}{4N}\left[\oint\limits_C dx_\mu \oint\limits_C dx'_\mu D_{m_D}(\vec{x}-\vec{x}')+\right.\right.$$
$$\left.\left.+\frac{m_D^2}{2}\int\limits_{\Sigma(C)} d\sigma_{\mu\nu}(\vec{x}(\xi))\int\limits_{\Sigma(C)} d\sigma_{\mu\nu}(\vec{x}(\xi'))D_{m_D}\left(\vec{x}(\xi)-\vec{x}(\xi')\right)\right]\right\}, \tag{4.13}$$

where $D_{m_D}(\vec{x}) = \mathrm{e}^{-m_D|\vec{x}|}/(4\pi|\vec{x}|)$ is the 3D Yukawa propagator. Note that the free photonic contribution completely cancels out of this expression, i.e. it is only the dual photon (of the mass m_D) which mediates the $C\times C$ and $\Sigma\times\Sigma$ interactions.

One can further expand the nonlocal string effective action

$$S_{\text{str}} = (gm_D)^2\frac{N-1}{8N}\int\limits_{\Sigma(C)} d\sigma_{\mu\nu}(\vec{x}(\xi))\int\limits_{\Sigma(C)} d\sigma_{\mu\nu}(\vec{x}(\xi'))D_{m_D}\left(\vec{x}(\xi)-\vec{x}(\xi')\right) \tag{4.14}$$

in powers of derivatives with respect to the world-sheet coordinates ξ_a's. Note that the actual parameter of this expansion is $1/(m_D R)^2$, where $R\sim\sqrt{\text{Area}(\Sigma)}$ is the size of Σ (see the discussion below). The resulting quasi-local action is (cf. Refs. [36, 65])

$$S_{\text{str}} = \sigma\int d^2\xi\sqrt{\mathsf{g}} + \alpha^{-1}\int d^2\xi\sqrt{\mathsf{g}}\mathsf{g}^{ab}(\partial_a t_{\mu\nu})(\partial_b t_{\mu\nu})+$$
$$+\kappa\int d^2\xi\sqrt{\mathsf{g}}\mathcal{R} + \mathcal{O}\left(\frac{\sigma}{m_D^4R^2}\right). \tag{4.15}$$

Here, $\partial_a \equiv \partial/\partial\xi^a$, and the following quantities characterize Σ: $\mathsf{g}_{ab}(\xi) = (\partial_a x_\mu(\xi))(\partial_b x_\mu(\xi))$ is the induced-metric tensor, $\mathsf{g} = \det\|\mathsf{g}^{ab}\|$, $t_{\mu\nu}(\xi) = \varepsilon^{ab}(\partial_a x_\mu(\xi))(\partial_b x_\nu(\xi))/\sqrt{\mathsf{g}}$ is the extrinsic-curvature tensor, $\mathcal{R} = \left(\partial^a\partial_a \ln\sqrt{\mathsf{g}}\right)/\sqrt{\mathsf{g}}$ is the expression for the scalar curvature in the conformal gauge $\mathsf{g}_{ab} = \sqrt{\mathsf{g}}\delta_{ab}$.

The third term on the r.h.s. of Eq. (4.15) is known to be a full derivative, and therefore it does not actually contribute to the string effective action, while the second term describes the so-called rigidity (or stiffness) of the string [31, 32]. The reason for the notation α^{-1}, introduced in Ref. [31], is that, as has been shown in that paper, it is α which is asymptotically free. This asymptotic freedom then indicates that the rigidity term can only be infrared relevant if the respective β-function has a zero in the infrared region. However, such a zero has not been found. Due to this fact, the rigidity term is not a good candidate to solve the old-standing problem of crumpling of large world sheets in Euclidean space-time. This necessitates seeking other possible solutions of this problem. One such solution, based on the string θ term, has been proposed in Ref. [31]. A possible derivation of such a term, within the SU(N) analogue of the 4D compact QED will be presented below.

The string coupling constants in Eq. (4.15) read

$$\sigma = 2\pi^2 \frac{N-1}{\sqrt{N}} \frac{\sqrt{\zeta}}{g_m} , \tag{4.16}$$

$$\alpha^{-1} = -3\kappa/2 = -\frac{\pi^2(N-1)}{4g_m^3 N^{3/2}\sqrt{\zeta}} . \tag{4.17}$$

A comment is in order regarding the negative sign of α. Up to a total derivative, the rigidity term reads

$$\int d^2\xi \sqrt{\mathsf{g}}\mathsf{g}^{ab}(\partial_a t_{\mu\nu})(\partial_b t_{\mu\nu}) = \int d^2\xi \sqrt{\mathsf{g}}(\Delta x_\mu)^2 ,$$

where the Laplacian associated with the metric g^{ab} acts on $x_\mu(\xi)$ as $\Delta x_\mu = \frac{1}{\sqrt{\mathsf{g}}}\partial_a\left(\sqrt{\mathsf{g}}\mathsf{g}^{ab}\partial_b x_\mu\right)$. In the conformal gauge, one thus readily gets for the rigidity term $\int d^2\xi \frac{1}{\sqrt{\mathsf{g}}}\left(\partial^2 x_\mu\right)^2$. For a nearly flat surface, $\mathsf{g} \simeq$ const $\equiv c^2$, and we have for the string propagator (in 2D Euclidean space),

$$\langle x_\mu(\xi)x_\nu(0)\rangle = \frac{\delta_{\mu\nu}}{\sigma}\int d^2p \, \frac{\mathrm{e}^{ip\xi}}{p^2\left(1+\frac{1}{\alpha c\sigma}p^2\right)} .$$

For $\alpha < 0$, this propagator has a tachyonic pole, whereas an unphysical mass pole shows up otherwise. We therefore conclude that the negative sign of α is important for the stability of strings.

In the weak-field (or low-density) approximation, we can thus fix the proportionality factor $2\pi^2$ at σ, that is close to the factor 8π of Eq. (3.17) which, however, could have not been fixed within the method of section 3. Another important fact is that (in the weak-field approximation) the string tension and higher string coupling constants are the same for all large enough surfaces $\Sigma(C)$. (The words "large enough" here mean the validity of the inequality $R \gg d$, i.e. the string length R should significantly exceed the exponentially large string thickness d. Indeed, the general n-th term of the derivative (or curvature) expansion of $S_{\rm str}$ is of the order of $\sigma m_D^2 R^4 (R/d)^{-2n}$.) It is, however, worth noting that some of the terms in the expansion (4.15) vanish at the surface of the minimal area corresponding to a given contour C (e.g. in the case of a flat surface considered in section 3). This concerns, for instance, the rigidity term.[l]

Let us finally demonstrate how to introduce the Kalb–Ramond field in a way alternative to its definition via the $\vec B_\mu$ field and incorporating automatically the free-photonic contribution to the Wilson loop. It is based on the direct combination of Eqs. (3.1) and (3.7), that yields

$$W_a = \sum_{\mathcal{N}=0}^{\infty} \frac{\zeta^{\mathcal{N}}}{\mathcal{N}!} \left\langle \exp\left[-\frac{g_m^2}{2}\int d^3x d^3y \vec\rho^{\,\mathcal{N}}(\vec x) D_0(\vec x-\vec y)\vec\rho^{\,\mathcal{N}}(\vec y) + \right.\right.$$
$$\left.\left. + i\vec\mu_a \int d^3x \vec\rho^{\,\mathcal{N}}\eta\right]\right\rangle_{\rm mon} = \frac{1}{\mathcal{Z}_{\rm mon}^N}\int \mathcal{D}\vec\varphi \times$$
$$\times \exp\left\{-\int d^3x \left[\frac12\left(\partial_\mu\vec\varphi - \frac{\vec\mu_a}{g_m}\partial_\mu\eta\right)^2 - 2\zeta\sum_i \cos\left(g_m\vec q_i\vec\varphi\right)\right]\right\}, \qquad (4.18)$$

where $\vec\varphi = \vec\chi + \frac{\vec\mu_a}{g_m}\eta$. One can further use the following identity, which shows explicitly how the Kalb–Ramond field unifies the monopole and the free-photonic contributions to the Wilson loop,

$$\exp\left[-\frac12\int d^3x\left(\partial_\mu\vec\varphi - \frac{\vec\mu_a}{g_m}\partial_\mu\eta\right)^2 - g^2\frac{N-1}{4N}\oint\limits_C dx_\mu \oint\limits_C dx'_\mu D_0(\vec x - \vec x')\right] =$$
$$= \int \mathcal{D}\vec h_{\mu\nu} \exp\left[-\int d^3x\left(\frac14 \vec h_{\mu\nu}^2 + \frac{i}{2}\vec\varphi\varepsilon_{\mu\nu\lambda}\partial_\mu\vec h_{\nu\lambda} - \frac{ig}{2}\vec\mu_a\vec h_{\mu\nu}\Sigma_{\mu\nu}\right)\right]. \qquad (4.19)$$

One can prove this by showing that both sides of this formula are equal to [m]

$$\exp\left[-\frac12\int d^3x\left(\partial_\mu\vec\varphi + g\vec\mu_a\Sigma_\mu\right)^2\right].$$

[l] Indeed, one can prove the equality $\mathsf{g}^{ab}(\partial_a t_{\mu\nu})(\partial_b t_{\mu\nu}) = (\mathcal{D}^a\mathcal{D}_a x_\mu)(\mathcal{D}^b\mathcal{D}_b x_\mu)$, where $\mathcal{D}_a\mathcal{D}_b x_\mu = \partial_a\partial_b x_\mu - \Gamma^c_{ab}\partial_c x_\mu$, and Γ^c_{ab} is the Christoffel symbol corresponding to the metric g^{ab}. On the other hand, the surface of the minimal area is defined by the equation $\mathcal{D}^a\mathcal{D}_a x_\mu(\xi) = 0$ together with the respective boundary condition at the contour C.

[m] Obviously, in the noncompact case, when monopoles are disregarded and $\vec h_{\mu\nu} = \partial_\mu \vec A_\nu - \partial_\nu \vec A_\mu$, the r.h.s. of Eq. (4.19) yields the free-photonic contribution to the Wilson loop, Eq. (4.12).

Inserting Eq. (4.19) into Eq. (4.18), we obtain

$$W_a \to W(C)_a^{\rm tot} = \frac{1}{\mathcal{Z}^{\rm tot}} \int \mathcal{D}\vec{h}_{\mu\nu} \mathcal{D}\vec{\varphi} \exp\Big[- \int d^3x \Big(\frac{1}{4}\vec{h}_{\mu\nu}^2 + \\ + \frac{i}{2}\,\vec{\varphi}\,\varepsilon_{\mu\nu\lambda}\partial_\mu \vec{h}_{\nu\lambda} - 2\zeta \sum_i \cos\left(g_m \vec{q}_i \vec{\varphi}\right) - \frac{ig}{2}\vec{\mu}_a \vec{h}_{\mu\nu}\Sigma_{\mu\nu}\Big)\Big] \,. \tag{4.20}$$

Denoting $\vec{\chi} \equiv -\vec{\varphi}$, we arrive back to Eq. (4.1) with $\mathcal{Z}_{\rm mon}^N \to \mathcal{Z}^{\rm tot}$ and the constraint $\partial_\mu \vec{h}_{\mu\nu} = 0$ abolished, as it should be.

4.2. *Adjoint representation*

Let us now extend the ideas of the previous subsection to the case of the Wilson loop in the adjoint representation. The charges of quarks in this representation are distributed along the roots, so that in the formulae for the Wilson loop, Eqs. (3.6) and (3.7), $\vec{\mu}_a$ should be replaced by $\vec{q}_i$. Noting that any root is a difference of two weights, we will henceforth in this subsection label roots by two indices running from 1 to N, e.g. $\vec{q}_{ab} = \vec{\mu}_a - \vec{\mu}_b$.[n] Equation (3.14) then leads to

$$\vec{q}_{ab}\vec{q}_{cd} = \frac{1}{2}\left(\delta_{ac} + \delta_{bd} - \delta_{ad} - \delta_{bc}\right), \tag{4.21}$$

according to which the product $\vec{q}_{ab}\vec{q}_{cd}$ may take the values $0, \pm\frac{1}{2}, \pm 1$. Owing to this fact, the ratio (3.8) is again equal to 1, i.e. for adjoint quarks, whose charge obeys the quantization condition $gg_m = 4\pi$, the Wilson loop is as surface independent, as it is for fundamental quarks.

The adjoint version of Eq. (4.1) (or of Eq. (4.20)) with the constraint $\partial_\mu \vec{h}_{\mu\nu} = 0$ abolished has the form

$$W(C)_{ab}^{\rm tot} = \frac{1}{\mathcal{Z}^{\rm tot}} \int \mathcal{D}\vec{h}_{\mu\nu} \mathcal{D}\vec{\chi} \exp\Big\{ \int d^3x \Big[- \frac{1}{4}\vec{h}_{\mu\nu}^2 + \\ + \frac{i}{2}\,\vec{\chi}\,\varepsilon_{\mu\nu\lambda}\partial_\mu \vec{h}_{\nu\lambda} + \zeta \sum_{c,d=1}^N \cos\left(g_m \vec{q}_{cd}\vec{\chi}\right)\Big] + \frac{ig}{2}\vec{q}_{ab} \int\limits_{\Sigma(C)} d\sigma_{\mu\nu}\vec{h}_{\mu\nu}\Big\} \,. \tag{4.22}$$

The saddle-point equation, emerging from the above formula in the course of integration over $\vec{\chi}$,

$$\frac{i}{2}\vec{\chi}\varepsilon_{\mu\nu\lambda}\partial_\mu \vec{h}_{\nu\lambda} = g_m \zeta \sum_{c,d=1}^N \vec{q}_{cd} \sin\left(g_m \vec{q}_{cd}\vec{\chi}\right), \tag{4.23}$$

[n] In particular, this makes explicit the number of positive roots, $\frac{N^2-N}{2}$.

is to be solved in a way similar to the one of the previous subsection, namely by using the Ansatz $\vec{h}_{\mu\nu} = \vec{q}_{ab} h_{\mu\nu}$, $\vec{\chi} = \vec{q}_{ab}\chi$. Multiplying both sides of Eq. (4.23) by $\vec{q}_{ab}$, we obtain

$$\frac{i}{2g_m\zeta}\varepsilon_{\mu\nu\lambda}\partial_\mu h_{\nu\lambda} = \sum_{c,d=1}^{N} \vec{q}_{ab}\vec{q}_{cd}\sin\left(g_m\vec{q}_{ab}\vec{q}_{cd}\chi\right). \tag{4.24}$$

Let us now compute the sum on the r.h.s. of this equation by virtue of Eq. (4.21). To this end, imagine the antisymmetric $N \times N$-matrix of roots $\vec{q}_{cd}$'s. Obviously, for a fixed root $\vec{q}_{ab}$, there is one root $\vec{q}_{cd}$ (equal to $\vec{q}_{ab}$) and a negative symmetric to it, whose scalar products with $\vec{q}_{ab}$ are equal to 1 and -1. Our aim is to calculate the number n of roots, whose scalar product with $\vec{q}_{ab}$ is equal to $\pm\frac{1}{2}$.[o] According to Eq. (4.21), these roots belong either to the rows $c = a$, $c = b$, or to the columns $d = a$, $d = b$, which in total contain $4N - 4$ elements. Among these, two are diagonal and other two are those which yield the scalar product ± 1. The remaining $n = 4(N-2)$ roots are precisely those which yield the scalar product $\pm\frac{1}{2}$, so that this product equals $\frac{1}{2}$ for $2(N-2)$ of them and $-\frac{1}{2}$ for the other $2(N-2)$. The sum on the r.h.s. of Eq. (4.24) thus takes the form

$$\sin(2\phi) - \sin(-2\phi) + 2(N-2)\left[\frac{1}{2}\sin\phi - \frac{1}{2}\sin(-\phi)\right] =$$
$$= 2\left[\sin(2\phi) + (N-2)\sin\phi\right],$$

where the field ϕ has been defined after Eq. (3.15). Let us now consider the limit $N \gg 1$, in which Eq. (4.24) takes the form $\sin\phi = iH_{ab}$, where (cf. Eq. (4.3))

$$H_{ab} \equiv \frac{1}{4Ng_m\zeta}\,\vec{q}_{ab}\,\varepsilon_{\mu\nu\lambda}\partial_\mu \vec{h}_{\nu\lambda}\ .$$

The analogue of the potential (4.2) in the same limit then reads

$$V\left[\vec{h}_{\mu\nu}\right] = 4N\zeta\int d^3x\left[H_{ab}\,\mathrm{arcsinh}\,H_{ab} - \sqrt{1 + H_{ab}^2}\right].$$

The string representation of the adjoint Wilson loop in the large-N limit is therefore given by Eq. (4.22) with the substitution

$$\int d^3x\left[\frac{i}{2}\vec{\chi}\varepsilon_{\mu\nu\lambda}\partial_\mu\vec{h}_{\nu\lambda} + \zeta\sum_{c,d=1}^{N}\cos\left(g_m\vec{q}_{cd}\vec{\chi}\right)\right] \Longrightarrow -V\left[\vec{h}_{\mu\nu}\right]$$

[o] For all other $N^2 - N - 2 - n$ roots, the scalar product vanishes, and so does the r.h.s. of Eq. (4.24).

(and the symbol "$\mathcal{D}\vec{\chi}$" removed).

We are finally interested in the adjoint counterparts of Eqs. (4.16), (4.17), one can obtain in the weak-field limit $|H_{ab}| \ll 1$. [p] In this limit, the formula (4.11) is reproduced with the substitution $\vec{\mu}_a \to \vec{q}_{ab}$. As a consequence, the ratios of adjoint case values of string couplings $\sigma_{\rm adj}$, $\alpha_{\rm adj}^{-1}$, and $\kappa_{\rm adj}$ to the respective fundamental case values, Eqs. (4.16) and (4.17), are equal to $\frac{2N}{N-1} \simeq 2$. In particular, for the string tensions this ratio coincides with the known leading large-N QCD result (see e.g. Ref. [66]). However, unlike the fundamental string, the adjoint one is unstable at large distances due to the production of W^+W^- pairs. The string-breaking distance, R_c, can be estimated from the balance of the string free energy, $\sigma_{\rm adj}R_c$, and the mass of the produced pair, $2m_W$ [50]. By virtue of Eq. (4.16), one obtains $R_c \propto d/\kappa$. Therefore, the breaking length of the adjoint string is a factor κ^{-1} larger than its thickness. However, since the very existence of the string implies that its length is much larger than its thickness (cf. the discussion in the paragraph preceding Eq. (4.18)), the question of (non-)existence of the adjoint string becomes purely numerical.

4.3. *k-strings*

The large-N ideas discussed in the Introduction have recently found a novel realization in the studies of the spectrum of k-strings in SU(N) gauge theories. A k-string is defined as the confining flux tube between sources in higher representations, carrying a charge k with respect to the center of the gauge group Z_N, i.e. representations with nonvanishing N-ality. These sources can be seen as the superposition of k fundamental charges, and charge conjugation exchanges k- and $(N-k)$-strings, so that non-trivial k-strings exist only for $N > 3$; [q] their string tensions σ_k can, and should, be used to constrain mechanisms of confinement [69, 70]. Results for the values of σ_k can be obtained by various approaches. Early results suggested the so-called "Casimir scaling" hypothesis for the ratio of string tensions [71],

$$R(k,N) \equiv \frac{\sigma_k}{\sigma_1} = \frac{k(N-k)}{N-1} \equiv C(k,N), \tag{4.25}$$

[p] Note that, using the formula $H_{ab} = \frac{1}{2N\zeta}\vec{q}_{ab}\vec{\rho}$ and the Cauchy inequality, leads to $|H_{ab}| \le \frac{|\vec{\rho}|}{2N\zeta}$ as the analogue of Eq. (4.9). This clearly leads to the same definition of the weak-field limit in terms of the low-density approximation, as in the fundamental case. Namely, the weak-field limit corresponds to densities $|\vec{\rho}|$, which are of the order N times smaller than the mean one (4.10).

[q] We will not consider here high-dimensional representations that are screened by gluons and do not yield a genuine asymptotic string tension.

where σ_1 is the fundamental string tension. This is based on the arguments of the Parisi–Sourlas dimensional reduction of the 4D QCD to the 2D one, which is due to stochastic vacuum fields, and the further use of the fact that in 2D confinement is produced by one-gluon exchange. Recent studies in supersymmetric Yang–Mills theories and M-theory suggest instead a "Sine scaling" formula,

$$R(k,N) = \frac{\sin(k\pi/N)}{\sin(\pi/N)} \ . \tag{4.26}$$

Corrections are expected to both formulae, but the form of such corrections is unknown for the physically relevant case of a four dimensional, nonsupersymmetric, SU(N) gauge theory.

In the large-N limit, where the interactions between flux tubes are suppressed by powers of $1/N$, the lowest-energy state of the system should be made of k fundamental flux tubes connecting the sources, hence

$$R(k,N) \ \overset{k-\text{fixed}}{\underset{}{\xrightarrow{N\to\infty}}} \ k \ . \tag{4.27}$$

Both the Casimir and the Sine scaling formulae satisfy this constraint; they also remain invariant under the replacement $k \to (N-k)$, which corresponds to the exchange of quarks with antiquarks. However, it has been argued in Refs. [72, 73] that the correction to the large-N behavior should occur as a power series in $1/N^2$ rather than $1/N$.[r] Clearly, such a behavior would exclude Casimir scaling as an *exact* description of the k-string spectrum.

Recent lattice calculations have provided new results for the spectrum of k-strings both in three and four dimensions [74–79]. They all confirm that Casimir scaling is a *good approximation* to the Yang–Mills results. To be more quantitative, one could say that all lattice results are within 10% of the Casimir scaling prediction, and that deviations from it are larger in four than they are in three dimensions, in agreement with strong-coupling predictions [78]. The taming of systematic errors is a crucial matter for such lattice calculations, and it can only be achieved by an intensive numerical analysis. In four dimensions, the higher statistics simulations presented in Ref. [78] show that corrections to the Casimir scaling formula are statistically significant, and actually favor the Sine scaling. Finally, it has been pointed out in Ref. [79] that higher-dimensional representations with common N-ality do yield the same string tension, as expected because of gluon screening.

[r] One should study with some care whether the arguments presented in Refs. [72, 73] hold independently of the space-time dimensionality.

These numerical results trigger a few comments on Casimir scaling. The original argument [71] was based on the idea that a 4D gauge theory in a random magnetic field could be described by a 2D theory without such a field. Besides the numerical results, there is little support for such an argument in QCD; moreover it is not clear that the same hypothesis could explain the approximate Casimir scaling observed in three dimensions. On the other hand, Casimir scaling appears "naturally" as the lowest-order result, both at strong-coupling in the case of k-strings in the Hamiltonian formulation of gauge theories, and in the case of the spectrum of bound states in chiral models. Corrections can be computed in the strong-coupling formulation and they turn out to be $\propto (D-2)/N$ – see e.g. Ref. [78] for a summary of results and references. While strong-coupling calculations are not directly relevant to describe the physics of the continuum theory, it is nonetheless instructive to have some quantitative analytic control within that framework. Last but not least, Casimir scaling also appears at the lowest order in the stochastic model of the QCD vacuum [80]. In view of these considerations, it is fair to say that approximate Casimir scaling should be a prerequisite for any model of confinement, that corrections should be expected, and that these corrections are liable to yield further information about the non-perturbative dynamics of strong interactions. Moreover, it would be very interesting to improve our understanding of some other aspects of the k-string spectrum, for example the origin of the Sine scaling for non-supersymmetric theories, or the structure of the corrections to this scaling form.

We will prove below that Casimir scaling occurs to a high accuracy in the low-density approximation of the SU(N) 3D GG model [81]. The k-string tension is defined by means of the k-th power of the fundamental Wilson loop. The surface-dependent part of the latter can be written, in terms of the dual-photon field, as

$$\langle W_k(\mathcal{C})\rangle_{\rm mon} = \sum_{a_1,\ldots,a_k=1}^{N}\left\langle \exp\left[-ig\left(\sum_{i=1}^{k}\vec{\mu}_{a_i}\right)\int d^3x\,\Sigma\left(\vec{x}\right)\vec{\chi}\left(\vec{x}\right)\right]\right\rangle_{\rm mon}, \tag{4.28}$$

where

$$\Sigma\left(\vec{x}\right) \equiv \int_{\Sigma(\mathcal{C})} d\sigma_\mu\left(\vec{x}(\xi)\right)\partial_\mu^x\delta\left(\vec{x}-\vec{x}(\xi)\right),$$

and $\Sigma(\mathcal{C})$ is an arbitrary surface bounded by the contour $\mathcal{C}$ and parametrized by the vector $\vec{x}(\xi)$.

The independence of Eq. (4.28) of the choice of $\Sigma(\mathcal{C})$ can readily be seen in the same way as for the case $k = 1$. The Σ-dependence rather appears in the weak-field, or low-density, approximation, which is equivalent to keeping only the quadratic term in the expansion of the cosine in Eq. (3.3). As has been demonstrated in subsection 4.1 (cf. the discussion around Eqs. (4.9), (4.10)), the notion "low-density" implies that the typical monopole density is related to the mean one, $\rho_{\rm mean} = \zeta N(N-1)$, by the sequence of inequalities

$$\rho_{\rm typical} \ll \zeta \cdot \mathcal{O}(N) \ll \rho_{\rm mean} = \zeta \cdot \mathcal{O}\left(N^2\right). \tag{4.29}$$

We will discuss in some more detail the correspondence between the low-density and large-N approximations after Eq. (4.36).

Denoting for brevity $\vec{a} \equiv -g\int d^3x \Sigma\left(\vec{x}\right)\vec{\chi}\left(\vec{x}\right)$, we can rewrite Eq. (4.28) as

$$\left\langle W_k(\mathcal{C})\right\rangle_{\rm mon} = \sum_{\substack{a_1,\ldots,a_k=1 \\ \left(\text{with possible repetitions}\right)}}^{N} \left\langle {\rm e}^{i\vec{a}\left(\vec{\mu}_{a_1}+\cdots+\vec{\mu}_{a_k}\right)}\right\rangle, \tag{4.30}$$

where in the low-density approximation the average is defined with respect to the action

$$\int d^3x \left[\frac{1}{2}\left(\partial_\mu\vec{\chi}\right)^2 + \frac{m_D^2}{2}\vec{\chi}^2\right]. \tag{4.31}$$

Similarly to the fundamental representation, in this approximation the string tension for a given k is the same for all surfaces $\Sigma(\mathcal{C})$, which are large enough in the sense $\sqrt{S} \gg d$, where S is the area of $\Sigma(\mathcal{C})$. In particular, the fundamental string tension found above is $\sigma_1 = \frac{N-1}{2N}\bar{\sigma}$, where $\bar{\sigma} \equiv 4\pi^2\frac{\sqrt{\zeta N}}{g_m}$, and the factor $\frac{N-1}{2N}$ is the square of a weight vector.

To evaluate Eq. (4.30) for $k > 1$, we should calculate expressions of the form

$$\left(n\vec{\mu}_{a_i} + \sum_{j=1}^{k-n}\vec{\mu}_{a_j}\right)^2, \tag{4.32}$$

where $(k-n)$ weight vectors $\vec{\mu}_{a_j}$'s are mutually different and also different from the vector $\vec{\mu}_{a_i}$. By virtue of the formula $\vec{\mu}_a\vec{\mu}_b = \frac{1}{2}\left(\delta_{ab} - \frac{1}{N}\right)$, we obtain for Eq. (4.32)

$$\frac{N-1}{2N}\left(n^2+k-n\right) - \frac{1}{2N}\left[2n(k-n) + 2\sum_{l=1}^{k-n-1} l\right] = \frac{k(N-k)}{2N} + \frac{1}{2}\left(n^2-n\right). \tag{4.33}$$

We should further calculate the number of times a term with a given n appears in the sum (4.32). In what follows, we will consider the case $k < N$, although k may be of the order of N. Then, $C_k^n \equiv \frac{k!}{n!(k-n)!}$ possibilities exist to choose out of k weight vectors n coinciding ones, whose index may acquire any values from 1 to N. The index of any weight vector out of the remaining $(k-n)$ ones may then acquire only $(N-1)$ values, and so on. Finally, the index of the last weight vector may acquire $(N-k+n)$ values. Therefore, the desired number of times a term with a given n appears in the sum (4.32) is

$$C_k^n N \cdot (k-n)(N-1) \cdot (k-n-1)(N-2) \cdots 1(N-k+n) =$$

$$= C_k^n A_N^{k-n+1}(k-n)! = \frac{k!N!}{n!(n+N-k-1)!}\ , \qquad (4.34)$$

where $A_N^{k-n+1} \equiv \frac{N!}{(N-k+n-1)!}$. Equations (4.33) and (4.34) together yield for the monopole contribution to the Wilson loop, Eq. (4.30),

$$\langle W_k(\mathcal{C})\rangle_{\rm mon} = k!N!\mathrm{e}^{-C\bar{\sigma}S} \cdot \sum_{n=1}^{k} \frac{1}{n!(n+N-k-1)!}\mathrm{e}^{-\frac{n^2-n}{2}\bar{\sigma}S}\ , \qquad (4.35)$$

where $C \equiv \frac{k(N-k)}{2N}$ is proportional to the Casimir of the rank-k antisymmetric representation of SU(N). We have thus arrived at a Feynman–Kac formula, where, in the asymptotic regime of interest, $S \to \infty$, only the first term in the sum is essential. The k-string tension is therefore $\sigma_k = C\bar{\sigma}$, yielding the Casimir-scaling law (4.25). It is interesting to note that the Casimir of the original unbroken SU(N) group is recovered. This is a consequence of the Dirac quantization condition [61], which distributes the quark charges along the weights of the fundamental representation and the monopole ones along the roots. The orthonormality of the roots then yields the action (4.31), which is diagonal in the dual magnetic variables; the sum of the weights squared is responsible for the Casimir factor, since $C = \left(\vec{\mu}_{a_1} + \cdots + \vec{\mu}_{a_k}\right)^2$, where all k weight vectors are different from each other. Therefore, terms where all k weight vectors are mutually different yield the dominant contribution to the sum (4.30). Their number in the sum is equal to $\frac{k!N!}{(N-k)!}$, that corresponds to the $(n=1)$ term in Eq. (4.35).

Let us further address the leading correction to Casimir scaling, which originates from the non-diluteness of plasma. Expanding the cosine up to the quartic term and using

$$\sum_{i=1}^{N(N-1)/2} q_i^\alpha q_i^\beta q_i^\gamma q_i^\delta = \frac{N}{2(N+1)}\left(\delta^{\alpha\beta}\delta^{\gamma\delta} + \delta^{\alpha\gamma}\delta^{\beta\delta} + \delta^{\alpha\delta}\delta^{\beta\gamma}\right),$$

which stems from the orthonormality of roots, we obtain the action

$$\int d^3x \left[\frac{1}{2}\left(\partial_\mu \vec{\chi}\right)^2 + \frac{m_D^2}{2}\left(\vec{\chi}^2 - \frac{g_m^2}{12(N+1)}\vec{\chi}^4\right)\right]. \tag{4.36}$$

By virtue of this formula, one can analyze the correspondence between the $1/N$-expansion and corrections to the low-density approximation. The natural choice for defining the behavior of the electric coupling constant in the large-N limit is the QCD-inspired one, $g = \mathcal{O}(N^{-1/2})$. To make some estimates, let us use an obvious argument that the $\vec{\chi}$ field configuration dominating in the partition function is the one where every term in the action (4.36) is of the order of unity. When applied to the kinetic term, this demand tells us that the characteristic wavelength l of the field $\vec{\chi}$ is related to the amplitude of this field as $l \sim |\vec{\chi}|^{-2}$. Substituting this estimate into the condition $l^3 m_D^2 |\vec{\chi}|^2 \sim 1$, we get $|\vec{\chi}|^2 \sim m_D$. The ratio of the quartic and mass terms, being of the order of $|\vec{\chi}|^2 g_m^2/N$, can then be estimated as $\frac{m_D g_m^2}{N} = g_m^3\sqrt{\frac{\zeta}{N}} \sim N\sqrt{\zeta}$. With exponentially high accuracy, this ratio is small, provided $N \lesssim \mathcal{O}\left(\mathrm{e}^{S_0/2}\right)$. [s] Therefore, the non-diluteness corrections are suppressed only for N's bounded from above by a certain parameter. However, due to the exponential largeness of this parameter, the constraint leaves enough space for N to be sufficiently large to provide the validity of the inequalities (4.29).

To proceed with the study of the non-diluteness correction, one needs to solve the saddle-point equation iteratively, corresponding to the average (4.30) taken with respect to the approximate action (4.36). Since it has been demonstrated above that the string tension is defined by the averages $\left\langle \mathrm{e}^{i\vec{a}\left(\vec{\mu}_{a_1}+\cdots+\vec{\mu}_{a_k}\right)}\right\rangle$, where all k weight vectors are mutually different, let us restrict ourselves to such terms in the sum (4.30) only. Solving the saddle-point equation with the Ansatz $\vec{\chi} = \vec{\chi}_0 + \vec{\chi}_1$, where $|\vec{\chi}_1| \ll |\vec{\chi}_0|$, we obtain

$$-\ln\left\langle \mathrm{e}^{i\vec{a}\left(\vec{\mu}_{a_1}+\cdots+\vec{\mu}_{a_k}\right)}\right\rangle = \frac{g^2}{2}\, C\int d^3x d^3y \Sigma\left(\vec{x}\right) D_{m_D}\left(\vec{x}-\vec{y}\right)\Sigma\left(\vec{y}\right) + \Delta\mathcal{S}\ . \tag{4.37}$$

The first term on the r.h.s. of Eq. (4.37) yields the string tension $\sigma_k = C\bar{\sigma}$,

[s] This constraint is similar to those which were derived at the end of sect. 2 as the necessary conditions for the stochasticity of the Higgs vacuum. We therefore conclude that the conditions of stochasticity of the Higgs vacuum and of the validity of the dilute-plasma approximation parallel each other.

while the second term yields the desired correction. This term reads

$$\Delta\mathcal{S} = -\frac{2\pi^2}{3}\frac{(gm_DC)^2}{N+1}\int d^3x\prod_{l=1}^{4}\left[\int d^3x_l D_{m_D}\left(\vec{x}-\vec{x}_l\right)\Sigma\left(\vec{x}_l\right)\right] =$$

$$= -\frac{\pi^2}{6}\frac{(gm_DC)^2}{N+1}\int d\sigma_{\mu\nu}\left(\vec{x}_1\right)d\sigma_{\mu\nu}\left(\vec{x}_2\right)d\sigma_{\lambda\rho}\left(\vec{x}_3\right)d\sigma_{\lambda\rho}\left(\vec{x}_4\right)\partial_\alpha^{x_1}\partial_\alpha^{x_2}\partial_\beta^{x_3}\partial_\beta^{x_4}I\ , \tag{4.38}$$

where $I \equiv \int d^3x\prod_{l=1}^{4}D_{m_D}\left(\vec{x}-\vec{x}_l\right)$. The action (4.38) can be represented in the form

$$\Delta\mathcal{S} = -\frac{(gC)^2}{(N+1)m_D^5}\int d\sigma_{\mu\nu}(\vec{x}_1)d\sigma_{\mu\nu}(\vec{x}_2)D(\vec{x}_1-\vec{x}_2)\ \times$$
$$\times\int d\sigma_{\lambda\rho}(\vec{x}_3)d\sigma_{\lambda\rho}(\vec{x}_4)D(\vec{x}_3-\vec{x}_4)\ \times\ G(\vec{x}_1-\vec{x}_3)\,. \tag{4.39}$$

Here, D and G are some positive functions, which depend on $m_D|\vec{x}_i-\vec{x}_j|$ and vanish exponentially at the distances $\gtrsim d$. They can be represented as $D(\vec{x}) = m_D^4\mathcal{D}(m_D|\vec{x}|)$, $G(\vec{x}) = m_D^4\mathcal{G}(m_D|\vec{x}|)$, where the functions $\mathcal{D}$ and $\mathcal{G}$ are dimensionless.[t]

The derivative expansion yields the Nambu-Goto action as leading term,

$$\int d\sigma_{\mu\nu}(\vec{x}_1)d\sigma_{\mu\nu}(\vec{x}_2)D(\vec{x}_1-\vec{x}_2) = \sigma_D\int d^2\xi\sqrt{\mathsf{g}(\vec{x}_1)}+\mathcal{O}\left(\frac{\sigma_D}{m_D^2}\right). \tag{4.40}$$

Here, $\sigma_D = 2m_D^2\int d^2z\mathcal{D}(|z|)$ (with z being dimensionless). Recalling that $d\sigma_{\alpha\beta}(\vec{x}) = \sqrt{\mathsf{g}(\vec{x})}t_{\alpha\beta}(\vec{x})d^2\xi$, we may take into account that we are interested in the *leading* term of the derivative expansion of the action $\Delta\mathcal{S}$, which corresponds to the so short distance $|\vec{x}_1-\vec{x}_3|$, that $t_{\alpha\beta}\left(\vec{x}_1\right)t_{\alpha\beta}\left(\vec{x}_3\right)\simeq 2$. (Higher terms of the derivative expansion contain derivatives of $t_{\alpha\beta}$ and do not contribute to the string tension.) This yields for the integral in Eq. (4.39)

$$\sigma_D^2\int d^2\xi d^2\xi'\sqrt{\mathsf{g}(\vec{x}_1)\mathsf{g}(\vec{x}_3)}\,G(\vec{x}_1-\vec{x}_3)\simeq$$

$$\simeq\frac{\sigma_D^2}{2}\int d\sigma_{\alpha\beta}\left(\vec{x}_1\right)d\sigma_{\alpha\beta}\left(\vec{x}_3\right)G\left(\vec{x}_1-\vec{x}_3\right) = \frac{\sigma_D^2}{2}\left[\sigma_G\int d^2\xi\sqrt{\mathsf{g}}+\mathcal{O}\left(\frac{\sigma_G}{m_D^2}\right)\right],$$

[t] Our investigations can readily be translated to the stochastic vacuum model of QCD [80] for the evaluation of a correction to the string tension, produced by the four-point irreducible average of field strengths. In that case, the functions D and G would be proportional to the gluonic condensate.

where $\sigma_G = 2m_D^2 \int d^2z\mathcal{G}(|z|)$. We finally obtain from Eq. (4.39) that

$$\Delta\sigma_k \simeq \frac{(gC)^2\sigma_D^2\sigma_G}{2(N+1)m_D^5} = \frac{\alpha}{4}\frac{(gC)^2 m_D}{N+1} = \alpha\frac{\bar{\sigma}C^2}{N+1},$$

where α is some dimensionless positive constant. This yields

$$\sigma_k + \Delta\sigma_k = \bar{\sigma}C\left(1 + \frac{\alpha C}{N+1}\right).$$

In the limit $k \sim N \gg 1$ of interest, the correction is $\mathcal{O}(1)$, while for $k = \mathcal{O}(1)$ it is $\mathcal{O}(1/N)$. The latter fact enables one to write down the final result for the leading correction to Eq. (4.25) due to the non-diluteness of monopole plasma,

$$R(k,N) + \Delta R(k,N) \equiv \frac{\sigma_k + \Delta\sigma_k}{\sigma_1 + \Delta\sigma_1} = C(k,N)\left[1 + \alpha\frac{(k-1)(N-k-1)}{2N(N+1)}\right].$$

This expression is invariant under the replacement $k \to (N-k)$ just as the expression (4.25), which does not account for non-diluteness. The fact that, at $k \sim N \gg 1$, the correction to the Casimir-scaling law is $\mathcal{O}(1)$ indicates that non-diluteness effects can significantly distort the Casimir-scaling behavior.

5. Generalization to the SU(N)-analogue of 4D compact QED with the θ term

In this section, we will consider the 4D case and introduce the field-theoretical θ term. [u] As was first found for compact QED in Refs. [33, 36] by means of the derivative expansion of the resulting nonlocal string effective action, this term generates the string θ term. Being proportional to the number of self-intersections of the world sheet, the latter might be important for the solution of the problem of crumpling of large world sheets [3, 31]. In this section, we will perform the respective analysis for the general SU(N) case under study, in the fundamental and in the adjoint representations, as well as for k-strings. It is worth noting that, in the lattice 4D compact QED, confinement holds only at strong coupling. On the other hand, the continuum counterpart [33, 36] (of the SU(N) version) which we are going to explore possesses confinement at arbitrary values of coupling. However, we will see that the solution to the problem of crumpling due to the θ term

[u] In the case when the ensemble of Abelian-projected monopoles is modeled by the magnetically-charged dual Higgs field, the effects produced by this term on the string effective action have been studied in Ref. [67].

is only possible in the strong-coupling regime, implied in a certain sense. Note also that the continuum sine-Gordon theory of the dual-photon field possesses an ultraviolet cutoff, Λ, which appears in taking the path-integral average over the shapes of monopole loops. However, the θ-parameter is dimensionless, and consequently its values, at which one may expect the disappearance of crumpling, will be cutoff-independent.

The full partition function, including the θ term and the average over free photons, is (cf. Eq. (3.1))

$$\mathcal{Z}^N = \int \mathcal{D}\vec{A}_\mu \mathrm{e}^{-\frac{1}{4}\int d^4x \vec{F}_{\mu\nu}^2} \sum_{\mathcal{N}=0}^{\infty} \frac{\zeta^{\mathcal{N}}}{\mathcal{N}!} \left\langle \exp\left\{-\mathcal{S}\left[\vec{j}_\mu^{\mathcal{N}}, \vec{A}_\mu\right]\right\}\right\rangle_{\mathrm{mon}}, \tag{5.1}$$

where

$$\mathcal{S}\left[\vec{j}_\mu^{\mathcal{N}}, \vec{A}_\mu\right] = \frac{1}{2}\int d^4x d^4y \vec{j}_\mu^{\mathcal{N}}(x) D_0(x-y) \vec{j}_\mu^{\mathcal{N}}(y) + \frac{i\theta g^2}{8\pi^2}\int d^4x \vec{A}_\mu \vec{j}_\mu^{\mathcal{N}}, \tag{5.2}$$

and $\vec{F}_{\mu\nu} = \partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu$. For $\mathcal{N} = 0$, the monopole current $j_\mu^{\mathcal{N}}$ is equal to zero, whereas for $\mathcal{N} \geq 1$, it is defined as $\vec{j}_\mu^{\mathcal{N}} = g_m \sum\limits_{k=1}^{\mathcal{N}} \vec{q}_{i_k} \oint dz_\mu^k(\tau) \delta\left(x - x^k(\tau)\right)$. The couplings g and g_m are now dimensionless and obey the same condition $gg_m = 4\pi$ as in the 3D case. We have also parametrized the trajectory of the k-th monopole by the vector $x_\mu^k(\tau) = y_\mu^k + z_\mu^k(\tau)$, where $y_\mu^k = \int\limits_0^1 d\tau x_\mu^k(\tau)$ is the position of the trajectory, whereas the vector $z_\mu^k(\tau)$ describes its shape, both of which should be averaged over. The fugacity of a single-monopole loop, ζ, entering Eq. (5.1), has the dimensionality $[\mathrm{mass}]^4$, $\zeta \propto \mathrm{e}^{-S_{\mathrm{mon}}}$, where the action of a single k-th loop, obeying the estimate $S_{\mathrm{mon}} \propto \frac{1}{g^2}\int\limits_0^1 d\tau \sqrt{(\dot{z}^k)^2}$, is assumed to be of the same order of magnitude for all loops. Finally, in Eq. (5.1), $D_0(x) = 1/(4\pi^2 x^2)$ is the 4D Coulomb propagator, and the average over monopole loops is defined similarly to Eq. (3.2) as

$$\langle \mathcal{O} \rangle_{\mathrm{mon}} = \prod_{n=0}^{\mathcal{N}} \int d^4 y_n \sum_{i_n = \pm 1, \ldots, \pm \frac{N(N-1)}{2}} \langle \mathcal{O} \rangle_{z_n(\tau)} .$$

The particular form of the path-integral average over the shapes of the loops, $z_n(\tau)$'s, is immaterial for the final (ultraviolet-cutoff dependent) expression for the partition function (see e.g. Ref. [68] for a similar situation in the plasma of closed dual strings in the Abelian Higgs models). The only thing which matters is the normalization $\langle 1 \rangle_{z_n(\tau)} = 1$, which will be implied henceforth. The analogue of the partition function (3.9), (3.10) then reads

$$\mathcal{Z}_{\rm mon}^N \left[\vec{A}_\mu\right] = \int \mathcal{D}\vec{j}_\mu \exp\left\{-\mathcal{S}\left[\vec{j}_\mu, \vec{A}_\mu\right]\right\} \times$$
$$\times \int \mathcal{D}\vec{\chi}_\mu \exp\left\{\int d^4x \left[2\zeta \sum_i \cos\left(\frac{\vec{q}_i\,|\vec{\chi}_\mu|}{\Lambda}\right) + i\vec{\chi}_\mu \vec{j}_\mu\right]\right\}, \qquad (5.3)$$

with the action $\mathcal{S}$ given by Eq. (5.2), while the full partition function is

$$\mathcal{Z}^N = \int \mathcal{D}\vec{A}_\mu \mathrm{e}^{-\frac{1}{4}\int d^4x \vec{F}_{\mu\nu}^2} \mathcal{Z}_{\rm mon}^N \left[\vec{A}_\mu\right].$$

In Eq. (5.3),

$$|\vec{\chi}_\mu| \equiv \left(\sqrt{\chi_\mu^1 \chi_\mu^1}, \ldots, \sqrt{\chi_\mu^{N-1}\chi_\mu^{N-1}}\right), \qquad (5.4)$$

and $\Lambda \sim \left|y^k\right| / \left(z^k\right)^2 \gg \left|z^k\right|^{-1}$ is the ultraviolet cutoff. Clearly, unlike the 3D case without the θ term, the $\vec{A}_\mu$ field is now coupled to $\vec{j}_\mu$, making $\mathcal{Z}_{\rm mon}^N$ $\vec{A}_\mu$-dependent. This eventually leads to the change of the mass of the dual photon $\vec{\chi}_\mu$, as well as to the appearance of the string θ term.

Let us address the fundamental case first. After the saddle-point integration over $\vec{\chi}_\mu$ [v], the analogue of Eqs. (4.1), (4.2) takes the form

$$W_a = \frac{1}{\mathcal{Z}^N} \int \mathcal{D}\vec{h}_{\mu\nu} \exp\left\{ - S\left[\vec{h}_{\mu\nu}\right] + \frac{ig}{2}\vec{\mu}_a \int\limits_{\Sigma(C)} d\sigma_{\mu\nu}\vec{h}_{\mu\nu}\right\}, \qquad (5.5)$$

where the constraint $\partial_\mu \vec{h}_{\mu\nu} = 0$ was already abolished. Here, the Kalb–Ramond action reads

$$S\left[\vec{h}_{\mu\nu}\right] = \int d^4x \left(\frac{1}{4}\vec{h}_{\mu\nu}^2 - \frac{i\theta g^2}{32\pi^2}\vec{h}_{\mu\nu}\tilde{\vec{h}}_{\mu\nu}\right) + V\left[\vec{h}_{\mu\nu}\right]. \qquad (5.6)$$

[v] Unlike the superrenormalizable 3D case, where loop corrections to the saddle point are rapidly converging, this is not necessarily the case in four dimensions. This fact is, however, clearly unimportant in the weak-field limit (Eq. (5.7) below), where the cosine in Eq. (5.3) is approximated by the quadratic term only, and the saddle-point integration over $\vec{\chi}_\mu$ becomes Gaussian. As we have seen in the previous section, this limit is already enough for the discussion of the string representation. One of the authors (D.A.) is grateful to H. Gies and E. Vicari for drawing his attention to this issue.

The potential V in this equation is given by Eq. (4.2) with the symbol $\int d^3x$ replaced by $\int d^4x$, and

$$H_a = \frac{g\Lambda}{\zeta(N-1)}\vec{\mu}_a \left|\partial_\mu \tilde{\vec{h}}_{\mu\nu}\right|.$$

In these formulae, $\tilde{\mathcal{O}}_{\mu\nu} \equiv \frac{1}{2}\varepsilon_{\mu\nu\lambda\rho}\mathcal{O}_{\lambda\rho}$, and the absolute value is defined in the same way as in Eq. (5.4), i.e. again with respect to the Lorentz indices only. Note that the form to which the θ term has been transformed is quite natural, since the respective initial expression of Eq. (5.2) can be rewritten modulo full derivatives as

$$\frac{i\theta g^2}{8\pi^2}\int d^4x \vec{A}_\mu \vec{j}_\mu^{\mathcal{N}} = -\frac{i\theta g^2}{32\pi^2}\int d^4x \left(\vec{F}_{\mu\nu} + \vec{F}_{\mu\nu}^{\mathcal{N}}\right)\left(\tilde{\vec{F}}_{\mu\nu} + \tilde{\vec{F}}_{\mu\nu}^{\mathcal{N}}\right),$$

where (cf. Eq. (3.4))

$$\vec{F}_{\mu\nu}^{\mathcal{N}}(x) = -\varepsilon_{\mu\nu\lambda\rho}\partial_\lambda \int d^4y D_0(x-y)\vec{j}_\rho^{\mathcal{N}}(y), \quad \partial_\mu \tilde{\vec{F}}_{\mu\nu}^{\mathcal{N}} = \vec{j}_\nu^{\mathcal{N}}.$$

Next, the mass of the Kalb–Ramond field, equal to the Debye mass of the dual photon, which follows from the action (5.6), is $m_D = \frac{g\eta}{4\pi}\sqrt{\left(\frac{4\pi}{g^2}\right)^2 + \left(\frac{\theta}{2\pi}\right)^2}$, where $\eta \equiv \sqrt{N\zeta}/\Lambda$. In the extreme strong-coupling limit, $g \to \infty$, this expression demonstrates the important difference of the case $\theta = 0$ from the case $\theta \neq 0$. Namely, since $\eta(g) \propto \mathrm{e}^{-\mathrm{const}/g^2} \to 1$, $m_D \to 0$ at $\theta = 0$, whereas $m_D \to \infty$ at $\theta \neq 0$. In another words, in the extreme strong-coupling limit, the correlation length of the vacuum, equal to d, goes large (small) at $\theta = 0$ ($\theta \neq 0$).

In the weak-field limit, $|H_a| \ll 1$, Eq. (5.5) yields (cf. Eq. (4.11))

$$W(C,\Sigma)^{\mathrm{tot},\ a}_{\mathrm{weak-field}} = \frac{1}{\mathcal{Z}^{\mathrm{tot}}}\int \mathcal{D}\vec{h}_{\mu\nu}\times$$

$$\times \exp\left\{-\int d^4x \left[\frac{g^2}{12\eta^2}\vec{H}^2_{\mu\nu\lambda} + \frac{1}{4}\vec{h}^2_{\mu\nu} - \frac{i\theta g^2}{32\pi^2}\vec{h}_{\mu\nu}\tilde{\vec{h}}_{\mu\nu} - \frac{ig}{2}\vec{\mu}_a\vec{h}_{\mu\nu}\Sigma_{\mu\nu}\right]\right\}, \quad (5.7)$$

where again $\mathcal{Z}^{\mathrm{tot}}$ is the same integral over $\vec{h}_{\mu\nu}$, but with $\Sigma_{\mu\nu}$ set to zero. Integrating over $\vec{h}_{\mu\nu}$, we then obtain

$$W(C,\Sigma)^{\mathrm{tot},\ a}_{\mathrm{weak-field}} = \exp\left\{-\frac{N-1}{4N}\left[g^2 \oint_C dx_\mu \oint_C dx'_\mu D_{m_D}(x-x') + \right.\right.$$

$$+\frac{\eta^2}{2}\int d^4x d^4x' D_{m_D}(x-x')\left(\Sigma_{\mu\nu}(x)\Sigma_{\mu\nu}(x') + \frac{i\theta g^2}{8\pi^2}\Sigma_{\mu\nu}(x)\tilde{\Sigma}_{\mu\nu}(x')\right)\Bigg]\Bigg]\Bigg\}, \tag{5.8}$$

where $D_m(x) \equiv mK_1(m|x|)/(4\pi^2|x|)$ is the 4D Yukawa propagator with K_1 denoting the modified Bessel function. (Note that, at $\theta = 0$, Eq. (5.8) takes the form of Eq. (4.13).) Further curvature expansion of the $\Sigma_{\mu\nu} \times \tilde{\Sigma}_{\mu\nu}$ interaction (analogous to the expansion of the $\Sigma_{\mu\nu} \times \Sigma_{\mu\nu}$ interaction in the action (4.14)) yields the string θ term equal to $ic_{\text{fund}}\nu$. Here, $\nu \equiv \frac{1}{2\pi}\int d^2\xi\sqrt{\mathsf{g}}\mathsf{g}^{ab}(\partial_a t_{\mu\nu})(\partial_b\tilde{t}_{\mu\nu})$ is the number of self-intersections of the world sheet, and the coupling constant c reads

$$c_{\text{fund}} = \frac{(N-1)\theta}{8N\left[\left(\frac{4\pi}{g^2}\right)^2 + \left(\frac{\theta}{2\pi}\right)^2\right]}\,. \tag{5.9}$$

As has already been discussed, c_{fund} is Λ-independent, since (similarly to the rigidity coupling constant α) it is dimensionless. We therefore see that, at

$$\theta_{\pm}^{\text{fund}} = \frac{\pi}{2}\left[\frac{N-1}{2N} \pm \sqrt{\left(\frac{N-1}{2N}\right)^2 - \left(\frac{16\pi}{g^2}\right)^2}\,\right], \tag{5.10}$$

c_{fund} becomes equal to π, and self-intersections are weighted in the string partition function with the factor $(-1)^\nu$, which might cure the problem of crumpling. This is only possible at $g \geq g^{\text{fund}} \equiv 4\sqrt{\frac{2\pi N}{N-1}}$, paralleling the strong-coupling regime, which should hold in the lattice version of the model (cf. the discussion in the beginning of this section).[w] In the extreme strong-coupling limit, understood in the sense $g \gg g^{\text{fund}}$, only one critical value, θ_+^{fund}, survives, $\theta_+^{\text{fund}} \to \pi\frac{N-1}{2N}$, whereas $\theta_-^{\text{fund}} \to 0$, i.e. θ_-^{fund} becomes a spurious solution, since $c_{\text{fund}}|_{\theta=0} = 0$.

With the use of the results of subsection 4.2, the adjoint case large-N counterpart of Eq. (5.10) can readily be found. Indeed, Eq. (5.9) in that case becomes

$$c_{\text{adj}} = \frac{\theta}{4\left[\left(\frac{4\pi}{g^2}\right)^2 + \left(\frac{\theta}{2\pi}\right)^2\right]},$$

so that c_{adj} is equal to π at

$$\theta_{\pm}^{\text{adj}} = \frac{\pi}{2}\left[1 \pm \sqrt{1 - \left(\frac{16\pi}{g^2}\right)^2}\,\right].$$

[w] Clearly, at $g = g^{\text{fund}}$, $\theta_+^{\text{fund}} = \theta_-^{\text{fund}} = \pi\,\frac{N-1}{4N}$.

Note that the respective lower bound for the critical value of g, [x] $g^{\rm adj}_{N\gg 1} = 4\sqrt{\pi}$, is only slightly different from the value $g^{\rm fund}_{N\gg 1} = 4\sqrt{2\pi}$. Similarly to the fundamental case, at $g \gg g^{\rm adj}_{N\gg 1}$, $\theta^{\rm adj}_{-}$ becomes a spurious solution, whereas $\theta^{\rm adj}_{+} \to \pi$. Thus, at $N \gg 1$ and in the strong-coupling limit, understood in the sense $g \gg g^{\rm fund}_{N\gg 1}$, the critical fundamental- and adjoint case values of θ, at which the problem of crumpling might be solved, are given by the simple formula $\theta^{\rm fund}_{+} = \frac{1}{2}\theta^{\rm adj}_{+} = \frac{\pi}{2}$.

Finally, for k-strings, the ratio $c_k/c_{\rm fund}$ is the same Casimir one, $C(k,N)$, as the ratio of string tensions, as long as $N \lesssim \mathcal{O}\left({\rm e}^{S_{\rm mon}/2}\right)$. [y] This enables one to readily find $\theta^k_{\pm}$ as well. Namely,

$$\theta^k_{\pm} = \frac{\pi}{2}\left[\frac{N-1}{2NC(k,N)} \pm \sqrt{\left(\frac{N-1}{2NC(k,N)}\right)^2 - \left(\frac{16\pi}{g^2}\right)^2}\,\right],$$

that is only possible at $g > g_k \equiv g^{\rm fund}\sqrt{C(k,N)} \geq g^{\rm fund}$.

6. Geometric aspects of confining strings: the physics of negative stiffness

The role of antisymmetric tensor field theories for confinement has been studied in detail in Ref. [33]. By analyzing compact antisymmetric tensor field theories of rank $h-1$ Quevedo and Trugenberger have shown that, starting from a "Coulomb" phase, the condensation of $(d-h-1)$-branes (where $D = d+1$ is the space-time dimension) leads to a generalized confinement phase for $(h-1)$-branes. Each phase in the model has two dual descriptions in terms of antisymmetric tensor of different ranks, massless for the Coulomb phase and massive for the confinement phase. Upon integration over the massive antisymmetric tensor field (in the case of an even number of dimensions – when the θ term is absent), one obtains the string effective action (4.14). The derivative expansion of this nonlocal string effective action, up to the term next to the rigidity one, produces the confining-string action,

$$S = \int d^2\xi \sqrt{\mathsf{g}}\; \mathsf{g}^{ab}\mathcal{D}_a x_\mu \left(T - s\mathcal{D}^2 + \frac{1}{M^2}\mathcal{D}^4\right)\mathcal{D}_b x_\mu. \qquad (6.1)$$

[x] As in the previous footnote, at the particular value of g, $g = g^{\rm adj}_{N\gg 1}$, $\theta^{\rm adj}_{+} = \theta^{\rm adj}_{-} = \frac{\pi}{2}$.

[y] In 4D, the ratio of string tensions indeed equals $C(k,N)$ as in 3D, as long as the dual-photon field can be treated as a free massive field, since the rest of the derivation of Eq. (4.35) is based on the properties of fundamental weights only.

Here, $\mathcal{D}_a$ is the covariant derivative with respect to the induced metric $\mathsf{g}_{ab} = (\partial_a x_\mu)(\partial_b x_\mu)$ on the surface $\vec{x}\,(\xi_0, \xi_1)$.

In (6.1), the first term provides a bare surface tension $2T$, while the second accounts for rigidity with stiffness parameter s. The last term can be written (up to surface terms) as a combination of the fourth power and the square of the gradient of the extrinsic curvature matrices, with M being a new mass scale. It thus suppresses world-sheet configurations with rapidly changing extrinsic curvature; due to its presence, the stiffness s may not necessarily be negative, as in Eq. (4.17) above, but also positive. We analyze the model (6.1) in the large-D approximation. To this end, we introduce a Lagrange multiplier matrix L^{ab} to impose the constraint $\mathsf{g}_{ab} = (\partial_a x_\mu)(\partial_b x_\mu)$, extending the action (6.1) to

$$S + \int d^2\xi \sqrt{\mathsf{g}} \;\; L^{ab} \left(\partial_a x_\mu \partial_b x_\mu - \mathsf{g}_{ab}\right) \; . \tag{6.2}$$

Then we parametrize the world-sheet in a Gauss map by $x_\mu(\xi) = \big(\xi_0, \xi_1, \phi^i(\xi)\big)$, $(i = 2, \ldots, D-1)$, where $-\beta/2 \le \xi_0 \le \beta/2$, $-R/2 \le \xi_1 \le R/2$, and $\phi^i(\xi)$ describe the $D-2$ transverse fluctuations. With the usual homogeneity and isotropy Ansatz $\mathsf{g}_{ab} = \rho\delta_{ab}$, $L^{ab} = L\mathsf{g}^{ab}$ of infinite surfaces $(\beta, R \to \infty)$ at the saddle point, we obtain

$$S \; = 2\int d^2\xi \left[T + L(1-\rho)\right] + \int d^2\xi \;\, \partial_a\phi^i V\left(T, s, M, L, \mathcal{D}^2\right) \partial_a\phi^i \;\; , \tag{6.3}$$

where

$$V\left(T, s, M, L, \mathcal{D}^2\right) = T + L - s\mathcal{D}^2 + \frac{1}{M^2}\mathcal{D}^4. \tag{6.4}$$

Integrating over the transverse fluctuations, in the infinite-area limit, we get the effective action

$$S^{\rm eff} = 2A_{\rm ext}\left[T + L(1-\rho)\right] +$$

$$+A_{\rm ext}\frac{D-2}{8\pi^2}\rho\int d^2p \, \ln\left[p^2 V\left(T, s, M, L, p\right)\right] \;\; , \tag{6.5}$$

where $A_{\rm ext} = \beta R$ is the extrinsic, physical, space-time area. For large D, the fluctuations of L and ρ are suppressed and these variables take their "classical values", determined by the two saddle-point equations

$$0 = f\left(T, s, M, L\right) \;\; , \quad \rho = \frac{1}{f'\left(T, s, M, L\right)} \;\; , \tag{6.6}$$

where the prime denotes a derivative with respect to L and the "saddle-function" f is defined by

$$f\,(T,s,M,L) \equiv L - \frac{D-2}{8\pi}\int dp\; p\; \ln\left[p^2 V\,(T,s,M,L,p)\right] . \qquad (6.7)$$

Using (6.6) in (6.5) we get $S^{\rm eff} = 2\,(T+L)\,A_{\rm ext}$ showing that $\mathcal{T} = 2\,(T+L)$ is the physical string tension.

The stability condition for the Euclidean surfaces is that $V\,(T,s,M,L,p)$ be positive for all $p^2 \geq 0$. However, we will require the same condition also for $p^2 \leq 0$, so that strings propagating in Minkowski space-time are not affected by the propagating states of negative norm which plague rigid strings. The stability condition becomes thus $\sqrt{T+L} \geq |sM/2|$, which allows us to introduce the real variables R and I defined by

$$R^2 \equiv \frac{M}{2}\sqrt{T+L} + \frac{sM^2}{4} \;, \quad I^2 \equiv \frac{M}{2}\sqrt{T+L} - \frac{sM^2}{4} \;. \qquad (6.8)$$

In terms of these, the kernel V can be written as

$$M^2 V\,(T,s,M,L,p) = \left(R^2+I^2\right)^2 + 2\left(R^2-I^2\right)p^2 + p^4 \;. \qquad (6.9)$$

In order to analyze the geometric properties of the string model (6.1) we will study two correlation functions. First, we consider the orientational correlation function

$$g_{ab}(\xi-\xi') \equiv \langle \partial_a \phi^i(\xi)\; \partial_b \phi^i(\xi')\rangle$$

for the normal components of tangent vectors to the world-sheet.

Secondly, we compute the scaling law of the distance d_E in embedding space between two points on the world-sheet when changing its projection d on the reference plane. The exact relation between the two lengths is

$$d_E^2 = d^2 + \sum_i \langle |\phi^i(\xi) - \phi^i(\xi')|^2\rangle \;. \qquad (6.10)$$

When $d_E^2 \propto d^2$ this implies that the Hausdorff dimension of the surfaces, defined as $d_E^2 = (d^2)^{2/D_H}$, is equal to 2, and the surface is smooth.

In order to establish the properties of our model, we analyze the saddle-point function $f\,(T,s,M,L)$ in (6.7). To this end, we must prescribe a regularization for the ultraviolet divergent integral. We use dimensional regularization, computing the integral in $(2-\epsilon)$ dimensions. For small ϵ, this leads to

$$f\,(T,s,M,L)=L+\frac{1}{16\pi^3}\left(R^2-I^2\right)\ln\frac{R^2+I^2}{\Lambda^2}-\frac{1}{8\pi^3}RI\left(\frac{\pi}{2}+\arctan\frac{I^2-R^2}{2RI}\right)\,, \tag{6.11}$$

where $\Lambda\equiv\mu\,\exp(2/\epsilon)$ and μ is a reference scale which must be introduced for dimensional reasons. The scale Λ plays the role of an ultraviolet cutoff, diverging for $\epsilon\to 0$.

The saddle-point function above is best studied by introducing the dimensionless couplings $t\equiv T/\Lambda^2$, $m\equiv M/\Lambda$ and $l\equiv L/\Lambda^2$. We will study in detail the case $s=0$, in which the saddle-point equations can be solved analytically since $R=I$. This choice is not too restrictive since, as we will show, $s=0$ is the infrared fixed point. We get

$$l\ =\frac{m^2c^2}{2}\ \left(1+\sqrt{1+\frac{4t}{m^2c^2}}\right)\ , \tag{6.12}$$

$$\rho\ =\left(1-\frac{mc}{2\sqrt{t+l}}\right)^{-1}\ , \tag{6.13}$$

where $c\equiv 1/32\pi^2$. This shows that the point ($t^*=0$, $s^*=0$, $m^*=0$) constitutes an infrared-stable fixed-point with vanishing physical string tension $\mathcal{T}$. This point is characterized by long-range correlations $g(d)=2\pi^2/a$, with a constant a, and by the scaling law

$$d_E^2=\frac{\pi^2}{a}\rho^*d^2\ ,\quad \rho^*\equiv\left(1-\frac{1}{2a}\right)^{-1}\ , \tag{6.14}$$

which shows that the Hausdorff dimension of world sheets is $D_H=2$. For $s=0$, the constant a can be computed analytically,

$$a^2=\lim_{\substack{t\to 0\\ m\to 0}}\frac{1+(2t/m^2c^2)+\sqrt{1+4t/m^2c^2}}{2}\ , \tag{6.15}$$

from which we recognize that $1\le\rho^*\le 2$.

At the infrared fixed point we can remove the cutoff. The renormalization of the model is easily obtained by noting that the effective action for transverse fluctuations to quadratic order decouples from other modes and is identical with the second term in (6.3) with $\mathcal{D}^2=\partial^2/\rho$ and ρ taking its saddle-point value. From here we identify the physical tension, stiffness and mass as

$$\mathcal{T}=\Lambda^2(t+l)\ ,\quad \mathcal{S}=\frac{s}{\rho}\ ,\quad \mathcal{M}=\Lambda m\rho\ . \tag{6.16}$$

For $s = 0$, we can compute analytically the corresponding γ functions,

$$\gamma_t \equiv -\Lambda \frac{d}{d\Lambda} \ln t = 2 + O\left(\frac{t}{m^2}\right) , \tag{6.17}$$

$$\gamma_m \equiv -\Lambda \frac{d}{d\Lambda} \ln m = 1 + O\left(\frac{t^2}{m^4}\right) . \tag{6.18}$$

The vicinity of the infrared fixed-point defines a new theory of smooth strings for which the range of the orientational correlations in embedding space is always of the same order or bigger than the length scale $1/\sqrt{\mathcal{T}}$ associated with the tension. The naively irrelevant term $\mathcal{D}^4/M^2$ in (6.1) becomes relevant in the large-D approximation since it generates a string tension proportional to M^2 which takes over the control of the fluctuations after the orientational correlations die off. Note moreover, that it is exactly this new quartic term which guarantees that the spectrum $p^2 V(T, s, M, L, p)$ has no other pole than $p = 0$, contrary to the rigid string, which necessarily has a ghost pole at $p^2 = -T/s$.

By studying the finite-size scaling [87] of the Euclidean model (6.1) on a cylinder of (spatial) circumference R it is possible to determine the universality class of confining strings. In the limit of large R, the effective action on the cylinder takes the form [41]

$$\lim_{\beta\to\infty} \frac{S^{\rm eff}}{\beta} = \mathcal{T}R - \frac{\pi c(D-2)}{6R} + \dots , \tag{6.19}$$

for $(D-2)$ transverse degrees of freedom, the universality class being encoded in the pure number c. This suggests that the effective theory describing the infrared behavior is a conformal field theory (CFT) with central charge c. In this case the number c also fixes the Lüscher term [88] in the quark-antiquark potential:

$$V(R) = \mathcal{T}R - \frac{\pi c(D-2)}{24R} + \dots . \tag{6.20}$$

In [41] it has been shown that confining strings are characterized by $c = 1$. Although they share the same value of c, confining strings are clearly different $c = 1$ theories from Nambu-Goto or rigid strings. Indeed, the former are smooth strings on any scale, while the latter crumple and fill the ambient space, at least in the infrared region. Our result $c = 1$ is in agreement with recent precision numerical determinations of Lüscher and Weisz [30] of this constant.

Having established that the model (6.1) describes smooth strings with $c = 1$, the question arises as to how much these results depend on the truncation of the original non-local action after the $\mathcal{D}^4$ term. In [41] it was proved that the answer to this question is no; the value of c and the smooth geometric properties are independent of an infinite set of truncations, provided that a solution for the polynomial "gap" equation exists. These properties are presumably common to a large class of non-local world-sheet interactions.

7. High temperature behavior of confining strings

The high-temperature behavior of large-N QCD has been studied by Polchinski in [46], where he shows that the deconfining transition in QCD is due to the condensation of Wilson lines, and the partition function of QCD flux tubes can be continued above the deconfining transition; this high-temperature continuation can be evaluated perturbatively. So, any string theory that is equivalent to QCD *must* reproduce this behavior. However, the Nambu–Goto action has the wrong temperature dependence, while the rigid string parametrically has the correct high-temperature behavior but with the wrong sign and an imaginary part signaling a world-sheet instability [89].

The high temperature behavior of the confining string model proposed in [40] has been studied in [45], where it has been shown that this model has a high-temperature behavior that agrees in temperature dependence, *sign and reality properties* with the large-N QCD result [46]. The starting point will be again (6.2). In the Gauss map, the value of the periodic coordinate ξ_0 is $-\beta/2 \leq \xi_0 \leq \beta/2$ with $\beta = 1/T$ and T the temperature. Note that, at high temperatures ($\beta \ll 1$), the scale M^2 can be temperature-dependent. This is not unusual in closed string theory as has been shown by Atick and Witten [90]. The value of ξ_1 is $-R/2 \leq \xi_1 \leq R/2$; $\phi^i(\xi)$ describe the $D-2$ transverse fluctuations. We look for a saddle-point solution with a diagonal metric $g_{ab} = \mathrm{diag}\ (\rho_0, \rho_1)$, and a Lagrange multiplier of the form $\lambda^{ab} = \mathrm{diag}\ (\lambda_0/\rho_0, \lambda_1/\rho_1)$.

After integration over transverse fluctuations we obtain, in the limit $R \to \infty$, four gap equations:

$$\frac{1-\rho_0}{\rho_0} = 0\ , \tag{7.1}$$

$$\frac{1}{\rho_1} = 1 - \frac{D-2}{2}\frac{1}{4\beta}\frac{\sqrt{2M}}{(\lambda_1 + t)^{3/4}}\ , \tag{7.2}$$

$$\frac{1}{2}(t-\lambda_1) + \frac{1}{2\rho_1}(\lambda_1 + t) - t - \lambda_0 + \frac{D-2}{2}\frac{\pi}{2\beta^2} = 0\ , \tag{7.3}$$

$$(t-\lambda_1)-\frac{1}{\rho_1}(\lambda_1+t)+\frac{D-2}{2}\frac{1}{\beta}\left[\sqrt{2M}\,(\lambda_1+t)^{1/4}-\frac{\pi}{\beta}\right]=0\ , \quad (7.4)$$

and a simplified form of the effective action,

$$S^{\rm eff}=A_{\rm ext}\mathcal{T}\sqrt{\frac{1}{\rho_1}}\ , \quad (7.5)$$

with $\mathcal{D}=2(\lambda_1+t)$ representing the physical string tension. By inserting (7.2) into (7.4), we obtain an equation for (λ_1+t) alone,

$$(\lambda_1+t)-\frac{D-2}{2}\frac{5}{8\beta}\sqrt{2M}\,(\lambda_1+t)^{1/4}+\frac{D-2}{2}\frac{\pi}{2\beta^2}-t=0\ . \quad (7.6)$$

Without loss of generality we set

$$(\lambda_1+t)^{1/4}=\frac{\sqrt{2M}}{\gamma}\ , \quad (7.7)$$

where γ is a dimensionless parameter. It is possible to show that, at high temperatures, when

$$t\beta^2\ll\frac{D-2}{2}\ , \quad (7.8)$$

we can completely neglect t in (7.6). Indeed, as we now show, λ_1 is proportional to $(D-2)^2/\beta^2$. We can thus rewrite (7.6) as

$$\lambda_1-\frac{D-2}{2}\,\frac{5}{8\beta}\,\gamma\,\lambda_1^{1/2}+\frac{D-2}{2}\,\frac{\pi}{2\beta^2}=0\ . \quad (7.9)$$

We now restrict ourselves to the regime

$$G(\gamma,D)=\frac{25}{64}\gamma^2\left(\frac{D-2}{2}\right)^2-2\pi\,\frac{D-2}{2}>0\ , \quad (7.10)$$

for which (7.9) admits two real solutions:

$$(\lambda_1^1)^{1/2}\ =\frac{5}{16\beta}\,\gamma\,\frac{D-2}{2}+\ \frac{1}{2\beta}\sqrt{G(\gamma,D)}\ , \quad (7.11)$$

$$(\lambda_1^2)^{1/2}\ =\frac{5}{16\beta}\,\gamma\,\frac{D-2}{2}-\ \frac{1}{2\beta}\sqrt{G(\gamma,D)}\ . \quad (7.12)$$

In both cases, λ_1 is proportional to $(D-2)/\beta^2$, which justifies neglecting t in (7.6) and implies that the scale M^2 must be chosen proportional to $1/\beta^2$. Moreover, since the physical string tension is real we are guaranteed that $M^2>0$, as required by the stability of our model. Any complex solutions for $\mathcal{D}$ would have been incompatible with the stability of the truncation.

Let us start by analyzing the first solution (7.11). Inserting (7.11) into (7.3), we obtain

$$\frac{1}{\rho_1} = 1 - \frac{4}{5 + \sqrt{25 - \frac{128\pi}{\gamma^2 \frac{D-2}{2}}}} \,. \tag{7.13}$$

Owing to the condition (7.10), $1/\rho_1$ is positive and, since λ_1^2 is real, the squared free energy is also positive,

$$F^2(\beta) \equiv \frac{S_{\rm eff}^2}{R^2} = \frac{1}{\beta^2}\left(\frac{5}{16}\,\gamma\,\frac{D-2}{2} - \frac{1}{2}\sqrt{G(\gamma,D)}\right)^4 \left(1 - \frac{4}{5 + \sqrt{25 - \frac{128\pi}{\gamma^2 \frac{D-2}{2}}}}\right) \,. \tag{7.14}$$

In this case the high-temperature behavior is the same as in QCD, but the sign is wrong, exactly as for the rigid string. There is, however, a crucial difference; (7.14) is real, while the squared free energy for the rigid string is imaginary, signaling an instability in the model.

If we now look at the behavior of ρ_1 at low temperatures, below the deconfining transition [41], we see that $1/\rho_1$ is positive. The deconfining transition is indeed determined by the vanishing of $1/\rho_1$ at $\beta = \beta_{\rm dec}$. In the case of (7.11) this means that $1/\rho_1$ is positive below the Hagedorn transition, touches zero at $\beta_{\rm dec}$ and remains positive above it. Exactly the same will happen also for F^2, which is positive below $\beta_{\rm dec}$, touches zero at $\beta_{\rm dec}$ and remains positive above it. This solution thus describes an unphysical "mirror" of the low-temperature behavior of the confining string, without a real deconfining Hagedorn transition. For this reason we discard it.

Let us now study the solution (7.12). Again, by inserting (7.12) into (7.3), we obtain

$$\frac{1}{\rho_1} = 1 - \frac{4}{5 - \sqrt{25 - \frac{128\pi}{\gamma^2 \frac{D-2}{2}}}} \,. \tag{7.15}$$

In this case, when

$$\gamma > 4\sqrt{\frac{\pi}{3}}\left(\frac{D-2}{2}\right)^{-1/2} \,, \tag{7.16}$$

$1/\rho_1$ becomes negative. The condition (7.16) is consistent with (7.10) and will be taken to fix the values of the range of parameter γ that enters (7.7). We will restrict to those that satisfy (7.16). Since $\rho_0 = 1$ and λ_1 is real and proportional to $1/\beta^2$, we obtain the squared free energy,

$$F^2(\beta) = -\frac{1}{\beta^2}\left(\frac{5}{16}\,\gamma\,\frac{D-2}{2} - \frac{1}{2}\sqrt{G(\gamma, D)}\right)^4 \left(\frac{4}{5-\sqrt{25-\frac{128\pi}{\gamma^2\frac{D-2}{2}}}} - 1\right) . \tag{7.17}$$

In the range defined by (7.16) this is *negative*. For this solution, thus, both $1/\rho_1$ and F^2 pass from positive values at low temperatures to negative values at high temperatures, exactly as one would expect for a string model undergoing the Hagedorn transition at an intermediate temperature. In fact, this is also what happens in the rigid string case, but there, above the Hagedorn transition, there is a second transition above which, at high temperature, λ_1 becomes large and essentially imaginary, giving a positive squared free energy. This second transition is absent in our model.

Let us now compare the result (7.17) with the corresponding one for large-N QCD [46],

$$F^2(\beta)_{\mathrm{QCD}} = -\frac{2g^2(\beta)N}{\pi^2\beta^2} , \tag{7.18}$$

where $g^2(\beta)$ is the QCD coupling constant. First of all let us simplify our result by choosing values of γ satisfying

$$\gamma \gg \sqrt{\frac{128\pi}{25}}\left(\frac{D-2}{2}\right)^{-1/2} .$$

In this case (7.17) reduces to

$$F^2(\beta) = -\frac{1}{\beta^2}\,\frac{8\pi^3}{125}\,\frac{D-2}{\gamma^2} . \tag{7.19}$$

This corresponds *exactly* to the QCD result (7.18) with the identifications

$$\begin{aligned} g^2 &\propto \frac{1}{\gamma^2} , \\ N &\propto D-2 . \end{aligned}$$

The weak β-dependence of the QCD coupling $g^2(\beta)$ can be accommodated in the parameter γ. Note that our result is valid at large values of γ, i.e. small values of g^2, as it should be for QCD at high temperatures [91]. Note also the interesting identification between the order of the gauge group and the number of transverse space-time dimensions. Moreover, since the sign of λ_1 does not change at high temperatures, the field x_μ is not unstable. The opposite happens in the rigid string case [89], where the change of sign of λ_1 gives rise to a world-sheet instability.

8. The influence of matter fields on the deconfinement phase transition in the 3D GG model at finite temperature

8.1. *Introduction*

The phase structure of the 3D GG model at finite temperature has for the first time been addressed in Ref. [51], where it was shown that, in the absence of W bosons, the weakly coupled monopole plasma undergoes the Berezinsky–Kosterlitz–Thouless (BKT) [82] phase transition into the molecular phase at the temperature $T_{\rm BKT} = g^2/2\pi$. Then, in Ref. [52], it was shown that the true phase transition, which takes into account W bosons, occurs at approximately half this temperature. Let us first discuss in some more detail the above-mentioned BKT phase transition, which takes place in the absence of W bosons (i.e. in the continuum limit of the 3D lattice compact QED extended by the Higgs field).

At finite temperature $T \equiv 1/\beta$, equations of motion of the fields χ and ψ, entering the partition function (2.2),

$$\mathcal{Z}_{\rm mon} \equiv \int \mathcal{D}\chi \mathcal{D}\psi \exp\left\{-\int d^3x \mathcal{L}[\chi, \psi | g_m, \zeta]\right\},$$

should be supplemented by the periodic boundary conditions in the temporal direction, with the period equal to β. Because of that, the lines of magnetic field emitted by a monopole cannot cross the boundary of the one-period region. Consequently, at distances larger than β in the direction perpendicular to the temporal one, magnetic field lines approaching the boundary should run almost parallel to it. Therefore, monopoles separated by such distances interact via the 2D Coulomb potential, rather than the 3D one. Since the average distance between monopoles in the plasma is of the order of $\zeta^{-1/3}$, we see that at $T \gtrsim \zeta^{1/3}$, the monopole ensemble becomes two-dimensional. Owing to the fact that ζ is exponentially small in the weak-coupling regime under discussion, the idea of dimensional reduction is perfectly applicable at the temperatures of the order of the above-mentioned critical temperature $T_{\rm BKT}$.

Note that, due to the T-dependence of the strength of the monopole-antimonopole interaction, which is a consequence of the dimensional reduction, the BKT phase transition in the 3D Georgi–Glashow model is inverse with respect to the standard one of the 2D XY model. Namely, monopoles exist in the plasma phase at the temperatures below $T_{\rm BKT}$ and in the molecular phase above this temperature. As has already been discussed, the analogy with the 2D XY-model established in Ref. [62] is that spin waves of that model correspond to free photons of the 3D GG model at zero temperature, while vortices correspond to magnetic monopoles. At finite temperature, disorder is rather produced by the thermally excited W bosons [52], whereas monopoles order the system, binding W bosons into pairs. However, the analogy is still true in the case of the continuum version of 3D compact QED (with the Higgs field) under discussion.

Let us therefore briefly discuss the BKT phase transition, occurring in the 2D XY-model at a certain critical temperature $T = T_{\rm BKT}^{\rm XY}$. At $T < T_{\rm BKT}^{\rm XY}$, the spectrum of the model is dominated by massless spin waves, and the periodicity of the angular variable is unimportant in this phase. The spin waves are unable to disorder the spin-spin correlation functions, and these decrease at large distances by some power law. On the contrary, at $T > T_{\rm BKT}^{\rm XY}$, the periodicity of the angular variable becomes important. This leads to the appearance of topological singularities (vortices) of the angular variable, which, contrary to spin waves, have nonvanishing winding numbers. Such vortices condense and disorder the spin-spin correlation functions, so that those start decreasing exponentially with the distance. Thus, the nature of the BKT phase transition is the condensation of vortices at $T > T_{\rm BKT}^{\rm XY}$. In another words, at $T > T_{\rm BKT}^{\rm XY}$, free vortices do exist and mix in the ground state (vortex condensate) of an indefinite global vorticity. Contrary to that, at $T < T_{\rm BKT}^{\rm XY}$, free vortices cannot exist, and they rather mutually couple into bound states of vortex-antivortex pairs. Such vortex-antivortex molecules are small-sized and short-living (virtual) objects. Their dipole-type fields are short-ranged and therefore cannot disorder significantly the spin-spin correlation functions. However, when the temperature starts rising, the size of these molecules grows, until at $T = T_{\rm BKT}^{\rm XY}$ it diverges, corresponding to the dissociation of the molecules into pairs. Therefore, one of the methods to determine the critical temperature of the BKT phase transition is to evaluate the mean squared separation in the molecule and to find the temperature at which it starts diverging.

Let us now return to the continuum limit of the finite-temperature 3D compact QED, extended by the Higgs field, and determine there the mean squared separation in the monopole-anti-monopole molecule. One can then see that, up to exponentially small corrections, the respective critical temperature is unaffected by the finiteness of the Higgs boson mass. Indeed, the mean squared separation reads [z]

$$\left\langle L^2\right\rangle = \frac{\int\limits_{|\mathbf{x}|>m_W^{-1}} d^2\mathbf{x}|\mathbf{x}|^{2-\frac{8\pi T}{g^2}}\exp\left[\frac{4\pi T}{g^2}K_0\left(m_H|\mathbf{x}|\right)\right]}{\int\limits_{|\mathbf{x}|>m_W^{-1}} d^2\mathbf{x}|\mathbf{x}|^{-\frac{8\pi T}{g^2}}\exp\left[\frac{4\pi T}{g^2}K_0\left(m_H|\mathbf{x}|\right)\right]}\,, \tag{8.1}$$

where K_0 denotes the modified Bessel function. Disregarding the exponential factors in the numerator and denominator of this equation, we obtain $\left\langle L^2\right\rangle \simeq \frac{4\pi T-g^2}{2m_W^2(2\pi T-g^2)}$, yielding the BKT critical temperature $T_{\rm BKT}=g^2/2\pi$. Besides that, we see that, as long as T does not tend to $T_{\rm BKT}$, and in the weak-coupling regime under study, the value of $\sqrt{\langle L^2\rangle}$ is exponentially smaller than the characteristic distance in the monopole plasma, $\zeta^{-1/3}$, i.e. molecules are very small-sized with respect to that distance.

The factor β at the action of the dimensionally-reduced theory, $S_{\rm d.-r.}=\beta\int d^2x\mathcal{L}[\chi,\psi|g_m,\zeta]$, can be removed (and this action can be cast to the original form of Eq. (2.2) with the substitution $d^3x\to d^2x$) by the obvious rescaling,

$$S_{\rm d.-r.}=\int d^2x\mathcal{L}\left[\chi^{\rm new},\psi^{\rm new}|\sqrt{K},\beta\zeta\right]. \tag{8.2}$$

Here, $K\equiv g_m^2T$, $\chi^{\rm new}=\sqrt{\beta}\chi$, $\psi^{\rm new}=\sqrt{\beta}\psi$, and in what follows we will denote for brevity $\chi^{\rm new}$ and $\psi^{\rm new}$ simply as χ and ψ, respectively. Averaging over the field ψ with the use of the cumulant expansion we find that the action is

$$S_{\rm d.-r.}\simeq\int d^2x\left[\frac12(\partial_\mu\chi)^2-2\xi\cos\left(\sqrt{K}\chi\right)\right]-$$
$$-2\xi^2\int d^2xd^2y\cos\left(\sqrt{K}\chi(\mathbf{x})\right)\mathcal{K}^{(2)}(\mathbf{x}-\mathbf{y})\cos\left(\sqrt{K}\chi(\mathbf{y})\right). \tag{8.3}$$

In this expression, similarly to Eq. (2.6), we have disregarded all the cumulants higher than the quadratic one, and the limits of applicability of this bilocal approximation will be discussed below. Further, in Eq. (8.3), $\mathcal{K}^{(2)}(\mathbf{x})\equiv {\rm e}^{KD_{m_H}^{(2)}(\mathbf{x})}-1$, where $D_{m_H}^{(2)}(\mathbf{x})\equiv K_0(m_H|\mathbf{x}|)/2\pi$ is the 2D Yukawa propagator, and $\xi\equiv\beta\zeta\exp\left[\frac{K}{2}D_{m_H}^{(2)}\left(m_W^{-1}\right)\right]$ denotes the monopole fugacity modified by the interaction of monopoles via the Higgs field. The exponential factor entering ξ reads

$$\xi\propto\exp\left[-\frac{4\pi}{g^2}\left(m_W\epsilon+T\ln\left(\frac{{\rm e}^\gamma}{2}c\right)\right)\right].$$

[z] In this section, 3-vectors are denoted as $\vec{a}$, whereas 2-vectors are denoted as $\mathbf{a}$.

Here, we have introduced the notation $c \equiv m_H/m_W$, $c<1$, and $\gamma \simeq 0.577$ is the Euler constant, so that $\frac{\mathrm{e}^\gamma}{2} \simeq 0.89 < 1$. We see that the modified fugacity remains exponentially small, provided that

$$T < -\frac{m_W \epsilon}{\ln\left(\frac{\mathrm{e}^\gamma}{2} c\right)}. \tag{8.4}$$

This constraint should be updated by another one, which would provide the convergence of the cumulant expansion. Analogously to the zero-temperature case, the divergence of the cumulant expansion would indicate that the Higgs vacuum loses its normal stochastic property and becomes a coherent one. In order to get the new constraint, notice that the parameter of the cumulant expansion reads $\xi I^{(2)}$, where $I^{(2)} \equiv \int d^2x \mathcal{K}^{(2)}(\mathbf{x})$. Evaluation of the integral $I^{(2)}$ yields [53]

$$I^{(2)} \simeq \frac{2\pi}{m_H^2}\left[\frac{1}{2}\left(c^2 - 1 + \left(\frac{2}{\mathrm{e}^\gamma}\right)^{8\pi T/g^2} \frac{1 - c^{2-\frac{8\pi T}{g^2}}}{1 - \frac{4\pi T}{g^2}}\right) + \mathrm{e}^{a/\mathrm{e}} - 1 + \frac{a}{\mathrm{e}}\right]. \tag{8.5}$$

(Note that at $T \to g^2/4\pi$, $\frac{1-c^{2-\frac{8\pi T}{g^2}}}{1-\frac{4\pi T}{g^2}} \to -2\ln c$, i.e. $I^{(2)}$ remains finite.) In the derivation of this expression, the parameter $a \equiv 4\pi\sqrt{2\pi}T/g^2$ was assumed to be of the order of unity. That is because the temperatures we are working at are of the order of T_{BKT}. Due to the exponential term in Eq. (8.5), the violation of the cumulant expansion may occur at high enough temperatures (which parallels the constraint (8.4)). The most essential, exponential, part of the parameter of the cumulant expansion thus reads

$$\xi I^{(2)} \propto \exp\left\{-\frac{4\pi}{g^2}\left[m_W\epsilon + T\left[\ln\left(\frac{\mathrm{e}^\gamma}{2}c\right) - \frac{\sqrt{2\pi}}{\mathrm{e}}\right]\right]\right\}.$$

Therefore, the cumulant expansion converges at the temperatures obeying the inequality

$$T < \frac{m_W\epsilon}{\frac{\sqrt{2\pi}}{\mathrm{e}} - \ln\left(\frac{\mathrm{e}^\gamma}{2}c\right)},$$

which strengthens the inequality (8.4). On the other hand, since we are working in the plasma phase, i.e. $T \leq T_{\mathrm{BKT}}$, it is enough to impose the upper bound

$$\kappa \leq \frac{2\pi\epsilon}{\frac{\sqrt{2\pi}}{\mathrm{e}} - \ln\left(\frac{\mathrm{e}^\gamma}{2}c\right)}$$

on the parameter of the weak-coupling approximation, κ. Note that, although this inequality is satisfied automatically at $\frac{\mathrm{e}^\gamma}{2}c \sim 1$ (or $c \sim 1$), since it then takes the form $\kappa \le \sqrt{2\pi}\mathrm{e}\epsilon$, this is not so automatic in the BPS limit, $c \ll 1$. Indeed, in this case, we have $\kappa \ln\left(\frac{2}{c\mathrm{e}^\gamma}\right) \le 2\pi\epsilon$. It is, however, quite feasible to obey this inequality since the logarithm is a weak function.

8.2. *Higgs-inspired corrections to the RG flow in the absence of W bosons*

Although we have seen in the previous subsection that the propagating Higgs field does not change the value of the critical temperature T_{BKT}, it is instructive to derive the corrections it produces to the RG flow. Such a derivation can also be extended to the SU(N) case (2.5). In that case [83], it is qualitatively clear that T_{BKT} should remain the same, since it only differs from that of the SU(2) case by the factor $\vec{q}_i^{\,2}$, which is equal to unity. However, some peculiarities of the SU(N) case at $N > 2$ become clear only upon a derivation of the RG equations, which will be given below.

Let us thus start with the SU(2) case (2.2), whose action after the dimensional reduction has the form (8.2). In what follows, we will use the usual RG strategy based on the integration over the high-frequency modes. Note that this procedure will be applied to *all* the fields, i.e. not only to χ, but also to ψ. Splitting the momenta into two ranges, $0 < p < \Lambda'$ and $\Lambda' < p < \Lambda$, one can define the high-frequency modes as $h = \chi_\Lambda - \chi_{\Lambda'}$, $\phi = \psi_\Lambda - \psi_{\Lambda'}$, where $\mathcal{O}_{\Lambda'}(\mathbf{x}) = \int\limits_{0<p<\Lambda'} \frac{d^2p}{(2\pi)^2}\mathrm{e}^{i\mathbf{px}}\mathcal{O}(\mathbf{p})$ and consequently $h(\mathbf{x}) = \int\limits_{\Lambda'<p<\Lambda} \frac{d^2p}{(2\pi)^2}\mathrm{e}^{i\mathbf{px}}\chi(\mathbf{p})$. The partition function,

$$\mathcal{Z}_\Lambda = \int\limits_{0<p<\Lambda} \mathcal{D}\chi(\mathbf{p})\mathcal{D}\psi(\mathbf{p}) \exp\left\{-S_{\mathrm{d.-r.}}\left[\chi_\Lambda, \psi_\Lambda\right]\right\},$$

can be rewritten

$$\mathcal{Z}_\Lambda = \int\limits_{0<p<\Lambda'} \mathcal{D}\chi(\mathbf{p})\mathcal{D}\psi(\mathbf{p}) \exp\left\{\frac{1}{2}\int d^2x\left[\chi_{\Lambda'}\partial^2\chi_{\Lambda'} + \psi_{\Lambda'}\left(\partial^2 - m_H^2\right)\psi_{\Lambda'}\right]\right\}\mathcal{Z}',$$

where

$$\mathcal{Z}' = \int\limits_{\Lambda'<p<\Lambda} \mathcal{D}\chi(\mathbf{p})\mathcal{D}\psi(\mathbf{p}) \exp\left\{\int d^2x\left[\frac{1}{2}h\partial^2 h + \frac{1}{2}\phi\left(\partial^2 - m_H^2\right)\phi + \right.\right.$$
$$\left.\left. + 2\xi\mathrm{e}^{\sqrt{K}(\psi_{\Lambda'}+\phi)}\cos\left(\sqrt{K}\left(\chi_{\Lambda'}+h\right)\right)\right]\right\},$$

and $\xi \equiv \beta\zeta$. Owing to the exponential smallness of the fugacity, $\mathcal{Z}'$ can further be expanded as

$$\mathcal{Z}' \simeq 1 + 2\xi \int d^2x \left\langle e^{\sqrt{K}(\psi_{\Lambda'}+\phi)}\right\rangle_\phi \left\langle \cos\left(\sqrt{K}\left(\chi_{\Lambda'}+h\right)\right)\right\rangle_h + 2\xi^2 \int d^2x d^2y \times$$
$$\times \left[\left\langle e^{\sqrt{K}(\psi_{\Lambda'}(\mathbf{x})+\phi(\mathbf{x}))} e^{\sqrt{K}(\psi_{\Lambda'}(\mathbf{y})+\phi(\mathbf{y}))}\right\rangle_\phi \left\langle \cos\left(\sqrt{K}\left(\chi_{\Lambda'}(\mathbf{x})+h(\mathbf{x})\right)\right) \times\right.\right.$$
$$\left.\times \cos\left(\sqrt{K}\left(\chi_{\Lambda'}(\mathbf{y})+h(\mathbf{y})\right)\right)\right\rangle_h - \left\langle e^{\sqrt{K}(\psi_{\Lambda'}(\mathbf{x})+\phi(\mathbf{x}))}\right\rangle_\phi \left\langle e^{\sqrt{K}(\psi_{\Lambda'}(\mathbf{y})+\phi(\mathbf{y}))}\right\rangle_\phi \times$$
$$\left.\times \left\langle \cos\left(\sqrt{K}\left(\chi_{\Lambda'}(\mathbf{x})+h(\mathbf{x})\right)\right)\right\rangle_h \left\langle \cos\left(\sqrt{K}\left(\chi_{\Lambda'}(\mathbf{y})+h(\mathbf{y})\right)\right)\right\rangle_h\right],$$

where

$$\langle\mathcal{O}\rangle_h \equiv \frac{\int\limits_{\Lambda'<p<\Lambda} \mathcal{D}\chi(\mathbf{p}) \exp\left(\frac12 \int d^2x h\partial^2 h\right)\mathcal{O}}{\int\limits_{\Lambda'<p<\Lambda} \mathcal{D}\chi(\mathbf{p}) \exp\left(\frac12 \int d^2x h\partial^2 h\right)},$$

$$\langle\mathcal{O}\rangle_\phi \equiv \frac{\int\limits_{\Lambda'<p<\Lambda} \mathcal{D}\psi(\mathbf{p}) \exp\left[\frac12 \int d^2x \phi\left(\partial^2 - m_H^2\right)\phi\right]\mathcal{O}}{\int\limits_{\Lambda'<p<\Lambda} \mathcal{D}\psi(\mathbf{p}) \exp\left[\frac12 \int d^2x \phi\left(\partial^2 - m_H^2\right)\phi\right]}.$$

Carrying out the averages we obtain

$$\mathcal{Z}' \simeq 1 + 2\xi A(0)B(0) \int d^2x e^{\sqrt{K}\psi_{\Lambda'}} \cos\left(\sqrt{K}\chi_{\Lambda'}\right) + (\xi A(0)B(0))^2 \int d^2x d^2y \times$$
$$\times e^{\sqrt{K}(\psi_{\Lambda'}(\mathbf{x})+\psi_{\Lambda'}(\mathbf{y}))} \sum_{k=\pm1}\left[A^{2k}(\mathbf{x}-\mathbf{y})B^2(\mathbf{x}-\mathbf{y}) - 1\right] \times$$
$$\times \cos\left[\sqrt{K}\left(\chi_{\Lambda'}(\mathbf{x}) + k\chi_{\Lambda'}(\mathbf{y})\right)\right], \tag{8.6}$$

where $A(\mathbf{x}) \equiv e^{-KG_h(\mathbf{x})/2}$, $B(\mathbf{x}) \equiv e^{KG_\phi(\mathbf{x})/2}$,

$$G_h(\mathbf{x}) = \int\limits_{\Lambda'<p<\Lambda} \frac{d^2p}{(2\pi)^2}\frac{e^{i\mathbf{px}}}{p^2}, \qquad G_\phi(\mathbf{x}) = \int\limits_{\Lambda'<p<\Lambda} \frac{d^2p}{(2\pi)^2}\frac{e^{i\mathbf{px}}}{p^2+m_H^2}.$$

Since in what follows we will take $\Lambda' = \Lambda - d\Lambda$, the factors $\left[A^{2k}(\mathbf{x}-\mathbf{y})B^2(\mathbf{x}-\mathbf{y}) - 1\right]$, $k = \pm1$, are small. Owing to this fact, it is convenient to introduce the coordinates $\mathbf{r} \equiv \mathbf{x} - \mathbf{y}$ and $\mathbf{R} \equiv \frac12(\mathbf{x}+\mathbf{y})$, and Taylor expand Eq. (8.6) in powers of $\mathbf{r}$. Clearly, this expansion should be performed up to the induced-interaction term

$\sim \xi^2 \int d^2R\, \mathrm{e}^{2\sqrt{K}\psi_{\Lambda'}(\mathbf{R})} \cos\left(2\sqrt{K}\chi_{\Lambda'}(\mathbf{R})\right)$, that itself should already be disregarded. Then we obtain

$$\mathcal{Z}_\Lambda = \int\limits_{0<p<\Lambda'} \mathcal{D}\chi(\mathbf{p})\mathcal{D}\psi(\mathbf{p}) \exp\left\{\int d^2x \left\{\frac{1}{2}\psi_{\Lambda'}\left(\partial^2 - m_H^2\right)\psi_{\Lambda'} + \right.\right.$$
$$+a_2(\xi A(0)B(0))^2\mathrm{e}^{2\sqrt{K}\psi_{\Lambda'}} - \frac{1}{2}\left[1 + a_1(\xi A(0)B(0))^2\frac{K}{2}\mathrm{e}^{2\sqrt{K}\psi_{\Lambda'}}\right](\partial_\mu\chi_{\Lambda'})^2 +$$
$$\left.\left. +2\xi A(0)B(0)\mathrm{e}^{\sqrt{K}\psi_{\Lambda'}}\cos\left(\sqrt{K}\chi_{\Lambda'}\right)\right\}\right\}, \tag{8.7}$$

where, for the sake of uniformity, we have replaced d^2R by d^2x and we have introduced the notation

$$a_1 \equiv \int d^2r r^2\left[A^{-2}(\mathbf{r})B^2(\mathbf{r}) - 1\right],\ a_2 \equiv \int d^2r\left[A^{-2}(\mathbf{r})B^2(\mathbf{r}) - 1\right]. \tag{8.8}$$

Taking into account that $\Lambda' = \Lambda - d\Lambda$ it is straightforward to get

$$a_1 = \alpha_1 K\frac{d\Lambda}{\Lambda^5}\left(1 + \frac{\Lambda^2}{m_H^2}\right), \qquad a_2 = \alpha_2 K\frac{d\Lambda}{\Lambda^3}\left(1 + \frac{\Lambda^2}{m_H^2}\right).$$

Here, $\alpha_{1,2}$ stand for some momentum-space-slicing dependent positive constants, whose concrete values will turn out to be unimportant for the final expressions describing the RG flow.

Next, since $a_{1,2}$ are infinitesimal (being proportional to $d\Lambda$), the terms containing these constants on the r.h.s. of Eq. (8.7) can be treated in the leading-order approximation of the cumulant expansion that we will apply for the average over ψ. In fact, we have

$$\int\limits_{0<p<\Lambda'} \mathcal{D}\psi(\mathbf{p}) \exp\left[\frac{1}{2}\int d^2x\psi_{\Lambda'}\left(\partial^2 - m_H^2\right)\psi_{\Lambda'}\right] \times$$
$$\times \exp\left(b\int d^2x\mathrm{e}^{2\sqrt{K}\psi_{\Lambda'}}f(\mathbf{x})\right) \simeq \exp\left[b\mathrm{e}^{2KG(0)}\int d^2x f(\mathbf{x}) + \right.$$
$$\left. + \frac{b^2}{2}\mathrm{e}^{4KG(0)}\int d^2x d^2y\left(\mathrm{e}^{4KG(\mathbf{x}-\mathbf{y})} - 1\right)f(\mathbf{x})f(\mathbf{y})\right],$$

where f is equal either to unity or to $(\partial_\mu\chi_{\Lambda'})^2$, $b \sim a_{1,2}(\xi A(0)B(0))^2$, and

$$G(\mathbf{x}) \equiv \int\limits_{0<p<\Lambda'} \frac{d^2p}{(2\pi)^2}\frac{\mathrm{e}^{i\mathbf{p}\mathbf{x}}}{p^2 + m_H^2},\ G(0) = \frac{1}{4\pi}\ln\left(1 + \frac{\Lambda'^2}{{m_H}^2}\right).$$

In order to estimate the parameter of the cumulant expansion, $\kappa \equiv b\mathrm{e}^{2KG(0)}\int d^2x\left(\mathrm{e}^{4KG(\mathbf{x})} - 1\right)$, note that we are working in the phase where monopoles form the plasma, i.e. below $T_{\rm BKT}$. Because of this fact, $4K|G(\mathbf{x})| \le 32\pi(\Lambda'/m_H)^2$, which, due to the factor $(\Lambda'/m_H)^2$, is generally much smaller than unity. Therefore we get

$$\kappa \simeq b\left(1+\frac{\Lambda'^2}{m_H{}^2}\right)^{\frac{K}{2\pi}} \cdot 4K\int d^2xG(\mathbf{x}) \simeq \frac{2bK}{\pi m_H^2}\left[1+\frac{K}{2\pi}\left(\frac{\Lambda'}{m_H}\right)^2\right].$$

Choosing for concreteness $b = a_2(\xi A(0)B(0))^2$ and taking into account that

$$A(0) \simeq 1-\frac{K}{2}G_h(0) = 1-\frac{K}{4\pi}\frac{d\Lambda}{\Lambda}\,, \tag{8.9}$$

$$B(0) \simeq 1+\frac{K}{2}G_\phi(0) \simeq 1+\frac{K}{4\pi}\left(\frac{\Lambda}{m_H}\right)^2\frac{d\Lambda}{\Lambda}\,, \tag{8.10}$$

we obtain $\kappa \simeq 2a_2K\xi^2/(\pi m_H^2)$ to leading order. This quantity possesses double smallness – firstly, because a_2 is infinitesimal and, secondly, due to the exponential smallness of ξ.

Such an extremely rapid convergence of the cumulant expansion enables one to replace $\mathrm{e}^{2\sqrt{K}\psi_{\Lambda'}}$ in the terms proportional to $a_{1,2}$ on the r.h.s. of Eq. (8.7) by the average value of this exponent equal to $\left(1+\frac{\Lambda'^2}{m_H{}^2}\right)^{\frac{K}{2\pi}}$. Comparing this expression with the initial one, we arrive at the following renormalizations of fields and parameters of the Lagrangian:

$$\chi_{\Lambda'}^{\rm new}=C\chi_{\Lambda'},\quad \psi_{\Lambda'}^{\rm new}=C\psi_{\Lambda'},\quad K^{\rm new}=\frac{K}{C^2},\quad \mu^{\rm new}=\frac{\mu}{C^2},\quad \xi^{\rm new}=A(0)B(0)\xi, \tag{8.11}$$

where $\mu \equiv m_H^2$,

$$C \equiv \left[1+\frac{Ka_1}{2}(\xi A(0)B(0))^2\left(1+\frac{\Lambda'^2}{m_H{}^2}\right)^{\frac{K}{2\pi}}\right]^{1/2} \simeq$$

$$\simeq \left[1+\frac{Ka_1}{2}(\xi A(0)B(0))^2\left(1+\frac{\Lambda^2}{m_H^2}\frac{K}{2\pi}\right)\right]^{1/2}. \tag{8.12}$$

Besides that, we obtain the shift of the free-energy density $F \equiv -\frac{\ln \mathcal{Z}'}{V}$,

$$F = F^{\rm new} - a_2(\xi A(0)B(0))^2\left(1+\frac{\Lambda'^2}{m_H{}^2}\right)^{\frac{K}{2\pi}} \simeq$$

$$\simeq F^{\rm new} - a_2(\xi A(0)B(0))^2\left(1+\frac{K}{2\pi}\frac{\Lambda'^2}{m_H{}^2}\right), \tag{8.13}$$

where V is the 2D-volume (i.e. area) of the system.

By making use of the relations (8.9), (8.10), it is then straightforward to derive from Eqs. (8.11)-(8.13) the RG equations in differential form. These are

$$d\xi = -\frac{K\xi}{4\pi}\left(1-\frac{\Lambda^2}{\mu}\right)\frac{d\Lambda}{\Lambda}\,, \quad dK = -\frac{\alpha_1}{2}K^3\xi^2\left[1+\left(\frac{K}{2\pi}+1\right)\frac{\Lambda^2}{\mu}\right]\frac{d\Lambda}{\Lambda^5}\,,$$

$$d\mu = -\frac{\alpha_1}{2}(K\xi)^2\mu\left[1+\left(\frac{K}{2\pi}+1\right)\frac{\Lambda^2}{\mu}\right]\frac{d\Lambda}{\Lambda^5}\,,$$

$$dF = \alpha_2 K\xi^2\left[1+\left(\frac{K}{2\pi}+1\right)\frac{\Lambda^2}{\mu}\right]\frac{d\Lambda}{\Lambda^3}\,.$$

Now let us change from the momentum scale to the real-space one; $\Lambda \to a \equiv 1/\Lambda$, $d\Lambda \to -d\Lambda$. This modifies the above equations to

$$d\xi = -\frac{K\xi}{4\pi}\frac{da}{a}\left(1-\frac{1}{\mu a^2}\right), \tag{8.14}$$

$$dK = -\frac{\alpha_1}{2}K^3\xi^2a^3da\left[1+\left(\frac{K}{2\pi}+1\right)\frac{1}{\mu a^2}\right], \tag{8.15}$$

$$d\mu = -\frac{\alpha_1}{2}(K\xi)^2\mu a^3da\left[1+\left(\frac{K}{2\pi}+1\right)\frac{1}{\mu a^2}\right], \tag{8.16}$$

$$dF = \alpha_2 K\xi^2 a da\left[1+\left(\frac{K}{2\pi}+1\right)\frac{1}{\mu a^2}\right]. \tag{8.17}$$

Our main aim below is to derive from Eqs. (8.14)-(8.16) the leading-order corrections in $(\mu a^2)^{-1}$ to the BKT RG flow in the vicinity of the critical point, $K_c^{(0)} = 8\pi$ (that clearly corresponds to $T_{\rm BKT}$), $y_c^{(0)} = 0$, where $y \equiv \xi a^2$, and the superscript "$^{(0)}$" denotes the zeroth order in the $(\mu a^2)^{-1}$-expansion. These values of $K_c^{(0)}$ and $y_c^{(0)}$ will be derived below. Additionally it will be demonstrated that, in the critical region, μ is evolving very slowly. Owing to this fact, the initial assumption on the largeness of μ (namely, that it is of the order of m_W^2), will be preserved by the RG flow, at least in that region. This enables one to consider μ as almost a constant and seek corrections to the RG flow of $K^{(0)}$ in powers of $(\mu a^2)^{-1}$. The zeroth-order equation stemming from Eq. (8.15) then reads

$$dK^{(0)} = -\frac{\alpha_1}{2}K^{(0)\,3}\xi^2a^3da\,, \tag{8.18}$$

and the zeroth-order in $(\mu a^2)^{-1}$ equation for y has the form

$$dy^{(0)\,2} = 2\frac{da}{a}y^{(0)\,2}x, \tag{8.19}$$

where $x \equiv 2 - \frac{K^{(0)}}{4\pi}$. Equations (8.18) and (8.19) yield the above-mentioned leading critical value of K, $K_c^{(0)}$. Next, with this value of $K_c^{(0)}$, Eq. (8.18) can be rewritten in the vicinity of the critical point as

$$dx = (8\pi)^2 \alpha_1 \xi^2 a^3 da. \tag{8.20}$$

Introducing the new variable $z = (8\pi)^2 \alpha_1 y^2$ and performing the rescaling $a^{\rm new} = a\sqrt{8\pi\alpha_1/\alpha_2}$ we get from Eqs. (8.17), (8.19), and (8.20) the system of equations,

$$dz^{(0)} = 2\frac{da}{a} x z^{(0)},\ dx = z^{(0)}\frac{da}{a},\ dF^{(0)} = z^{(0)}\frac{da}{a^3}. \tag{8.21}$$

These equations yield the standard RG flow in the vicinity of the critical point, $x = z^{(0)} = 0$, which has the form [82, 84] $z^{(0)} - x^2 = \tau$, where $\tau \propto (T_{\rm BKT} - T)/T_{\rm BKT}$ is some constant. In particular, $x \simeq \sqrt{z}$ at $T \to T_{\rm BKT} - 0$. Owing to the first of Eqs. (8.21), this relation yields $\left(z_{\rm in}^{(0)}\right)^{-1/2} - \left(z^{(0)}\right)^{-1/2} = \ln(a/a_{\rm in})$, where the subscript "${}_{\rm in}$" means the initial value. Taking into account that $z_{\rm in}^{(0)}$ is exponentially small, while $z^{(0)} \sim 1$ (the value at which the growth of $z^{(0)}$ stops), when $x_{\rm in} \le \sqrt{\tau}$ we obtain

$$\ln(a/a_{\rm in}) \sim \left(z_{\rm in}^{(0)}\right)^{-1/2} \sim \tau^{-1/2}.$$

According to this relation, at $T \to T_{\rm BKT} - 0$, the correlation length diverges with an essential singularity as $a(\tau) \sim \exp(\text{const}/\sqrt{\tau})$. (At $T < T_{\rm BKT}$, $a \equiv d$, while at $T > T_{\rm BKT}$, the correlation length becomes infinite due to the short-rangeness of molecular fields.) As far as the leading part of the free-energy density is concerned, it scales as $F^{(0)} \sim a^{-2}$ and therefore remains continuous in the critical region. Moreover, the correction to this behavior stemming from the finiteness of the Higgs boson mass (the last term on the r.h.s. of Eq. (8.17)) is clearly of the same functional form, $\sim \exp(-\text{const}'/\sqrt{\tau})$, i.e. it is also continuous.

We are now a the position to address the leading-order (in $(\mu a^2)^{-1}$) corrections to the above-discussed BKT RG flow of $K^{(0)}$ and $z^{(0)}$. To this end, let us represent K and z as $K = K^{(0)} + K^{(1)}/(\mu a^2)$, $z = z^{(0)} + z^{(1)}/(\mu a^2)$ that, by virtue of Eqs. (8.14) and (8.15), leads to the novel equations,

$$dK^{(1)} - 2K^{(1)}\frac{da}{a} = -4\pi\frac{da}{a}\left(z^{(0)} + \frac{z^{(1)}}{\mu a^2}\right)\left(1 + \frac{K^{(0)}}{2\pi} + \frac{3K^{(1)}}{K^{(0)}}\right), \tag{8.22}$$

$$dz^{(1)} - 2z^{(1)}\frac{da}{a} = -2\frac{da}{a}\left[z^{(1)}\left(\frac{K^{(0)}}{4\pi} - 2\right) + \frac{z^{(0)}}{4\pi}\left(K^{(1)} - K^{(0)}\right)\right]. \tag{8.23}$$

In the vicinity of the critical point, we can insert into Eq. (8.23) the above-obtained critical values of $K^{(0)}$ and $z^{(0)}$ to get

$$dz^{(1)} = 2z^{(1)}\frac{da}{a}\ . \tag{8.24}$$

Therefore, $z^{(1)} = C_1 a^2$, where C_1 is the integration constant of dimensionality (mass)^2, $C_1 \ll \mu$. Inserting this solution into Eq. (8.22), considered in the vicinity of the critical point, one obtains

$$dK^{(1)} - 2K^{(1)}\frac{da}{a} = -\frac{4\pi C_1}{\mu}\frac{da}{a}\left(\frac{3K^{(1)}}{8\pi} + 5\right). \tag{8.25}$$

Its integration is straightforward and yields

$$K^{(1)} = C_2\left(\mu a^2\right)^{1-\frac{3C_1}{4\mu}} + \frac{40\pi C_1}{4\mu - 3C_1}\ , \tag{8.26}$$

where the dimensionless integration constant C_2 should be much smaller than $(\mu a^2)^{\frac{3C_1}{4\mu}}$. (Note that the last term in Eq. (8.26) is positive.) Therefore, the total correction, $K^{(1)}/(\mu a^2)$, scales approximately with a in the critical region as $\frac{40\pi C_1}{(4\mu - 3C_1)\mu a^2}$. We see that, at the critical point, this expression vanishes due to the divergence of the correlation length. Note also that for $z^{(1)} \ll \mu a^2$ Eq. (8.26) can obviously be rewritten as

$$\frac{K^{(1)}}{\mu a^2} = C_2\left(\mu a^2\right)^{-\frac{3z^{(1)}}{4\mu a^2}} + \frac{40\pi z^{(1)}}{\left(4\mu a^2 - 3z^{(1)}\right)\mu a^2}\ .$$

With the above-discussed critical behavior of the correlation length, $a(\tau)$, this relation determines the correction to the BKT RG flow, $z^{(0)} - \left(2 - \frac{K^{(0)}}{4\pi}\right)^2 = \tau$.

Finally, in order to justify the approximation that μ was considered constant, we should check that, under the RG flow, μ indeed evolves slowly. To this end, let us pass in Eq. (8.16), considered in the critical region, from the variable ξ to the variable z and again perform the rescaling $a \to a^{\rm new}$. This yields $\frac{d\mu}{\mu} = -\frac{z}{2}\frac{da}{a}$ or $d\mu = -\frac{C_1}{2}\frac{da}{a}$. Since $C_1 \ll \mu$, we conclude that $\frac{|d\mu|}{\mu} \ll \frac{da}{a}$. This inequality means that, in the vicinity of the BKT critical point, μ is really evolving slowly. This fact justifies our treatment of μ as a large (with respect to Λ^2) constant quantity, approximately equal to its initial value, of the order of m_W^2.

Let us now proceed with a similar RG analysis of the SU(N) version of the theory (2.2) given by Eq. (2.5). Applying to the respective dimensionally-reduced theory the above-described RG procedure, we arrive at the analogue of Eq. (8.7),

$$\mathcal{Z}_\Lambda = \int\limits_{0<p<\Lambda'} \mathcal{D}\vec{\chi}(\mathbf{p})\mathcal{D}\psi(\mathbf{p}) \exp\left\{\int d^2x \left\{\frac{1}{2}\,\psi_{\Lambda'}\left(\partial^2 - m_H^2\right)\psi_{\Lambda'} + \right.\right.$$
$$+(\xi A(0)B(0))^2 \mathrm{e}^{2\sqrt{K}\psi_{\Lambda'}} \sum_{ij} a_2^{ij} \cos\left[\sqrt{K}\left(\vec{q}_i - \vec{q}_j\right)\vec{\chi}_{\Lambda'}\right] + 2\xi A(0)B(0)\times$$
$$\times \mathrm{e}^{\sqrt{K}\psi_{\Lambda'}} \sum_i \cos\left(\sqrt{K}\vec{q}_i\vec{\chi}_{\Lambda'}\right) - \frac{1}{2}\left[\delta^{ab} + (\xi A(0)B(0))^2 \frac{K}{8}\mathrm{e}^{2\sqrt{K}\psi_{\Lambda'}}\times\right.$$
$$\left.\left.\left.\times\sum_{ij} a_1^{ij}\left(\vec{q}_i + \vec{q}_j\right)^\alpha \left(\vec{q}_i + \vec{q}_j\right)^\beta \cos\left[\sqrt{K}\left(\vec{q}_i - \vec{q}_j\right)\vec{\chi}_{\Lambda'}\right]\right]\left(\partial_\mu \chi_{\Lambda'}^\alpha\right)\left(\partial_\mu \chi_{\Lambda'}^\beta\right)\right\}\right\}. \tag{8.27}$$

Here, $\alpha, \beta = 1, \ldots, (N-1)$, and we have introduced notation similar to (8.8),

$$a_1^{ij} \equiv \int d^2r r^2 \left[B^2(\mathbf{r})\mathrm{e}^{K\vec{q}_i\vec{q}_j G_h(\mathbf{r})} - 1\right],\ a_2^{ij} \equiv \int d^2r \left[B^2(\mathbf{r})\mathrm{e}^{K\vec{q}_i\vec{q}_j G_h(\mathbf{r})} - 1\right],$$

so that, at $\Lambda' = \Lambda - d\Lambda$,

$$a_1^{ij} = \alpha_1 K \frac{d\Lambda}{\Lambda^5}\left(\vec{q}_i\vec{q}_j + \frac{\Lambda^2}{m_H^2}\right),\ a_2^{ij} = \alpha_2 K \frac{d\Lambda}{\Lambda^3}\left(\vec{q}_i\vec{q}_j + \frac{\Lambda^2}{m_H^2}\right).$$

The main difference of Eq. (8.27) from Eq. (8.7) is due to the terms containing $\cos\left[\sqrt{K}\left(\vec{q}_i - \vec{q}_j\right)\vec{\chi}_{\Lambda'}\right]$, which violate the RG invariance. Nevertheless, this invariance does hold approximately, since the respective sums are dominated by the terms with $i = j$. Working within this approximation and making use of the identity $\sum\limits_i q_i^\alpha q_i^\beta = \frac{N}{2}\delta^{\alpha\beta}$, we obtain

$$\mathcal{Z}_\Lambda \simeq \exp\left[a_2 \frac{N(N-1)}{2}(\xi A(0)B(0))^2 \left(1 + \frac{\Lambda'^2}{{m_H}^2}\right)^{\frac{K}{2\pi}} V\right]\times$$
$$\times \int\limits_{0<p<\Lambda'} \mathcal{D}\vec{\chi}(\mathbf{p})\mathcal{D}\psi(\mathbf{p}) \exp\left\{\int d^2x \left\{\frac{1}{2}\psi_{\Lambda'}\left(\partial^2 - m_H^2\right)\psi_{\Lambda'} + \right.\right.$$
$$+2\xi A(0)B(0)\mathrm{e}^{\sqrt{K}\psi_{\Lambda'}} \sum_i \cos\left(\sqrt{K}\vec{q}_i\vec{\chi}_{\Lambda'}\right) -$$
$$\left.\left.-\frac{1}{2}\left[1 + \left(1 + \frac{\Lambda'^2}{{m_H}^2}\right)^{\frac{K}{2\pi}} \frac{NKa_1}{4}(\xi A(0)B(0))^2\right]\left(\partial_\mu \vec{\chi}_{\Lambda'}\right)^2\right\}\right\}. \tag{8.28}$$

This expression has again been derived in the leading order of the cumulant expansion applied in the average over ψ.

The shift of the free-energy density stemming from Eq. (8.28) (cf. Eq. (8.13)) is

$$F^{\rm new}-F\simeq a_2\frac{N(N-1)}{2}(\xi A(0)B(0))^2\left(1+\frac{K}{2\pi}\frac{\Lambda'^2}{{m_H}^2}\right). \qquad (8.29)$$

As far as the renormalization of fields and coupling constants is concerned, it is given by Eq. (8.11), where the first equation should be modified by ${\vec\chi_{\Lambda'}}^{\rm new}=C\vec\chi_{\Lambda'}$, and the parameter C from Eq. (8.12) now reads

$$C=\left[1+\frac{KNa_1}{4}(\xi A(0)B(0))^2\left(1+\frac{\Lambda'^2}{{m_H}^2}\right)^{\frac{K}{2\pi}}\right]^{1/2}\simeq$$

$$\simeq\left[1+\frac{KNa_1}{4}(\xi A(0)B(0))^2\left(1+\frac{\Lambda^2}{m_H^2}\frac{K}{2\pi}\right)\right]^{1/2}. \qquad (8.30)$$

From Eqs. (8.29) and (8.30) we deduce that, in the SU(N) case at $N>2$, the RG flow of couplings and of the free-energy density is identical to that of the SU(2) case. Indeed, all the N-dependence can be absorbed into the constants $\alpha_{1,2}$ by rescaling them to $\bar\alpha_1\equiv N\alpha_1/2$, $\bar\alpha_2=N(N-1)\alpha_2/2$ and further redefining (cf. the notation introduced after Eq. (8.20)) $\bar z=(8\pi)^2\bar\alpha_1y^2$ and $\bar a^{\rm new}=a\sqrt{8\pi\bar\alpha_1/\bar\alpha_2}$. In particular, the critical temperature $T_{\rm BKT}$ remains the same as in the SU(2) case. (As has already been mentioned, this also follows from the estimate of the mean squared separation in the monopole-anti-monopole molecule, if one takes into account that the square of any root vector is equal to unity.) Thus, the principal difference of the SU(N) case, $N>2$, from the SU(2) one is that, while in the SU(2) case the RG invariance is exact (modulo the negligibly small higher-order terms of the cumulant expansion applied to the average over ψ), in the SU(N) case it is only approximate, even in the limit $m_H\to\infty$.

8.3. *Finite-temperature 3D compact QED with massless fundamental fermions*

Let us consider the extension of the model (2.1) by fundamental dynamical quarks [55] (for the sake of simplicity, we omit the summation over the flavor indices, but consider the general case of an arbitrary number of flavors). The action is modified by

$$\Delta S=-i\int d^3x\bar\psi\left(\vec\gamma\vec D+h\frac{\tau^a}{2}\Phi^a\right)\psi\,,$$

where the Yukawa coupling h has the dimensionality $(\text{mass})^{1/2}$, $D_\mu\psi = \left(\partial_\mu - ig\frac{\tau^a}{2}A_\mu^a\right)\psi$, $\bar{\psi} = \psi^\dagger\beta$, the Euclidean Dirac matrices are defined as $\vec{\gamma} = -i\beta\vec{\alpha}$ with $\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\vec{\alpha} = \begin{pmatrix} 0 & \vec{\tau} \\ \vec{\tau} & 0 \end{pmatrix}$, and $\vec{\tau}$ denote the Pauli matrices. As will be demonstrated, at $T > T_{\rm BKT}$, quark zero modes in the monopole field lead to the additional attraction of a monopole and an anti-monopole (M and $\bar{M}$ for shortness) in the molecule. In particular, when the number of these modes (equal to the number of massless flavors) is sufficiently large, the molecule shrinks so strongly that its size becomes of the order of the inverse W boson mass. Another factor which defines the size of the molecule is the characteristic range of localization of zero modes. Namely, it can be shown that the stronger zero modes are localized in the vicinity of the monopole center, the smaller the molecular size is. We will consider the case when the Yukawa coupling h of quarks with the Higgs field vanishes, and massless quarks do not acquire any mass. This means that zero modes are maximally delocalized. Such a weakness of the quark-mediated interaction of monopoles opens the possibility for molecules to undergo eventually a phase transition into the plasma phase at the temperatures of the order of $T_{\rm BKT}$. However, this will be shown to occur only provided that the number of flavors is equal to one, whereas at any larger number of flavors, the critical temperature becomes exponentially small. This means that the interaction mediated by such a number of zero modes is already strong enough to maintain the molecular phase at any larger temperature.

For a short while, let us consider the general case $h \neq 0$. One can then see that the Dirac equation in the field of the third isotopic component of the 't Hooft–Polyakov monopole [27] decomposes into two equations for the components of the SU(2) doublet ψ. The masses of these components stemming from such equations are equal to each other and given by $m_q = h\eta/2$. Next, the Dirac equation in the full monopole potential has been shown [85] to possess a zero mode whose asymptotic behavior at $r \equiv |\vec{x}\,| \gg m_q^{-1}$ has the form

$$\chi^+_{\nu\,n} = \mathcal{N}\,\frac{{\rm e}^{-m_q r}}{r}\left(s^+_\nu s^-_n - s^-_\nu s^+_n\right), \quad \chi^-_{\nu\,n} = 0. \tag{8.31}$$

Here, $\chi^\pm_n$ are the upper and the lower components of the mode, i.e. $\psi = \binom{\chi^+_n}{\chi^-_n}$, $n = 1, 2$ is the isotopic index, $\nu = 1, 2$ is the Dirac index, $s^+ = \binom{1}{0}$, $s^- = \binom{0}{1}$, and $\mathcal{N}$ is the normalization constant.

It is a well known fact that, in 3D, the 't Hooft–Polyakov monopole is actually an instanton [3, 26]. Therefore, we can use the results of Ref. [86] on the quark contribution to the effective action of the instanton-anti-instanton molecule in QCD. Let us thus recapitulate the analysis of Ref. [86] adopting it to our model. To this end, we fix the gauge $\Phi^a = \eta\delta^{a3}$ and define the analogue of the free propagator S_0 by the relation $S_0^{-1} = -i\left(\vec{\gamma}\vec{\partial} + m_q\tau^3\right)$. Next, we define the propagator S_M in the field of a monopole located at the origin, $\vec{A}^{a\,M}$ $[A_i^{a\,M} \to \varepsilon^{aij}x^j/\left(gr^2\right)$ at $r \gg m_W^{-1}]$, by the formula $S_M^{-1} = S_0^{-1} - g\vec{\gamma}\frac{\tau^a}{2}\vec{A}^{a\,M}$. Obviously, the propagator $S_{\bar{M}}$ in the field of an anti-monopole located at a certain point $\vec{R}$, $\vec{A}^{a\,\bar{M}}(\vec{x}) = -\vec{A}^{a\,M}\left(\vec{x} - \vec{R}\right)$, is defined by the equation for S_M^{-1} with the replacement $\vec{A}^{a\,M} \to \vec{A}^{a\,\bar{M}}$. Finally, one can consider the molecule made out of the monopole and anti-monopole and define the total propagator S in the field of such a molecule, $\vec{A}^a = \vec{A}^{a\,M} + \vec{A}^{a\,\bar{M}}$, by means of the equation for S_M^{-1} with $\vec{A}^{a\,M}$ replaced by $\vec{A}^a$.

One can further introduce the notation $|\psi_n\rangle$, $n = 0, 1, 2, \ldots$, for the eigenfunctions of the operator $-i\vec{\gamma}\vec{D}$ defined at the field of the molecule, namely $-i\vec{\gamma}\vec{D}\,|\psi_n\rangle = \lambda_n\,|\psi_n\rangle$, where $\lambda_0 = 0$. This yields the formal spectral representation for the total propagator S,

$$S(\vec{x}, \vec{y}) = \sum_{n=0}^{\infty} \frac{|\psi_n(\vec{x})\rangle\,\langle\psi_n(\vec{y})|}{\lambda_n - im_q\tau^3}\ .$$

Next, it is worth recalling the mean-field approximation, discussed in section 3, within which zero modes dominate in the quark propagator, i.e.

$$S(\vec{x}, \vec{y}) \simeq \frac{|\psi_0(\vec{x})\rangle\,\langle\psi_0(\vec{y})|}{-im_q\tau^3} + S_0(\vec{x}, \vec{y})\,. \tag{8.32}$$

The approximation (8.32) remains valid for the molecular phase near the phase transition (i.e. when the temperature approaches $T_{\rm BKT}$ from above), we will be interested in. That is merely due to the fact that, in this regime, molecules become very much inflated being about to dissociate.

One now has $S = \left(S_M^{-1} + S_{\bar{M}}^{-1} - S_0^{-1}\right)^{-1} = S_{\bar{M}}\mathcal{S}^{-1}S_M$, where

$$\mathcal{S} = S_0 - (S_M - S_0)\,S_0^{-1}\,(S_{\bar{M}} - S_0) = S_0 - \frac{\left|\psi_0^M\right\rangle\left\langle\psi_0^M\right|}{-im_q\tau^3}\,S_0^{-1}\,\frac{\left|\psi_0^{\bar{M}}\right\rangle\left\langle\psi_0^{\bar{M}}\right|}{-im_q\tau^3}\ ,$$

and $\left|\psi_0^M\right\rangle$, $\left|\psi_0^{\bar{M}}\right\rangle$ are the zero modes of the operator $-i\vec{\gamma}\vec{D}$ defined in the field of a monopole and an anti-monopole respectively. Denoting further $a = \left\langle\psi_0^{\bar{M}}\middle|g\vec{\gamma}\frac{\tau^a}{2}\vec{A}^{a\,M}\middle|\psi_0^M\right\rangle$, it is straightforward to see, by the definition of the zero mode, that $a = \left\langle\psi_0^{\bar{M}}\right| - i\vec{\gamma}\,\vec{\partial}\left|\psi_0^M\right\rangle = \left\langle\psi_0^{\bar{M}}\middle|S_0^{-1}\middle|\psi_0^M\right\rangle$. This yields $\mathcal{S} = S_0 + (a^*/m_q^2)\left|\psi_0^M\right\rangle\left\langle\psi_0^{\bar{M}}\right|$, where the star stands for complex conjugation, and therefore $\det \mathcal{S} = \left[1 + (|a|/m_q)^2\right] \cdot \det S_0$. Finally, defining the desired effective action as $\Gamma = \ln[\det S^{-1}/\det S_0^{-1}]$, we obtain $\Gamma = \text{ const } + N_f \ln\left(m_q^2 + |a|^2\right)$ in the general case with N_f flavors. The constant in this formula, standing for the sum of effective actions defined at the monopole and at the anti-monopole, cancels out in the normalized expression for the mean squared separation in the $M\bar{M}$ molecule.

Let us now set h equal to zero, and so m_q is equal to zero as well. Notice first of all that, although in this case the direct Yukawa interaction of the Higgs boson with quarks is absent, they keep interacting with each other via the gauge field. Owing to this fact, the problem of finding a quark zero mode in the monopole field is still legitimate. (Note that, according to Eq. (8.31), this mode will be non-normalizable in the sense of a discrete spectrum. However, it is clear that, in the gapless case $m_q = 0$ under discussion, the zero mode, which lies exactly on the border of the two contiguous Dirac seas, should be treated not as an isolated state of a discrete spectrum, but rather as a state of a continuum spectrum. This means that it should be understood as follows: $|\psi_0(\vec{x})\rangle \sim \lim\limits_{p\to 0}\left(\mathrm{e}^{ipr}/r\right)$, where $p = |\vec{p}|$. Once considered in this way, zero modes are normalizable by the standard condition of normalization of the radial parts of spherical waves, R_{pl}, $\int\limits_0^\infty dr r^2 R_{p'l} R_{pl} = 2\pi\delta(p' - p)$ [92].) The dependence of the absolute value of the matrix element a on the distance R between a monopole and an anti-monopole can now be readily found. Indeed, we have $|a| \propto \int d^3r/\left(r^2\left|\vec{r} - \vec{R}\right|\right) \simeq -4\pi\ln(\mu R)$, where μ stands for the infrared cutoff.

We can now switch on the temperature and explore a possible modification of the BKT critical temperature, T_{BKT}, due to the zero-mode mediated interaction. As has been discussed, this can be done upon the evaluation of the mean squared separation in the $M\bar{M}$ molecule and further finding the temperature below which it starts diverging. In this way we should take into account that, in the dimensionally-reduced theory, the usual Coulomb interaction of monopoles (without loss of generality, we consider the molecule with the temporal component of $\vec{R}$ equal to zero) $R^{-1} = \sum\limits_{n=-\infty}^{+\infty}\left(\mathcal{R}^2 + (\beta n)^2\right)^{-1/2}$ goes over into $-2T\ln(\mu\mathcal{R})$, where $\mathcal{R}$ denotes the absolute value of the 2-vector $\mathbf{R}$. This statement can be checked e.g. by virtue of the Euler–MacLaurin formula. As far as the novel logarithmic interaction, $\ln(\mu R) = \sum\limits_{n=-\infty}^{+\infty}\ln\left[\mu\left(\mathcal{R}^2 + (\beta n)^2\right)^{1/2}\right]$, is concerned, it transforms into

$$\pi T\mathcal{R} + \ln\left[1 - \exp(-2\pi T\mathcal{R})\right] - \ln 2 \; . \tag{8.33}$$

Let us prove this statement. We make use of the formula [93]

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + x^2} = \frac{1}{2x}\left[\pi\coth(\pi x) - \frac{1}{x}\right].$$

This yields

$$x\sum_{n=-\infty}^{+\infty} \frac{1}{x^2 + (2\pi n/a)^2} = \frac{1}{x} + \frac{xa^2}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2 + (xa/2\pi)^2} = \frac{a}{2}\coth\left(\frac{ax}{2}\right).$$

On the other hand, the l.h.s. of this expression can be written as

$$\frac{1}{2}\frac{d}{dx}\sum_{n=-\infty}^{+\infty}\ln\left[x^2 + \left(\frac{2\pi n}{a}\right)^2\right].$$

Integrating over x, we get

$$\sum_{n=-\infty}^{+\infty}\ln\left[x^2 + \left(\frac{2\pi n}{a}\right)^2\right] = a\int dx \coth\left(\frac{ax}{2}\right) =$$
$$= 2\ln\sinh\left(\frac{ax}{2}\right) = ax + 2\ln\left(1 - \mathrm{e}^{-ax}\right) - 2\ln 2 \; .$$

Setting $\frac{2\pi}{a} = \mu\beta$ and $x = \mu\mathcal{R}$ we arrive at Eq. (8.33).

Thus, the statistical weight of the quark-mediated $M\bar{M}$ interaction in the molecule at high temperatures has the form $\exp(-2N_f\ln|a|) \propto [\pi T\mathcal{R} + \ln[1-\exp(-2\pi T\mathcal{R})] - \ln 2]^{-2N_f}$. Accounting for both, (former) logarithmic and Coulomb, interactions we find that the mean squared separation $\langle L^2\rangle$ in the molecule as a function of T, g, and N_f is given by

$$\left\langle L^2\right\rangle = \frac{\int\limits_{m_W^{-1}}^{\infty} d\mathcal{R}\mathcal{R}^{3-\frac{8\pi T}{g^2}}\left[\pi T\mathcal{R} + \ln\left[1-\exp(-2\pi T\mathcal{R})\right] - \ln 2\right]^{-2N_f}}{\int\limits_{m_W^{-1}}^{\infty} d\mathcal{R}\mathcal{R}^{1-\frac{8\pi T}{g^2}}\left[\pi T\mathcal{R} + \ln\left[1-\exp(-2\pi T\mathcal{R})\right] - \ln 2\right]^{-2N_f}} \; . \tag{8.34}$$

At large $\mathcal{R}$, $\ln 2 \ll \pi T\mathcal{R}$ and $\left|\ln\left[1-\exp(-2\pi T\mathcal{R})\right]\right| \simeq \exp(-2\pi T\mathcal{R}) \ll \pi T\mathcal{R}$. Consequently, we see that $\left\langle L^2\right\rangle$ is finite at $T > T_{\rm BKT}^{N_f} = (2-N_f)g^2/4\pi$, which reproduces $T_{\rm BKT}$ at $N_f = 0$. For $N_f = 1$, the plasma phase is still present at $T < g^2/4\pi$. Instead, when $N_f \geq 2$, at temperatures which are parametrically larger than the temperature of dimensional reduction,

$\mathcal{O}(\zeta^{1/3})$, the monopole ensemble exists only in the molecular phase. At such temperatures, massless dynamical quarks produce their own deconfinement. They also destroy the monopole-based confinement of any other fundamental matter. Clearly, the temperature $T_{\rm BKT}^{(N_f=1)}$ is exponentially larger than $\zeta^{1/3}$ just as $T_{\rm BKT}$ is, which fully validates the idea of dimensional reduction. Note also that, at $N_f \gg \max\left\{1, 4\pi T/g^2\right\}$, $\sqrt{\langle L^2\rangle} \to m_W^{-1}$, which means that such a large number of zero modes shrinks the molecule to the minimal admissible size. In conclusion, notice that, in the presence of W bosons, the influence of *heavy* fundamental scalar matter to the dynamics of the deconfining phase transition has been studied in Ref. [56].

8.4. *3D Georgi–Glashow model and its supersymmetric generalization at finite temperature*

The crucial difference in the finite-temperature behavior of the 3D GG model [52] from that of 3D compact QED [51] stems from the presence in the former of the heavy charged $W^\pm$ bosons. Although being practically irrelevant at zero temperature due to their heaviness, at finite temperature these bosons, due to their thermal excitation, form a plasma whose density is compatible to that of monopoles. The density of the $W^\pm$ plasma is given by (see e.g. Ref. [94])

$$\rho_W = -6T\left\{\frac{\partial}{\partial\tilde\mu}\int\frac{d^2p}{(2\pi)^2}\ln\left[1-{\rm e}^{\beta(\tilde\mu-\varepsilon(\mathbf{p}))}\right]\right\}\Bigg|_{\tilde\mu=0} = 6\int\frac{d^2p}{(2\pi)^2}\frac{1}{{\rm e}^{\beta\varepsilon(\mathbf{p})}-1} =$$
$$= \frac{3m_W^2}{\pi}\int\limits_1^\infty\frac{dzz}{{\rm e}^{m_W\beta z}-1} \simeq \frac{3m_W^2}{\pi}\int\limits_1^\infty dzz{\rm e}^{-m_W\beta z} = \frac{3m_WT}{\pi}\left(1+\frac{T}{m_W}\right){\rm e}^{-m_W\beta}.$$

Here, $\tilde\mu$ stands for the chemical potential, $\varepsilon(\mathbf{p}) = \sqrt{\mathbf{p}^2+m_W^2}$, and the factor "6" represents the total number of spin states of W^+ and W^- bosons. We have also denoted $z \equiv \varepsilon(\mathbf{p})/m_W$ and taken into account that, since we want to stay in the plasma phase of monopoles and the temperatures under consideration should not exceed $T_{\rm BKT}$, $T \ll m_W$. Therefore, up to small corrections, $\rho_W = \frac{3m_WT}{\pi}{\rm e}^{-m_W\beta}$.

It has been found in [52], further elaborated in [53, 56], etc., and reviewed in [95] that, when the finite-temperature 3D compact QED is promoted, upon the incorporation of W bosons, to the finite-temperature 3D GG model, the critical temperature $T_{\rm BKT}$ and the U(1) universality class of the deconfining phase transition are changed to $T_c = \frac{g^2}{4\pi\epsilon}$ and the 2D-Ising universality class, respectively. In what follows, we will survey the main ideas of Ref. [52], proceeding further to the analysis of the SUSY 3D GG model at finite temperature [54].

The Lagrangian of 3D compact QED, rewritten in terms of the vortex operator $V = \mathrm{e}^{-ig_m\chi/2}$, is

$$\mathcal{L}_{3\mathrm{D}} = \frac{1}{2}(\partial_\mu\chi)^2 - 2\zeta\cos(g_m\chi) = \frac{2}{g_m^2}\,|\partial_\mu V|^2 - \zeta\left[V^2 + (V^*)^2\right],$$

and has explicit magnetic Z_2 symmetry [50]. At zero temperature, this symmetry is spontaneously broken, since $\langle V(\vec{x})V^*(0)\rangle \stackrel{|\vec{x}|\to\infty}{\longrightarrow} 1$. It can be shown [50] that the breakdown of the magnetic symmetry implies confinement. Since this symmetry is inherited in the 3D compact QED from the initial 3D GG model, the deconfinement phase transition, occurring in the latter at a certain temperature (equal to T_c), should be associated with the restoration of this very symmetry. Already this fact alone indicates that the phase transition in the finite-temperature 3D GG model should be of the same kind as that of the (Z_2 invariant) 2D Ising model, rather than the BKT phase transition of compact QED. It has been shown quantitatively in Ref. [52] that this is indeed the case if W bosons are additionally included in the compact QED Lagrangian. At finite temperature, this can be done by noting that W bosons are nothing but vortices of the χ field and they can be incorporated into the dimensionally-reduced Lagrangian, $\mathcal{L}_{2\mathrm{D}} = \frac{1}{2}(\partial_\mu\chi)^2 - 2\xi\cos(\sqrt{K}\chi)$, by adding to it the term $-2\mu\cos\tilde{\chi}$ (from now on, χ will denote the dual-photon field in the *dimensionally-reduced* theory). The field $\tilde{\chi}$, dual to the field χ, is defined through the relation $i\partial_\mu\tilde{\chi} = g\sqrt{\beta}\varepsilon_{\mu\nu}\partial_\nu\chi$. The fugacity of W bosons, μ, is proportional to their density ρ_W, therefore $\mu \propto m_W T\mathrm{e}^{-m_W\beta}$. (Note also that an alternative way to introduce W bosons into $\mathcal{L}_{2\mathrm{D}}$ has been proposed in Ref. [96]. Within that approach, the above definition of $\tilde{\chi}$ through χ is abolished. Instead, an extra interaction of these, now independent, fields of the type $i(\partial_{x_1}\chi)(\partial_{x_2}\tilde{\chi})$ appears.)

Owing to the novel cosine term, even in the absence of the monopole plasma, the dual photon never becomes massless [52]. Rather, its mass m_D increases, and the vacuum correlation length d decreases, with the increase of the temperature. Consequently, contrary to what we had in the case of the inverse BKT phase transition, where the correlation length was becoming infinite at $T > T_{\mathrm{BKT}}$, now it never becomes infinite. This result parallels the general expectation that, with the increase of the temperature in any local field theory, the degree of disorder becomes higher, and the correlation length decreases. Indeed, in the absence of monopoles (i.e. at $\xi = 0$),

$$\mathcal{L}_{2\mathrm{D}} = \frac{1}{2}(\partial_\mu \phi)^2 - 2\mu \cos\left(g\sqrt{\beta}\phi\right), \tag{8.35}$$

where $\phi \equiv \sqrt{T}\tilde{\chi}/g$. Therefore, $m_D^2(\beta) = 2\mu\beta g^2 \propto m_W g^2 \mathrm{e}^{-m_W \beta}$. We see that m_D grows with the decrease of β, i.e. with the increase of T, proving the above statement.

Evaluating the mean squared separation in the W^+W^- molecule in the same way as we did earlier for the $M\bar{M}$ molecule (cf. Eqs. (8.1) and (8.34)), we have $\left\langle L^2 \right\rangle \propto \int d^2\mathbf{R}\, \mathcal{R}^{2-\frac{g^2}{2\pi T}}$. The convergence of this integral at $T < T_{\mathrm{XY}} \equiv \frac{g^2}{8\pi}$ means that, in the absence of monopoles, with the increase of temperature W bosons pass from the molecular phase into the plasma one at $T = T_{\mathrm{XY}}$. In the absence of dynamical monopoles, a *static* $M\bar{M}$ pair becomes linearly confined at $T > T_{\mathrm{XY}}$. Indeed, the $M\bar{M}$ potential, $\mathcal{V}$, is related to the correlation function of two vortex operators as $\langle V(\mathbf{x})V^*(\mathbf{y})\rangle = \mathrm{e}^{-\beta\mathcal{V}(\mathbf{x}-\mathbf{y})}$. In the phase where static monopoles are confined, one has

$$\langle V(\mathbf{x})V^*(\mathbf{y})\rangle \stackrel{|\mathbf{x}-\mathbf{y}|\to\infty}{\longrightarrow} \mathrm{e}^{-\beta\sigma|\mathbf{x}-\mathbf{y}|}, \tag{8.36}$$

and the magnetic Z_2 symmetry is restored. Let us clarify the origin of Eq. (8.36) and get an idea of the mass scale which appears in σ. To this end, notice that we are dealing with the situation dual to finite-temperature 3D compact QED below T_{BKT} [51]. In the latter case, external static quarks, represented by Polyakov loops, are linearly confined. Therefore, just as the correlation function of two Polyakov loops in that theory decreases with the distance at the inverse soliton mass, so does the correlation function of two vortex operators in our case. In fact, it can be shown that vortex operators create solitons in the theory (8.35). Solitons in this theory carry a unit topological charge, corresponding to the U(1) symmetry generated by the topological current $j_\mu = \frac{g\sqrt{\beta}}{2\pi}\varepsilon_{\mu\nu}\partial_\nu\phi$. In the dimensionally-reduced theory, the 3D vortex operator, $V(\vec{x}) = \mathrm{e}^{-ig_m\chi(\vec{x})/2}$, is replaced by $V(\mathbf{x}) = \mathrm{e}^{-i\sqrt{K}\chi(\mathbf{x})/2}$, or equivalently,

$$V(\mathbf{x}) = \exp\left(\frac{\sqrt{K}}{2}\varepsilon_{\mu\nu}\int\limits_{\mathbf{x}} dx_\mu \partial_\nu \phi\right). \tag{8.37}$$

The integration contour here goes either to infinity or to a point where there is a vortex operator of the opposite magnetic charge,

$$V^*(\mathbf{y}) = \exp\left(\frac{\sqrt{K}}{2}\varepsilon_{\mu\nu}\int^{\mathbf{y}} dx_\mu \partial_\nu \phi\right). \tag{8.38}$$

One can further demonstrate that the operators (8.37) and (8.38) have topological charges 1 and -1 and are nothing but the creation operators of a soliton and an anti-soliton respectively. This fact stems from the relations $\langle QV(0)\rangle = \langle V(0)\rangle$ and $\langle QV^*(0)\rangle = -\langle V^*(0)\rangle$, where $Q = \varepsilon_{\mu\nu}\oint_C dx_\mu j_\nu(\mathbf{x})$ is the topological charge operator with the contour C encircling the origin anticlockwise. These relations stem from others,

$$\langle j_\mu(\mathbf{x})V(0)\rangle \simeq -\frac{x_\mu}{2\pi\mathbf{x}^2}\langle V(0)\rangle, \qquad \langle j_\mu(\mathbf{x})V^*(0)\rangle \simeq \frac{x_\mu}{2\pi\mathbf{x}^2}\langle V^*(0)\rangle$$

(where "$\simeq$" means "neglecting correlation functions of topological currents higher than the two-point one"), that themselves can be obtained from the two-point correlation function of topological currents,

$$\langle j_\mu(\mathbf{x}j_\nu(0)\rangle = \frac{g^2\beta}{(2\pi)^3}\left(2\frac{x_\mu x_\nu}{\mathbf{x}^2} - \delta_{\mu\nu}\right)\frac{1}{\mathbf{x}^2}.$$

Owing to the fact that the operator (8.37) ((8.38)) creates a soliton (anti-soliton), the large-distance behavior of the correlation function (8.36) is determined by the lightest state with unit topological charge, i.e. by the soliton. One can therefore identify $\beta\sigma$ on the r.h.s. of Eq. (8.36) with the soliton mass $m_{\rm sol}$, therefore $\sigma = Tm_{\rm sol}$.

We have considered above the two idealistic situations, namely when either dynamical monopoles or W bosons are absent. In reality, when both are present and interact with each other, the deconfining phase transition temperature is neither $T_{\rm BKT}$ nor $T_{\rm XY}$, but is rather T_c, which in fact lies between the two. The deconfining phase transition takes place when the densities of monopoles and W bosons become equal, i.e. $2\xi = \rho_W$. Up to inessential pre-exponential factors, this happens when the monopole action S_0 is equal to $m_W\beta$, i.e. at $T = T_c$. At this temperature, the thickness of the string connecting a W^+ and a W^- boson is of the order of its length. As has been discussed (cf. e.g. after Eq. (3.17)) the thickness of the string is the inverse Debye mass of the dual photon, $d \propto \zeta^{-1/2}$. The length of the string is clearly of the order of the average distance between W's, that is $\mu^{-1/2}$. Therefore, again up to preexponential factors, the thickness and the length of the string are equal at $T = T_c$. Note that these qualitative arguments can be formalized by the RG procedure [52]. In the presence of

W bosons, the RG equations possess three fixed points. The first two are the zero- and infinite-temperature ones. The third fixed point of the RG flow is a nontrivial infrared unstable one, $T = T_c$, $\xi = \mu$, which corresponds to the phase transition. (At this point, however, both fugacities become infinite. It has been demonstrated in Ref. [96] that a correcting factor $(\epsilon + 2)/(2\epsilon + 1)$, by which T_c should be multiplied, appears if one demands that the RG flow of the fugacities should stop when at least one of them becomes of the order of unity.)

Note that an independent condition for the deconfining phase transition stems from the coincidence of the scaling dimensions of the operators $:\cos\left(\sqrt{K}\chi\right):$ and $:\cos\left(g\sqrt{\beta}\phi\right):$ at the critical temperature. The scaling dimensions are equal $\frac{K}{4\pi}$ and $\frac{g^2\beta}{4\pi}$ respectively, so that the monopole cosine term is relevant at $T < T_{\rm BKT}$, whereas the cosine term of the W bosons is relevant at $T > T_{\rm XY}$. The two scaling dimensions become equal when $g_m^2 T = g^2\beta$, which yields the critical temperature $\frac{g^2}{4\pi}$. As we see, this value is indeed equal to T_c, up to the factor $\frac{1}{\epsilon}$, which is of the order of unity and is generated by the loop corrections. At the temperature $\frac{g^2}{4\pi}$, both scaling dimensions are equal to unity, therefore both cosine terms in $\mathcal{L}_{\rm 2D}$ are relevant at this temperature. In fact, in the whole region of temperatures $T_{\rm XY} < T < T_{\rm BKT}$, both terms are relevant.

The supersymmetric version of the 3D GG model [54] contains, in addition to the dual photon, its superpartner – an adjoint fermion, which we will call the photino. This model possesses a discrete parity symmetry, which should lead to the masslessness of photinos. At zero temperature, this is however not the case, since the parity is spontaneously broken via a non-vanishing photino condensate. Thus, at finite temperature, one can anticipate two phase transitions – one related to the vanishing of the photino condensate and the other one due to deconfinement. These two transitions could either be distinct and happen at different temperatures, or could coincide. In this respect, the model is similar to QCD with adjoint quarks, where a similar question can be asked about the (non-)coincidence of deconfinement and restoration of discrete chiral symmetry.

The Lagrangian of the model contains the bosonic fields of the nonsupersymmetric GG model: the massless photon, the heavy $W^{\pm}$ vector bosons and the massive Higgs field, as well as their superpartners – photino, winos and Higgsino. It has been shown in [97] that, just like in the GG model, the monopole effects render the dual photon massive, although the mass in this case is parametrically smaller, since it is due to the contribution from a two-monopole sector, rather than a single-monopole sector as in the GG model. The low-energy sector of the theory is described by the supersymmetric sine-Gordon model. Its Euclidean action in the superfield notation is (we adopt here the notations of Ref. [98], in particular $\int d^2\theta\bar{\theta}\theta = 1$)

$$S = -\int d^3x d^2\theta \left[\frac{1}{2}\,\Phi\bar{D}_\alpha D_\alpha\Phi + \bar{\zeta}\cos\left(g_m\Phi\right)\right]. \tag{8.39}$$

In this equation, the scalar supermultiplet and supercovariant derivatives have the form

$$\Phi(\vec{x},\theta) = \chi + \bar{\theta}\lambda + \frac{1}{2}\bar{\theta}\theta F, \quad D_\alpha = \frac{\partial}{\partial\bar{\theta}_\alpha} - (\hat{\partial}\theta)_\alpha, \quad \bar{D}_\alpha = \frac{\partial}{\partial\theta_\alpha} - (\bar{\theta}\hat{\partial})_\alpha\ .$$

Here, χ again denotes the dual-photon field (real scalar), λ is the photino field, which is the two-component Majorana spinor ($\bar{\lambda} = \lambda^T\tau^2$), F is an auxiliary scalar field, $\hat{\partial} \equiv \gamma_i\partial_i$, and $\vec{\gamma} = \vec{\tau}$. The monopole fugacity $\bar{\zeta}$ has dimensionality $[\text{mass}]^2$ and is exponentially small. In terms of the disorder operator, the action (8.39) in component notation can be rewritten, up to an inessential constant, as

$$S = \int d^3x \left[\frac{2}{g_m^2}\,|\partial_\mu V|^2 - \frac{1}{2}\,\bar{\lambda}\hat{\partial}\lambda - \frac{g_m^2\zeta}{2}\left(V^2 + V^{*2}\right)\bar{\lambda}\lambda - \frac{(g_m\zeta)^2}{2}\left(V^4 + V^{*4}\right)\right],$$

where $\zeta = \bar{\zeta}/4$. Besides the magnetic Z_2 symmetry, this action has an additional discrete parity symmetry inherited from the full supersymmetric GG action,

$$V(x_1,x_2,x_3) \to iV(-x_1,x_2,x_3), \quad \lambda(x_1,x_2,x_3) \to \tau^3\lambda(-x_1,x_2,x_3)\ .$$

The photino mass term is odd under the parity transformation. Thus, the photino can acquire a mass only if parity is spontaneously broken, which is indeed the case. The breaking of parity results in a non-vanishing photino condensate $\langle\bar{\lambda}\lambda\rangle \sim g_m^2\zeta m_W$ and leads to a non-vanishing photino mass $m = 2g_m^2\zeta$ (equal to the Debye mass of the dual photon). It can be proved [54] that the equality of the dual-photon and the photino masses is preserved on the quantum level as well.

At finite temperatures, one can integrate photinos out in the dimensionally-reduced theory. Again including W bosons, we arrive at a theory similar to the one we had in the non-supersymmetric case with action,

$$S_{\text{d.-r.}} = \int d^2x \left[\frac{1}{2}(\partial_\mu\chi)^2 - 2\bar{\xi}\cos\left(2\sqrt{K}\chi\right) - 2\mu\cos\left(g\sqrt{\beta}\phi\right)\right], \tag{8.40}$$

where $\bar{\xi} \propto \beta\zeta^2$ is a certain positive and exponentially small fugacity. The crucial difference from the non-supersymmetric GG model is due to the factor "2" in the term $\cos(2\sqrt{K}\chi)$, which makes the model (8.40) Z_4 invariant. Similarly to the non-supersymmetric case, up to higher-loop corrections, the condition for the determination of the deconfinement phase transition temperature reads $4g_m^2 T = g^2\beta$, yielding $T_c = \frac{g^2}{8\pi}$. The universality class of the phase transition is therefore Z_4 rather than Z_2, as it was in the non-supersymmetric case. Further, it has been argued in Ref. [54] that the parity order parameter, $\langle V^2 \rangle + \langle V^{*2} \rangle$, vanishes at $T > T_c$. Therefore, the parity restoring phase transition takes place at the same temperature T_c as the deconfining transition. While this is an interesting phenomenon, it seems to be somewhat non-generic. In particular, in 4D gauge theory with adjoint fermions there is no reason to expect the deconfining and chiral-symmetry restoring phase transitions to coincide. The physical order parameter for deconfinement is the 't Hooft loop V [99], while for chiral symmetry it is the fermionic bilinear form $\bar{\lambda}\lambda$. In four dimensions, the two have a very different nature. While $\bar{\lambda}\lambda$ is a local field, V is a string-like object. It is thus difficult to imagine these two order parameters combining into a single one as it is the case in the 3D theory discussed above. The lattice results indeed indicate that, at least in the SU(3) theory, in four dimensions the two transitions are distinct [100].

9. Summary

Below we would like to mention once again some specific issues discussed in this review.

- 3D GG model at $T = 0$, $N \geq 2$
  - $m_H < \infty \Rightarrow N$ must be smaller than $\exp[(\text{certain constant}) \times m_W/g^2]$, otherwise the Higgs vacuum is not stochastic;
  - $m_H = \infty \Rightarrow$ the string tension of the flat Wilson loop in the fundamental representation is obtained; it possesses an ambiguity in the numerical factor due to the exponentially large thickness of the string;
  - the Kalb–Ramond field, which incorporates both monopoles and free photons $\Rightarrow$ the numerical factor at all the string coupling constants (in particular, at the string tension) is fixed for an arbitrarily shaped surface, in the weak-field (low-density) approximation; a generalization of the theory of confining strings to the adjoint case and to k-strings: $\frac{\sigma_{\rm adj}}{\sigma_{\rm fund}} \simeq 2$ at $N \gg 1$; $\frac{\sigma_k}{\sigma_{\rm fund}} = \frac{k(N-k)}{N-1}$ (Casimir scaling) in

the low-density approximation; the leading non-diluteness correction is derived, such that, at $k \sim N \gg 1$, it can significantly distort the Casimir scaling;

– 4D case with the field-theoretical θ term in the strong-coupling regime $\Rightarrow$ the string θ term; fundamental, adjoint, and k case critical values of θ, at which crumpling might disappear; a modification of the vacuum structure due to the θ term.

- 3D GG model at $T \geq T_{\rm d.r.} \sim \zeta^{1/3}$, effects of W bosons are neglected

– $N = 2$, $m_H < \infty \Rightarrow \frac{g^2}{m_W}$ may not be larger than a certain function of $\frac{m_H}{m_W}$, otherwise the Higgs vacuum is not stochastic;

– Higgs-induced corrections to the BKT RG flow in the leading order in m_H^{-1}; m_H itself evolves very slowly in the vicinity of the BKT critical point; $N > 2 \Rightarrow$ unlike the SU(2) case, the RG invariance holds only modulo the approximation $\sum\limits_{ij} a_{ij} \cos[(\vec{q}_i - \vec{q}_j)\vec{b}] \simeq \sum\limits_i a_{ii}$, even at $m_H \to \infty$;

– in the presence of N_f dynamical fundamental quark flavors, at $N_f = 1$ and $m_q = 0$, $T_{\rm BKT} = \frac{g^2}{4\pi}$; at $N_f \geq 2$ and/or $m_q \neq 0$, any fundamental matter (including quarks themselves) is deconfined at $T \geq T_{\rm d.r.}$;

- W bosons are taken into account, supersymmetric generalization $\Rightarrow$ the deconfining and the discrete-parity restoring phase transitions occur at the same temperature, $\frac{g^2}{8\pi}$; the universality class of the deconfining phase transition is the same as in Z_4 invariant spin models.

- confining string

– the presence of the negative-stiffness term forces the introduction of a term into the string action that suppresses the formation of spikes and, thus, prevents the crumpling of the world-sheet. This action, in the large-D limit, has an infrared stable fixed point at zero stiffness, which corresponds to a tensionless smooth string with Hausdorff dimension 2;

– the effective theory describing the infrared behavior of the confining string is a conformal field theory with central charge $c = 1$;

– at high temperature, the free energy of the confining string goes like $F^2(\beta) \propto -1/\beta^2$ with $\beta = 1/T$ and it agrees in sign, temperature behavior, and the reality property with the large-N QCD result obtained by Polchinski.

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